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SIMULTANEOUS SOLUTION OF LINEAR EQUATIONS AND INEQUALITIES IN MAX-ALGEBRA

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Let $a \oplus b = \max(a, b)$ and $a \otimes b = a + b$ for $a, b \in \mathbb{R}$. Max-algebra is an analogue of linear algebra developed on the pair of operations $(\oplus, \otimes)$ extended to matrices and vectors. The system of equations $A \otimes x = b$ and inequalities $C \otimes x \leq d$ have each been studied in the literature. We consider a problem consisting of these two systems and present necessary and sufficient conditions for its solvability. We also develop a polynomial algorithm for solving max-linear program whose constraints are max-linear equations and inequalities.

Keywords: max-algebra, linear equations and inequalities, max-linear programming

Classification: 15A06, 15A39, 90C26, 90C27

1. INTRODUCTION

Consider the following 'multi-machine interactive process' (MMIPP).

Products $P_1, \ldots, P_m$ are prepared using n machines, every machine contributing to the completion of each product by producing a semi-product. It is assumed that every machine can work on all products simultaneously and that all these actions on a machine start as soon as the machine is ready to work. Let a_{ij} be the duration of the work of the jth machine needed to complete the semi-product for P_i $(i = 1, \ldots, m; j = 1, \ldots, n)$. Let us denote by x_j the starting time of the jth machine $(j = 1, \ldots, n)$. Then, all semi-products for P_i $(i = 1, \ldots, m; j = 1, \ldots, n)$ will be ready at time $\max(a_{i1} + x_1, \ldots, a_{in} + x_n)$. If the completion times $b_1, \ldots, b_m$ are given for each product then the starting times have to satisfy the following system of equations:

$$\max(a_{i1} + x_1, \ldots, a_{in} + x_n) = b_i \text{ for all } i \in M.$$

Using the notation $a \oplus b = \max(a, b)$ and $a \otimes b = a + b$ for $a, b \in \mathbb{R}$ extended to matrices and vectors in the same way as in linear algebra, then this system can be written as

$$A \otimes x = b. \tag{1}$$

Any system of the form (1) is called 'one-sided max-linear system'. Also, if the requirement is that each product is to be produced on or before the completion times $b_1, \ldots, b_m$, then the starting times have to satisfy

$$\max(a_{i1} + x_1, \ldots, a_{in} + x_n) \leq b_i \text{ for all } i \in M,$$

which can also be written as

$$A \otimes x \leq b. \tag{2}$$

The system of inequalities (2) is called 'one-sided max-linear system of inequalities'. System of equation and inequalities were each discussed in the first papers that deal with max-algebra [7, 10] and [12]. Also see [2, 4] and an excellent monograph [3].

Since one-sided systems of linear equations and systems of inequalities in max-algebra have each received some attention in the past, we consider combination of the two as one system and the aim is to discuss the existence and uniqueness of solution to such system. We also present a method for solving max-linear programs whose constraints are equations and inequalities.

We introduce the following notations

$$\begin{aligned}
M &= \{1, 2, \ldots, m\}, \\
N &= \{1, 2, \ldots, n\}, \\
a^{-1} &= -a, \text{ for all } a \in \mathbb{R}, \\
S(A, b) &= \{x \in \mathbb{R}^n; A \otimes x = b\}, \\
M_j &= \left\{ k \in M; b_k \otimes a_{kj}^{-1} = \min_{i \in M} \left(b_i \otimes a_{ij}^{-1} \right) \right\} \text{ for all } j \in N, \\
\bar{x}_j &= \min_{i \in M} (b_i \otimes a_{ij}^{-1}) \text{ for all } j \in N, \\
\bar{x} &= (\bar{x}_1, \ldots, \bar{x}_n)^T, \\
S(A, b, \leq) &= \{x \in \mathbb{R}^n; A \otimes x \leq b\}.
\end{aligned}$$

The following theorems show how system of equation and inequalities can each be solved.

Theorem 1.1. (Zimmermann [12]) Let $A = (a_{ij}) \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Then $x \in S(A, b)$ if and only if

(i) $x \leq \bar{x}$ and

(ii) $\bigcup_{j \in N_x} M_j = M$ where $N_x = \{j \in N; x_j = \bar{x}_j\}$.

Theorem 1.2. (Zimmermann [12], Green [7]) $x \in S(A, b, \leq)$ if and only if $x \leq \bar{x}$.

