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STABILITY CRITERIA OF LINEAR NEUTRAL SYSTEMS WITH DISTRIBUTED DELAYS

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In this paper, stability of linear neutral systems with distributed delay is investigated. A bounded half circular region which includes all unstable characteristic roots, is obtained. Using the argument principle, stability criteria are derived which are necessary and sufficient conditions for asymptotic stability of the neutral systems. The stability criteria need only to evaluate the characteristic function on a straight segment on the imaginary axis and the argument on the boundary of a bounded half circular region. If there are no characteristic roots on the imaginary axis, the number of unstable characteristic roots can be obtained. The results of this paper extend those in the literature. Numerical examples are given to illustrate the presented results.

Keywords: neutral systems, distributed delay, stability criteria

Classification: 65L07,34K06

1. INTRODUCTION

We are concerned with the asymptotic stability of linear neutral systems with distributed delay [7] described by

$$\begin{cases} \dot{x}(t) = \int_{-\tau}^{0} [\,\mathrm{d}R(\theta)]x(t+\theta) + \int_{-\tau}^{0} \mathrm{d}[K(\theta)]\dot{x}(t+\theta), \quad t \ge 0, \\ x(\theta) = \varphi(\theta), \quad -\tau < \theta \le 0, \end{cases}$$

where $x(t) \in \mathcal{R}^n$, $R(\theta), K(\theta) \in \mathcal{R}^{n\times n}$, and $\varphi \in C^1(-\tau, 0]$. The integrals in (1) are Rieman–Stiltjes ones. Here the entries r_{ij} of the matrix R are functions of bounded variation and with bounded first moments, i. e.,

$$\int_{-\tau}^{0} |\theta|\, |\,\mathrm{d}r_{ij}(\theta)| < \infty, i, j = 1, \ldots, n. \tag{1}$$

Also, by definition,

$$\int_{-\tau}^{0} [\,\mathrm{d}K(\theta)]\dot{x}(t+\theta) = \sum_{j=1}^{m} L_j \dot{x}(t-h_j) + \int_{-\tau}^{0} M(\theta)\dot{x}(t+\theta)\,\mathrm{d}\theta, \tag{2}$$

where the constants τ and $h_j \geq 0$, the matrices L_j and the absolutely integrable matrix $M(\theta)$ with entries m_{ij} satisfy

$$\sum_{j=1}^{m} \|L_j\| + \int_{-\tau}^{0} \|M(\theta)\| \, \mathrm{d}\theta \leq \alpha < 1. \tag{3}$$

The associated difference equation for (1) is as follows

$$x(t) = \sum_{j=1}^{m} L_j x(t - h_j) + \int_{-\tau}^{0} M(\theta) x(t + \theta) \, \mathrm{d}\theta. \tag{4}$$

When matrices $R(\theta)$ and $K(\theta)$ in system (1) are piece-wise constant on $(-\tau, 0]$, i. e., they are step functions with finite number of discontinuities, system (1) reduces to a linear neutral system with discrete delays as follows

$$\dot{x}(t) = Ax(t) + \sum_{j=1}^{m} [B_j x(t - \tau_j) + C_j \dot{x}(t - \tau_j)], \tag{5}$$

where $x(t) \in \mathcal{R}^n$, A, B_j and $C_j \in \mathcal{R}^{n \times n}$ are constant matrices and τ_j for $j = 1, \ldots, m$ stand for positive constant delays. The associated difference equation for (5) is as follows

$$x(t) = \sum_{j=1}^{m} [C_j x(t - \tau_j)]. \tag{6}$$

Since the spectrum of the neutral system tend to the spectrum of the associated difference equation at high frequencies, the neutral system may have infinitely many unstable roots. The stability of the neutral system may be very sensitive to small changes in the delay. In order to discuss the sensitiveness of the stability, the concept of strong stability has been introduced by [2, 3]. If the associated difference equation is strongly stable, the neutral system has at most finitely many unstable roots [3, 10, 12]. If the neutral system is stable and the associated difference equation is strongly stable, the stability of the neutral system is insensitive to small changes in delays [3, 10, 12].