The existence of a unique solution to the max-linear system $A \otimes x = b$ is described by the following result.

Theorem 1.3. (Butkovič [4]) Let $A = (a_{ij}) \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Then $S(A, b) = \{\bar{x}\}$ if and only if

(i) $\bigcup_{j \in N} M_j = M$ and

(ii) $\bigcup_{j \in N'} M_j \neq M$ for any $N' \subseteq N, N' \neq N$.

2. A SYSTEM OF EQUATIONS AND INEQUALITIES

Let $A = (a_{ij}) \in \mathbb{R}^{k\times n}$, $C = (c_{ij}) \in \mathbb{R}^{r\times n}$, $b = (b_1, \ldots, b_k)^T \in \mathbb{R}^k$ and $d = (d_1, \ldots, d_r)^T \in \mathbb{R}^r$. A *one-sided max-linear system with both equations and inequalities* is of the form:

$$\begin{aligned} A \otimes x &= b \\ C \otimes x &\leq d. \end{aligned} \tag{3}$$

We shall use the following notation throughout this paper

$$\begin{aligned}
R &= \{1, 2, \ldots, r\}, \\
S(A, C, b, d) &= \{x \in \mathbb{R}^n; A \otimes x = b \text{ and } C \otimes x \leq d\}, \\
S(C, d, \leq) &= \{x \in \mathbb{R}^n; C \otimes x \leq d\}, \\
\bar{\bar{x}}_j &= \min_{i\in R}(d_i \otimes c_{ij}^{-1}) \text{ for all } j \in N, \\
\bar{\bar{x}} &= (\bar{\bar{x}}_1, \ldots, \bar{\bar{x}}_n)^T, \\
K &= \{1, \ldots, k\}, \\
K_j &= \left\{ k \in K; b_k \otimes a_{kj}^{-1} = \min_{i\in K} \left(b_i \otimes a_{ij}^{-1}\right) \right\} \text{ for all } j \in N, \\
\bar{x}_j &= \min_{i\in K}(b_i \otimes a_{ij}^{-1}) \text{ for all } j \in N, \\
\bar{x} &= (\bar{x}_1, \ldots, \bar{x}_n)^T, \\
J &= \{j \in N; \bar{\bar{x}}_j \geq \bar{x}_j\} \text{ and} \\
L &= N \setminus J.
\end{aligned}$$

We also define the vector $\hat{x} = (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n)^T$, where

$$\hat{x}_j \equiv \begin{cases} \bar{x}_j & \text{if } j \in J \\ \bar{\bar{x}}_j & \text{if } j \in L, \end{cases} \tag{4}$$

and $N_{\hat{x}} = \{j \in N; \hat{x}_j = \bar{x}_j\}$.

Theorem 2.1. Let $A = (a_{ij}) \in \mathbb{R}^{k\times n}$, $C = (c_{ij}) \in \mathbb{R}^{r\times n}$, $b = (b_1, \ldots, b_k)^T \in \mathbb{R}^k$ and $d = (d_1, \ldots, d_r)^T \in \mathbb{R}^r$. Then the following three statements are equivalent:

(i) $S(A, C, b, d) \neq \emptyset$

(ii) $\hat{x} \in S(A, C, b, d)$

(iii) $\bigcup_{j\in J} K_j = K$.

Proof. *(i)* $\Longrightarrow$ *(ii)*. Let $x \in S(A, C, b, d)$, therefore $x \in S(A, b)$ and $x \in S(C, d, \leq)$. Since $x \in S(C, d, \leq)$, it follows from Theorem 1.2 that $x \leq \bar{\bar{x}}$. Now that $x \in S(A, b)$

and also $x \in S(C,d,\le)$, we need to show that $\bar{\bar{x}}_j \ge \bar{x}_j$ for all $j \in N_x$ (that is $N_x \subseteq J$). Let $j \in N_x$ then $x_j = \bar{x}_j$. Since $x \in S(C,d,\le)$ we have $x \le \bar{\bar{x}}$ and therefore $\bar{x}_j \le \bar{\bar{x}}_j$ thus $j \in J$. Hence, $N_x \subseteq J$ and by Theorem 1.1 $\bigcup_{j\in J} K_j = K$. This also proves $(i) \Longrightarrow (iii)$.

$(iii) \Longrightarrow (i)$. Suppose $\bigcup_{j\in J} K_j = K$. Since $\hat{x} \le \bar{\bar{x}}$ we have $\hat{x} \in S(C,d,\le)$. Also $\hat{x} \le \bar{x}$ and $N_{\hat{x}} \supseteq J$ gives $\bigcup_{j\in N_{\hat{x}}} K_j \supseteq \bigcup_{j\in J} K_j = K$. Hence $\bigcup_{j\in N_{\hat{x}}} K_j = K$, therefore $\hat{x} \in S(A,b)$ and $\hat{x} \in S(C,d,\le)$. Hence $\hat{x} \in S(A,C,b,d)$ (that is $S(A,C,b,d) \ne \emptyset$) and this also proves $(iii) \Longrightarrow (ii)$. □

Theorem 2.2. Let $A = (a_{ij}) \in \mathbb{R}^{k\times n}$, $C = (c_{ij}) \in \mathbb{R}^{r\times n}$, $b = (b_1,\dots,b_k)^T \in \mathbb{R}^k$ and $d = (d_1,\dots,d_r)^T \in \mathbb{R}^r$. Then $x \in S(A,C,b,d)$ if and only if

(i) $x \le \hat{x}$ and

(ii) $\bigcup_{j\in N_x} K_j = K$ where $N_x = \{j \in N\ ;\ x_j = \bar{x}_j\}$.

Proof. $(\Longrightarrow)$ Let $x \in S(A,C,b,d)$, then $x \le \bar{x}$ and $x \le \bar{\bar{x}}$. Since $\hat{x} = \bar{x} \oplus' \bar{\bar{x}}$ we have $x \le \hat{x}$. Also, $x \in S(A,C,b,d)$ implies that $x \in S(C,d,\le)$. It follows from Theorem 1.1 that $\bigcup_{j\in N_x} K_j = K$.

$(\Longleftarrow)$ Suppose that $x \le \hat{x} = \bar{x} \oplus' \bar{\bar{x}}$ and $\bigcup_{j\in N_x} K_j = K$. It follows from Theorem 1.1 that $x \in S(A,b)$, also by Theorem 1.2 $x \in S(C,d,\le)$. Thus $x \in S(A,b) \cap S(C,d,\le) = S(A,C,b,d)$. □

We introduce the symbol $|X|$ which stands for the number of elements of the set X.

Lemma 2.3. Let $A = (a_{ij}) \in \mathbb{R}^{k\times n}$, $C = (c_{ij}) \in \mathbb{R}^{r\times n}$, $b = (b_1,\dots,b_k)^T \in \mathbb{R}^k$ and $d = (d_1,\dots,d_r)^T \in \mathbb{R}^r$. If $|S(A,C,b,d)| = 1$ then $|S(A,b)| = 1$.

Proof. Suppose $|S(A,C,b,d)| = 1$, that is $S(A,C,b,d) = \{x\}$ for an $x \in \mathbb{R}^n$. Since $S(A,C,b,d) = \{x\}$ we have $x \in S(A,b)$ and thus $S(A,b) \ne \emptyset$. For contradiction, suppose $|S(A,b)| > 1$. We need to check the following two cases: (i) $L \ne \emptyset$ and (ii) $L = \emptyset$ where $L = N \setminus J$, and show in each case that $|S(A,C,b,d)| > 1$.

Proof of Case (i), that is $L \ne \emptyset$: Suppose that L contains only one element say $n \in N$ i.e $L = \{n\}$. Since $x \in S(A,C,b,d)$ it follows from Theorem 2.1that $\hat{x} \in S(A,C,b,d)$. That is $x = \hat{x} = (\bar{x}_1, \bar{x}_2, \dots, \bar{x}_{n-1}, \bar{\bar{x}}_n) \in S(A,C,b,d)$. It can also be seen that, $\bar{\bar{x}}_n < \bar{x}_n$ and any vector of the form $z = (\bar{x}_1, \bar{x}_2, \dots, \bar{x}_{n-1}, \alpha) \in S(A,C,b,d)$, where $\alpha \le \bar{\bar{x}}_n$. Hence $|S(A,C,b,d)| > 1$. If L contains more than one element, then the proof is done in a similar way.