Stability of system (1) has been investigated in [7] via the characteristic function. Several stability criteria for system (5) have been given in [2, 4, 5, 6, 9, 11]. Recently for system (5) bounded regions which include all unstable characteristic roots are obtained in [5], and based on the bounded regions several stability criteria have been obtained.

This work is motivated by [7]. Following the same line as in [5, 7], for system (1), a bounded half circular region which includes all unstable characteristic roots, is obtained using matrix norms. Then stability criteria for system (1) are presented using the argument principle. Furthermore, if there are no characteristic roots on the imaginary axis, the number of unstable characteristic roots can be obtained.

Throughout the paper, $|x|$ and $||F||$ stand for the norm of a vector x and the norm of a matrix F, respectively. The jth eigenvalue of F is denoted by $\lambda_j(F)$, $\Re z$ and $\Im z$ stand for the real part and the imaginary part of a complex number z, respectively. The symbol $\sup_{t \in \omega} f(t)$ stands for the supremum of the numbers $f(t)$ where $t \in \omega$.

2. PRELIMINARIES

In this section, some definitions and lemmas are reviewed which will be used to state and prove the main results of the paper.

Assume that the conditions (1)–(3) hold for system (1) and the initial function $\varphi(\theta)$ is continuously differentiable

$$\sup_{-\tau\le t\le 0} |\varphi(t)| < \infty, \qquad \sup_{-\tau\le t\le 0} |\dot{\varphi}(t)| < \infty \tag{7}$$

and

$$\|\varphi\|_1 = |\varphi(0)| + |\dot{\varphi}(0)| + \Big[\int_0^\infty |F_0|^2(t)\,\mathrm{d}t\Big]^{1/2} + \Big[\int_0^\infty |F_1|^2(t)\,\mathrm{d}t\Big]^{1/2} < \infty, \tag{8}$$

where

$$F_0(t) = \int_{-\tau}^{-t} [\,\mathrm{d}R(\theta)]\varphi(t+\theta) \quad 0 \le t < \infty, \tag{9}$$

and

$$F_1(t) = \int_{-\tau}^{-t} [\,\mathrm{d}K(\theta)]\varphi(t+\theta) \quad 0 \le t < \infty. \tag{10}$$

Under these assumptions, i.e., (1)–(3), (7) and (8), problem (1) has the unique solution $x(t,\varphi)$ and there exists a Laplace transform of the solution [7]. The characteristic equation of system (1) is

$$P(z) = \det[zI - z\bar{K}(z) - \bar{R}(z)] = 0, \tag{11}$$

whose root is called a characteristic root, where

$$\begin{cases} \bar{R}(z) = \displaystyle\int_{-\tau}^{0} \exp(z\theta)\,\mathrm{d}R(\theta), \\ \bar{K}(z) = \displaystyle\int_{-\tau}^{0} \exp(z\theta)\,\mathrm{d}K(\theta). \end{cases}$$

Now we consider the asymptotic stability of system (1). Our results are related to the work of Kolmanovskii and Myshkis, see [7]. The following lemmas will be used to prove the main results.

Lemma 2.1. (Kolmanovskii and Myshkis [7]) Let characteristic equation (11) has no zeros in the half-plane $\Re z \ge 0$ and kernels R and K satisfy conditions (1)–(3), respectively. Then system (1) is asymptotically stable.

Remark 2.2. The condition (3) is a sufficient condition for strong stability of difference system (4). The above lemma [7] also required condition (3). In the case of finite delays, i.e., for difference system (6), condition (3) becomes $\sum_{j=1}^{m} \|C_j\| < 1$. We have to point out that condition (3) is very conservative. It is a further topic to improve condition (3) for obtaining a bounded region including all the unstable roots.

Lemma 2.3. (Lancaster [8]) If $\|\cdot\|$ denotes any matrix norm for which $\|I\| = 1$, and if $\|F\| < 1$, then $(I - F)^{-1}$ exists and

$$\|(I - F)^{-1}\| \leq \frac{1}{1 - \|F\|}.$$

The argument principle can be stated as follows.

Lemma 2.4. (Brown [1]) Suppose that

(i) a function $G(s)$ is analytic throughout the domain D except for poles, the domain D is interior to a positively oriented simple closed counter γ;

(ii) $G(s)$ is analytic and nonzero on γ;

(iii) counting multiplicities, Z is the number of zeros and Y is the number of poles of $G(s)$ inside γ.

Then

$$\frac{1}{2\pi} \triangle_\gamma \arg G(s) = Z - Y, \tag{12}$$

where, change of the argument of $G(s)$ along the closed line γ is defined by

$$\triangle_\gamma \arg G(s) = \arg G(\gamma_2) - \arg G(\gamma_1),$$

where γ_1 and γ_2 stand for the starting point and final point of γ, respectively.

3. MAIN RESULTS

In this section, the main results of the present paper will be derived. First we will consider the asymptotic stability of system (1) with conditions (1)–(3). If there exist unstable characteristic roots, i. e., $P(z) = 0$ for $\Re z \geq 0$, we will give a bounded half circular region in the complex plane which includes all the unstable characteristic roots of (11).

Theorem 3.1. Assume that conditions (1)–(3) hold and for the kernel R

$$\int_{-\tau}^{0} \| \mathrm{d}R(\theta)\| \leq \beta, \tag{13}$$

where β is a positive constant. Let z be a characteristic root of Eq. (11) with $\Re z \geq 0$, then

$$|z| \leq \frac{\beta}{1 - \alpha}, \tag{14}$$

where α is given by (3).

P r o o f. Since z is a characteristic root of Eq. (11) with $\Re z \geq 0$, we have

$$P(z) = \det[zI - z\bar{K}(z) - \bar{R}(z)] = 0,$$

where $\bar{R}$ and $\bar{K}$ are defined by

$$\begin{aligned}\bar{R}(z) &= \int_{-\tau}^{0} \exp(z\theta)\, \mathrm{d}R(\theta),\\ \bar{K}(z) &= \int_{-\tau}^{0} \exp(z\theta)\, \mathrm{d}K(\theta).\end{aligned}$$

By condition (2), we have

$$\begin{aligned}\bar{K}(z) &= \int_{-\tau}^{0} \exp(z\theta)\, \mathrm{d}K(\theta)\\ &= \sum_{j=1}^{m} L_j \exp(-zh_j) + \int_{-\tau}^{0} M(\theta)\exp(z\theta)\, \mathrm{d}\theta.\end{aligned}$$

From condition (3) and $\Re z \geq 0$,

$$\begin{aligned}|\bar{K}(z)\| &= \Big\| \sum_{j=1}^{m} L_j \exp(-zh_j) + \int_{-\tau}^{0} M(\theta)\exp(z\theta)\, \mathrm{d}\theta \Big\| && (15)\\ &\leq \sum_{j=1}^{m} \|L_j\| + \int_{-\tau}^{0} \|M(\theta)\|\, \mathrm{d}\theta && (16)\\ &\leq \alpha. && (17)\end{aligned}$$

According to Lemma 2.3 and $\|\bar{K}(z)\| \leq \alpha < 1$, for $\Re z \geq 0$ the matrix $(I - \bar{K}(z))^{-1}$ exists. Therefore,

$$\begin{aligned}P(z) &= \det[zI - z\bar{K}(z) - \bar{R}(z)]\\ &= \det[I - \bar{K}(z)]\det[zI - (I - \bar{K}(z))^{-1}\bar{R}(z)]\\ &= 0,\end{aligned}$$

which means

$$\det[zI - (I - \bar{K}(z))^{-1}\bar{R}(z)] = 0 \qquad (18)$$

holds. By condition (13), for $\Re z \geq 0$ we have

$$\begin{aligned}\|\bar{R}(z)\| &= \Big\| \int_{-\tau}^{0} \exp(z\theta)\, \mathrm{d}R(\theta) \Big\| && (19)\\ &\leq \int_{-\tau}^{0} \|\, \mathrm{d}R(\theta)\| && (20)\\ &\leq \beta. && (21)\end{aligned}$$

Let matrix

$$W(z) = (I - \bar{K}(z))^{-1}\bar{R}(z),$$

there exists an integer j, $1 \leq j \leq n$ such that

$$z = \lambda_j(W(z)). \qquad (22)$$

From (22), (17), (21) and Lemma 2.3, we obtain that

$$\begin{aligned} |z| &= |\lambda_j(W(z))| \\ &\leq \|(I-\bar{K}(z))^{-1}\bar{R}(z)\| \\ &\leq \|(I-\bar{K}(z))^{-1}\|\|\bar{R}(z)\| \\ &\leq \frac{1}{1-\|\bar{K}(z)\|}\|\bar{R}(z)\| \\ &\leq \frac{\beta}{1-\alpha}. \end{aligned}$$

Thus the proof is completed. □

We need the following definition to present an equivalent version of Theorem 3.1.