Proof of Case (ii), that is $L = \emptyset$ $(J = N)$: Suppose that $J = N$. Then we have $\hat{x} = \bar{x} \le \bar{\bar{x}}$. Suppose without loss of generality that $x, x' \in S(A,b)$ such that $x \ne x'$. Then $x \le \bar{x} \le \bar{\bar{x}}$ and also $x' \le \bar{x} \le \bar{\bar{x}}$. Thus, $x, x' \in S(C,d,\le)$. Consequently, x, $x' \in S(A,C,b,d)$ and $x \ne x'$. Hence $|S(A,C,b,d)| > 1$. □

Theorem 2.4. Let $A = (a_{ij}) \in \mathbb{R}^{k\times n}$, $C = (c_{ij}) \in \mathbb{R}^{r\times n}$, $b = (b_1, \dots, b_k)^T \in \mathbb{R}^k$ and $d = (d_1, \dots, d_r)^T \in \mathbb{R}^r$. If $|S(A, C, b, d)| = 1$ then $J = N$.

Proof. Suppose $|S(A, C, b, d)| = 1$. It follows from Theorem 2.1 that $\bigcup_{j\in J} K_j = K$. Also, $|S(A, C, b, d)| = 1$ implies that $|S(A, b)| = 1$ (Lemma 2.3). Moreover, $|S(A, b)| = 1$ implies that $\bigcup_{j\in N} K_j = K$ and $\bigcup_{j\in N'} K_j \neq K, N' \subseteq N, N' \neq N$ (Theorem 1.3). Since $J \subseteq N$ and $\bigcup_{j\in J} K_j = K$, we have $J = N$. $\square$

Corollary 2.5. Let $A = (a_{ij}) \in \mathbb{R}^{k\times n}$, $C = (c_{ij}) \in \mathbb{R}^{r\times n}$, $b = (b_1, \dots, b_k)^T \in \mathbb{R}^k$ and $d = (d_1, \dots, d_r)^T \in \mathbb{R}^r$. If $|S(A, C, b, d)| = 1$ then $S(A, C, b, d) = \{\bar{x}\}$.

Proof. The statement follows from Theorem 2.4 and Lemma 2.3. $\square$

Corollary 2.6. Let $A = (a_{ij}) \in \mathbb{R}^{k\times n}$, $C = (c_{ij}) \in \mathbb{R}^{r\times n}$, $b = (b_1, \dots, b_k)^T \in \mathbb{R}^k$ and $d = (d_1, \dots, d_r)^T \in \mathbb{R}^k$. Then, the following three statements are equivalent:

(i) $|S(A, C, b, d)| = 1$.

(ii) $|S(A, b)| = 1$ and $J = N$.

(iii) $\displaystyle\bigcup_{j\in J} K_j = K$ and $\displaystyle\bigcup_{j\in J'} K_j \neq K$, for every $J' \subseteq J$, $J' \neq J$, and $J = N$.

Proof. $(i) \Longrightarrow (ii)$ Follows from Lemma 2.3 and Theorem 2.4.

$(ii) \Longrightarrow (i)$ Let $J = N$, therefore $\bar{x} \leq \bar{\bar{x}}$ and thus $S(A, b) \subseteq S(C, d, \leq)$. Therefore we have $S(A, C, b, d) = S(A, b) \cap S(C, d, \leq) = S(A, b)$. Hence $|S(A, C, b, d)| = 1$.

$(ii) \Longrightarrow (iii)$ Suppose that $S(A, b) = \{x\}$ and $J = N$. It follows from Theorem 1.3 that $\bigcup_{j\in N} K_j = K$ and $\bigcup_{j\in N'} K_j \neq K, N' \subseteq N, N' \neq N$. Since $J = N$ the statement now follows from Theorem 1.3.

$(iii) \Longrightarrow (ii)$ It is immediate that $J = N$ and the statement now follows from Theorem 1.3. $\square$

Theorem 2.7. Let $A = (a_{ij}) \in \mathbb{R}^{k\times n}$, $C = (c_{ij}) \in \mathbb{R}^{r\times n}$, $b = (b_1, \dots, b_k)^T \in \mathbb{R}^k$ and $d = (d_1, \dots, d_r)^T \in \mathbb{R}^k$. If $|S(A, C, b, d)| > 1$ then $|S(A, C, b, d)|$ is infinite .