Definition 3.2. Assume that conditions (1)–(3) hold. Define the constant $r = \frac{\beta}{1-\alpha}$. Let (ρ, θ) be polar coordinates. Let l_a be the straight segment which is on the imaginary axis, whose two terminal points are $d_1 = (r, \pi/2)$ and $d_2 = (r, -\pi/2)$, respectively. Let l_b be the half circumference on the right half plane defined by

$$l_b = \{(\rho, \theta) : \rho = r, -\pi/2 \leq \theta \leq \pi/2\}. \tag{23}$$

Furthermore, let $l = l_a \bigcup l_b$, and D stands for the set of a bounded half circular region surrounded by l. The boundary of D is l and $\bar{D} = D \bigcup l$. See Figure.

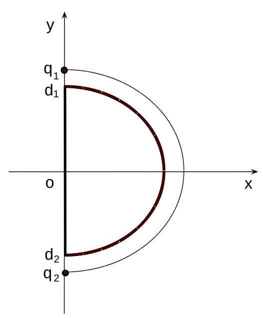

Fig. The regions D and Q.

Theorem 3.1 can be rewritten by means of Definition 3.2 as follows.

Corollary 3.3. Assume that conditions (1)–(3) hold. Let z be a characteristic root of Eq. (11) with $\Re z \geq 0$. Then

$$z \in \bar{D} \quad \text{for} \quad \Re z \geq 0,$$

where $\bar{D}$ is given by Definition 3.2, see Figure.

We need the following definition to present a simplified stability criterion.

Definition 3.4. Assume that conditions (1)–(3) hold. On the complex plane we take two points $q_1 = (r+\varepsilon, \pi/2)$ and $q_2 = (r+\varepsilon, -\pi/2)$, respectively, where $r = \frac{\beta}{1-\gamma}$ and $\varepsilon > 0$. On the imaginary axis, we have one straight segment $l'_a = q_1 q_2$. Let l'_b be the half circumference on the right half plane defined by

$$l'_b = \{(\rho, \theta) : \rho = r + \varepsilon, -\pi/2 \le \theta \le \pi/2\}. \tag{24}$$

Furthermore, let $l' = l'_a \bigcup l'_b$, and Q stands for the set of a bounded half circular region surrounded by l'. The boundary of Q is l' and $\bar{Q} = Q \bigcup l'$. It is obvious that $D \subset Q$, see Figure.

The following theorem will exclude all the unstable characteristic root of Eq. (11) from the set $\bar{D}$. Necessary and sufficient condition for asymptotic stability of system (1) are given by the argument principle.

Theorem 3.5. (i) Assume that conditions (1)–(3) hold. Then system (1) is asymptotically stable if and only if

$$P(z) \ne 0 \quad \text{for} \quad z \in l \tag{25}$$

and

$$\triangle_l \arg P(z) = 0 \tag{26}$$

hold, where $\arg P(z)$ stands for the argument of $P(z)$ and $\triangle_l \arg P(z)$ change of the argument of $P(z)$ along the closed half circle l.

A simplified version of the above stability criterion is as follows.

(ii) Assume that conditions (1)–(3) hold. Then system (1) is asymptotically stable if and only if

$$P(z) \ne 0 \quad \text{for} \quad z \in d_1 o \tag{27}$$

and

$$\triangle_{l'} \arg P(z) = 0 \tag{28}$$

hold, where o stands for the origin, i. e., $o = (0, 0)$, see Figure.

P r o o f. The proof of (i). Suppose system (1) is asymptotically stable. All zeros of $P(z)$ are on the left half plane. By Lemma 2.4 (the argument principle), we have that $P(z) \ne 0$ for $z \in l$ and (26) hold.

Conversely, assume that $P(z) \ne 0$ for $z \in l$ and (26) hold. Since $P(z)$ is an entire function, it has at most a finite number zeros in any bounded region. According to Lemma 2.1, for the asymptotic stability of (1) with conditions (1)–(3), we need to check whether $P(z) = 0$ for z with nonnegative real part. By means of Corollary 3.3, it is sufficient to check whether $P(z) = 0$ for $z \in \bar{D}$. Using Lemma 2.4, the proof of (i) is completed.

The proof of (ii). The simplification is obvious and straightforward. We only mention that the characteristic roots of Eq. (11) are symmetric with respect to the real axis since matrices R and K are real. It is sufficient to check whether $P(z) = 0$ for $z \in d_1 o$, the upper half part of $d_1 d_2$. □

Remark 3.6. Theorem 3.5 is related to the work of Kolmanovskii and Myshkis [7]. It is a simplified version of the work of Kolmanovskii and Myshkis. Although we only discuss the case of finite distributed delay, based on the result of Kolmanovskii and Myshkis [7], Theorem 3.5 can be extended to the case of infinite delay.

Remark 3.7. The regions D and Q in Theorem 3.5 are delay-independent. Since $d_1 o$ is located in the imaginary axis and a part of the boundary of region D, the computational effort for checking condition (27) is much less than checking condition (25). Furthermore, the computational effort for checking condition (28) is almost the same as checking condition (26) for a sufficiently small $\varepsilon > 0$.

Now we investigate the number of the unstable characteristic roots when there are no characteristic roots on the imaginary axis.

Theorem 3.8. Assume that conditions (1)–(3) hold and system (1) is unstable. If

$$P(z) \neq 0 \quad \text{for} \quad z \in d_1 o \tag{29}$$

and

$$\frac{1}{2\pi} \triangle_{l'} \arg P(z) = Z, \tag{30}$$

then the number of the unstable characteristic roots is Z, see Figure.

Proof. The proof is similar to Theorem 3.5. The characteristic roots of Eq. (11) are symmetric with respect to the real axis since matrices R and K are real. It is sufficient to check whether $P(z) = 0$ for $z \in d_1 o$, the upper half part of $d_1 d_2$. We have that

$$P(z) \neq 0 \quad \text{for} \quad z \in d_1 o \quad \text{implies} \quad P(z) \neq 0 \quad \text{for} \quad z \in l'. \tag{31}$$

From condition (29), we have (31), i. e., $P(z) \neq 0$ for $z \in l'$.

Since $P(z)$ is an entire function, it has at most a finite number zeros in any bounded region. According to Corollary 3.3, unstable roots are located in the region $\bar{D}$. Since $D \subset Q$, and (31), it is sufficient to compute the number of the unstable roots in Q. Using Lemma 2.4, the argument principle, the number of unstable characteristic roots is given by (30). The proof is completed. □

Remark 3.9. Theorem 3.8 shows that the number of unstable characteristic roots is given by the argument principle if there are no characteristic roots on the imaginary axis. The qualitative information is useful to compute numerically all the unstable characteristic roots.

Remark 3.10. Now we describe an algorithm to evaluate the argument over the half circle. The positive direction of l' is $q_1 \to d_1 \to o \to d_2 \to q_2 \to q_1$. Two sufficiently small positive constants ε and h are taken by a given accuracy.

On the imaginary axis, we have one straight segment $l'_a = q_1 q_2$. The interval $[-r - \varepsilon, r + \varepsilon]$ is divided into N_1 parts. The length of each part is h. We have

$$N_1 = 2(r + \varepsilon)/h.$$

To achieve the given accuracy, the h must be sufficiently small. We can check if $P(i\omega) = 0$ at each node. Then the argument of $P(i\omega)$ is evaluated for ω from $r + \varepsilon$ to $-r - \varepsilon$ which increases h successively.

The half circumference on the right half plane defined by

$$l'_b = \{(\rho, \theta) : \rho = r + \varepsilon, -\pi/2 \leq \theta \leq \pi/2\}.$$

then the $\rho = r + \varepsilon$ is fixed and the θ is the variable. The interval $[-\pi/2, \pi/2]$ is divided into N_2 parts. The length of the each part is h. We have

$$N_2 = \pi/h.$$

The argument of $P(\rho, \theta)$ is evaluated for θ from $-\pi/2$ to $\pi/2$ which increases h successively.