Proof. Suppose $|S(A, C, b, d)| > 1$. By Corollary 2.6 we have $\bigcup_{j\in J} K_j = K$, for some $J \subseteq N$, $J \neq N$(that is $\exists j \in N$ such that $\bar{x}_j > \bar{\bar{x}}_j$). Now $J \subseteq N$ and $\bigcup_{j\in J} K_j = K$, Theorem 2.2 implies that any vector $x = (x_1, x_2, \dots, x_n)^T$ of the form

$$x_j \equiv \begin{cases} \bar{x}_j & \text{if } j \in J \\ y \leq \bar{\bar{x}}_j & \text{if } j \in L \end{cases}$$

is in $S(A, C, b, d)$, and the statement follows. $\square$

Remark 2.8. From Theorem 2.7 we can say that the number of solutions to the one-sided system containing both equations and inequalities can only be $0, 1, \text{or } \infty$.

The vector $\hat{x}$ plays an important role in the solution of the one-sided system containing both equations and inequalities. This role is the same as that of the principal solution $\bar{x}$ to the one-sided max-linear system $A \otimes x = b$, see [1] for more details.

3. MAX-LINEAR PROGRAM WITH EQUATION AND INEQUALITY CONSTRAINTS

Suppose that the vector $f = (f_1, f_2, \ldots, f_n)^T \in \mathbb{R}^n$ is given. The task of minimizing [maximizing]the function $f(x) = f^T \otimes x = \max(f_1 + x_1, f_1 + x_2 \ldots, f_n + x_n)$ subject to (3) is called max-linear program with one-sided equations and inequalities and will be denoted by $\text{MLP}_{\leq}^{\min}$ and [$\text{MLP}_{\leq}^{\max}$]. We denote the sets of optimal solutions by $S^{\min}(A, C, b, d)$ and $S^{\max}(A, C, b, d)$, respectively.

Lemma 3.1. (Butkovič and Aminu [5]) Suppose $f \in \mathbb{R}^n$ and let $f(x) = f^T \otimes x$ be defined on $\mathbb{R}^n$. Then,

(i) $f(x)$ is max-linear, i. e. $f(\lambda \otimes x \oplus \mu \otimes y) = \lambda \otimes f(x) \oplus \mu \otimes f(x)$ for every $x, y \in \mathbb{R}^n$.

(ii) $f(x)$ is isotone, i. e. $f(x) \leq f(y)$ for every $x, y \in \mathbb{R}^n$, $x \leq y$.

Note that it would be possible to convert equations to inequalities and conversely but this would result in an increase of the number of constraints or variables and thus increasing the computational complexity. The method we present here does not require any new constraint or variable.

We denote by

$$(A \otimes x)_i = \max_{j \in N}(a_{ij} + x_j).$$

A variable x_j will be called *active* if $x_j = f(x)$, for some $j \in N$. Also, a variable will be called *active* on the constraint equation if the value $(A \otimes x)_i$ is attained at the term x_j respectively. It follows from Theorem 2.2 and Lemma 3.1 that $\hat{x} \in S^{\max}(A, C, b, d)$. We now present a polynomial algorithm which finds $x \in S^{\min}(A, C, b, d)$ or recognizes that $S^{\min}(A, B, c, d) = \emptyset$. Due to Theorem 2.1 either $\hat{x} \in S(A, C, b, d)$ or $S(A, C, b, d) = \emptyset$. Therefore, we assume in the following algorithm that $S(A, C, b, d) \neq \emptyset$ and also $S^{\min}(A, C, b, d) \neq \emptyset$.

Theorem 3.2. The algorithm ONEMLP-EI is correct and its computational complexity is $O((k + r)n^2)$.

Proof. The correctness follows from Theorem 2.2 and the computational complexity is computed as follows. In Step 1 $\bar{x}$ is $O(kn)$, while $\bar{\bar{x}}$, $\hat{x}$ and K_j can be determined in $O(rn)$, $O(k + r)n$ and $O(kn)$ respectively. The loop 3-7 can be repeated at most $n - 1$ times, since the number of elements in J is at most n and in Step 4 at least one element will be removed at a time. Step 3 is $O(n)$, Step 6 is $O(kn)$ and Step 7 is $O(n)$. Hence loop 3-7 is $O(kn^2)$. $\square$

Algorithm 1 ONEMLP-EI
(Max-linear program with one-sided equations and inequalities)

Input: $f = (f_1, f_2, \ldots, f_n)^T \in \mathbb{R}^n$, $b = (b_1, b_2, \ldots b_k)^T \in \mathbb{R}^k$, $d = (d_1, d_2, \ldots d_r)^T \in \mathbb{R}^r$, $A = (a_{ij}) \in R^{k\times n}$ and $C = (c_{ij}) \in R^{r\times n}$.
Output: $x \in S^{\min}(A, C, b, d)$.