From the above algorithm, we know that the characteristic function $P(s)$ is a function of one real variable on l'. On the imaginary axis, $P(i\omega)$ is a function of ω which is from $r + \varepsilon$ to $-r - \varepsilon$. On the right half plane, since $\rho = r + \varepsilon$ is fixed, $P(\rho, \theta)$ is a function of θ which is from $-\pi/2$ to $\pi/2$. The way of practical evaluation of the argument over the half circle is essentially the same as the imaginary axis since the evaluation only involves the single variable θ. The algorithm will be used in Section 4.

The difference between the current work and the previous work [4, 5] is discussed in the following remark.

Remark 3.11. Delay in system (1) is distributed. While delay in system (5) is discrete. System (5) can be viewed as a special case of system (1). Furthermore, in [4], only sufficient conditions for stability of linear neutral system with a single delay, a special case of system (5), have been discussed. In [5], stability criteria for system (5) are given. In this work, applying Lemma 2.1, we derive Theorem 3.5 for system (1) which extends the main result in [5]. However, a similarly simplified version of the main result in [5] is not given in [5]. In addition, the number of unstable characteristic roots is not investigated in [5].

Now we discuss the difference between the current work and that of [12].

Remark 3.12. Recently Mikhaylov criterion has been extended to the case of neutral delay systems (5) by [12]. It needs checking if $P(i\omega) = 0$ and evaluating the argument of $P(i\omega)$ for ω from 0 to ∞. While Theorems 3.5 and 3.8 need checking if $P(i\omega) = 0$ on l'_a which is a finite segment of the imaginary axis and evaluating the argument of $P(s)$ on the bounded boundary l'. In the work of [12], the case of distributed delay is not investigated.

For system (5), assume that $\sum_{j=1}^{m} \|C_j\| < 1$. We have that

$$r = \frac{\beta}{1 - \alpha}, \tag{32}$$

where

$$\beta = \|A\| + \sum_{j=1}^{m} \|B_j\|, \quad \text{and} \quad \alpha = \sum_{j=1}^{m} \|C_j\|.$$

4. NUMERICAL EXAMPLES

In this section, we give two examples to illustrate Theorems 3.5 and 3.8. The 2-matrix norm $\|A\| = \sqrt{\lambda_{\max}(A^*A)}$ is used. Take $\varepsilon = h = 0.001$. The algorithm described in Section 4 is used, see Remark 3.10.

Example 1. Consider the linear neutral system with five discrete delays

$$\dot{x}(t) = Ax(t) + \sum_{j=1}^{5}[B_j x(t-\tau_j) + C_j \dot{x}(t-\tau_j)], \tag{33}$$

with

$$A = \begin{bmatrix} -1.2 & 2 \\ 4 & -1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 2 & -1.9 \\ 0.9 & 1.3 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -0.2 & 0.1 \\ 0.4 & 0.3 \end{bmatrix},$$

$$B_3 = \begin{bmatrix} 0.2 & -0.1 \\ 0.6 & 0.7 \end{bmatrix}, \quad B_4 = \begin{bmatrix} 0.6 & 0.1 \\ 0.3 & -0.5 \end{bmatrix}, \quad B_5 = \begin{bmatrix} 0.02 & -0.01 \\ 0.03 & 0.01 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 0.2 & 0.1 \\ -0.1 & 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0.1 & 0.2 \\ -0.1 & 0.2 \end{bmatrix}, \quad C_3 = \begin{bmatrix} 0.2 & 0.1 \\ 0 & -0.1 \end{bmatrix},$$

$$C_4 = \begin{bmatrix} 0.1 & 0 \\ 0 & -0.1 \end{bmatrix} \quad \text{and} \quad C_5 = \begin{bmatrix} 0.06 & -0.04 \\ 0 & -0.03 \end{bmatrix}.$$

We have

$$\alpha = 0.9273 < 1, \quad \beta = 9.4310, \quad \text{and} \quad r = 129.7249.$$

When the delays $\{\tau_1, \tau_2, \tau_3, \tau_4, \tau_5\} = \{0.1, 0.2, 0.3, 0.4, 0.5\}$, $\{1.1, 1.2, 1.3, 1.4, 1.5\}$ and $\{1, 2, 3, 4, 5\}$, respectively, Theorem 3.5 is not satisfied and the system is not asymptotically stable. Using Theorem 3.8, we can know the number of unstable roots is 1. By Newton's method, for $\{\tau_1, \tau_2, \tau_3, \tau_4, \tau_5\} = \{1, 2, 3, 4, 5\}$, we obtain that the unstable root is 1.8749.

When the delays $\{\tau_1, \tau_2, \tau_3, \tau_4, \tau_5\} = \{11, 12, 13, 14, 15\}$, Theorem 3.5 is not satisfied and the system is not asymptotically stable. Using Theorem 3.8, we can know the number of unstable roots is 3. By Newton's method, we obtain that the 3 unstable roots are 1.7302 and $0.0178 \pm 0.2765i$, respectively.

When $\{\tau_1, \tau_2, \tau_3, \tau_4, \tau_5\} = \{51, 52, 53, 54, 55\}$, we can know the number of unstable roots is 11 using Theorem 3.8. By Newton's method, we obtain that the 11 unstable roots are $1.7302, 0.005 \pm 0.0611i, 0.0045 \pm 0.1834i, 0.0037 \pm 0.3057i, 0.0024 \pm 0.4281i$ and $0.0007 \pm 0.5505i$, respectively.

Example 2. Consider the linear neutral system with distributed delay

$$\dot{x}(t) = Ax(t) + Bx(t-\tau) + L\dot{x}(t-\tau) + M\int_{-\tau}^{0} \dot{x}(t+\theta)\,\mathrm{d}\theta. \tag{34}$$

When the parameters of the system are

$$A = -3, \quad B = 2, \quad L = 0.2, \quad M = -0.1, \quad \text{and} \quad \tau = 1,$$

we have

$$\alpha = 0.3 < 1, \quad \beta = 5, \quad \text{and} \quad r = 7.1429.$$

Theorem 3.5 is satisfied and the system is asymptotically stable.

When the parameters of the system are

$$A = 5, \quad B = 1, \quad L = 0.1, \quad M = 0.8 \quad \text{and} \quad \tau = 1,$$

we have

$$\alpha = 0.9 < 1, \quad \beta = 6, \quad \text{and} \quad r = 60.$$

Theorem 3.5 is not satisfied and the system is not asymptotically stable. Using Theorem 3.8, we can know the number of unstable roots is 1. By Newton's method, we obtain that the unstable root is 5.8024.

When the parameters of the system are

$$A = \begin{bmatrix} 0.1 & 1 \\ 0.2 & -0.1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}, \quad L = \begin{bmatrix} 0.1 & 0.2 \\ 0.3 & 0.1 \end{bmatrix},$$

$$M = \begin{bmatrix} -0.1 & 0.2 \\ -0.2 & 0.1 \end{bmatrix}, \quad \text{and} \quad \tau = 1,$$

we have

$$\alpha = 0.6618 < 1, \quad \beta = 3.6263, \quad \text{and} \quad r = 10.7225.$$

Theorem 3.5 is not satisfied and the system is not asymptotically stable. Using Theorem 3.8, we can know the number of unstable roots is 1. By Newton's method, we obtain that the unstable root is 1.2338.

5. CONCLUSIONS

For neutral system (1) with conditions (1)–(3), using matrix norms, a bounded half circular region which includes all the unstable characteristic roots, is obtained. We have to point out that condition (3) is very conservative. It is a further topic to improve (3) for obtaining a bounded region including all the unstable characteristic roots.

Using the argument principle, stability criteria are derived which are necessary and sufficient for asymptotic stability of neutral system (1). We have to mention that the work of Kolmanovskii and Myshkis [7], Lemma 2.1 is crucial to prove Theorem 3.5. It is difficult to check the conditions of Lemma 2.1. Theorem 3.5 gives simple ways to check Lemma 2.1.

If there exist no characteristic roots on the imaginary axis, the number of unstable characteristic roots can be obtained by Theorem 3.8 for neutral system (1) with conditions (1)–(3). The qualitative information is useful to compute numerically all the unstable characteristic roots.

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