1. Find $\bar{x}$, $\bar{\bar{x}}$, $\hat{x}$ and K_j, $j \in J$;$J = \{j \in N; \bar{\bar{x}}_j \geq \bar{x}_j\}$
2. $x := \hat{x}$
3. $H(x) := \{j \in N; f_j + x_j = f(x)\}$
4. $J := J \setminus H(x)$
5. If
$$\bigcup_{j\in J} K_j \neq K$$
then stop ($x \in S^{\min}(A, C, b, d)$)
6. Set x_j small enough (so that it is not active on any equation or inequality) for every $j \in H(x)$
7. Go to 3

3.1. AN EXAMPLE

Consider the following system max-linear program in which $f = (5, 6, 1, 4, -1)^T$,

$$A = \begin{pmatrix} 3 & 8 & 4 & 0 & 1 \\ 0 & 6 & 2 & 2 & 1 \\ 0 & 1 & -2 & 4 & 8 \end{pmatrix}, \quad b = \begin{pmatrix} 7 \\ 5 \\ 7 \end{pmatrix},$$

$$C = \begin{pmatrix} -1 & 2 & -3 & 0 & 6 \\ 3 & 4 & -2 & 2 & 1 \\ 1 & 3 & -2 & 3 & 4 \end{pmatrix} \quad \text{and} \quad d = \begin{pmatrix} 5 \\ 5 \\ 6 \end{pmatrix}.$$

We now make a record run of Algorithm ONEMLP-EI. $\bar{x} = (5, -1, 3, 3, -1)^T$, $\bar{\bar{x}} = (2, 1, 7, 3, -1)^T$, $\hat{x} = (2, -1, 3, 3, -1)^T$, $J = \{2, 3, 4, 5\}$ and $K_2 = \{1, 2\}$, $K_3 = \{1, 2\}$, $K_4 = \{2, 3\}$ and $K_5 = \{3\}$. $x := \hat{x} = (2, -1, 3, 3, -1)^T$ and $H(x) = \{1, 4\}$ and $J \not\subseteq H(x)$. We also have $J := J \setminus H(x) = \{2, 3, 5\}$ and $K_2 \cup K_3 \cup K_5 = K$. Then set $x_1 = x_4 = 10^{-4}$ (say) and $x = (10^{-4}, -1, 3, 10^{-4}, -1)^T$. Now $H(x) = \{2\}$ and $J := J \setminus H(x) = \{3, 5\}$. Since $K_3 \cup K_5 = K$ set $x_2 = 10^{-4}$(say) and we have $x = (10^{-4}, 10^{-4}, 3, 10^{-4}, -1)^T$. Now $H(x) = \{3\}$ and $J := J \setminus H(x) = \{5\}$. Since $K_5 \neq K$ then we stop and an optimal solution is $x = (10^{-4}, 10^{-4}, 3, 10^{-4}, -1)^T$ and $f^{\min} = 4$.

4. MAX-LINEAR PROGRAM WITH TWO-SIDED CONSTRAINTS: A SPECIAL CASE

Suppose $c = (c_1, c_2, \ldots, c_m)^T, d = (d_1, d_2, \ldots, d_m)^T \in \mathbb{R}^m, A = (a_{ij})$ and $B = (b_{ij}) \in \mathbb{R}^{m\times n}$ are given matrices and vectors. The system

$$A \otimes x \oplus c = B \otimes x \oplus d, \tag{5}$$

is called non-homogeneous two-sided max-linear system and the set of solutions of this system will be denoted by S. Two-sided max-linear systems have been studied in [6, 8, 9] and [11].

Optimization problems whose objective function is max-linear and constraint (5) are called max-linear programs (MLP). Max-linear programs are studied in [5] and solution methods for both minimization and maximization problems were developed. The methods are proved to be pseudopolynomial if all entries are integer.

Consider max-linear programs with two-sided constraints (minimization), MLP$^{\min}$

$$\begin{aligned} & f(x) = f^T \otimes x \longrightarrow \min \\ & \text{subject to} \\ & A \otimes x \oplus c = B \otimes x \oplus d, \end{aligned} \tag{6}$$

where $f = (f_1, \ldots, f_n)^T \in \mathbb{R}^n$, $c = (c_1, \ldots, c_m)^T$, $d = (d_1, \ldots, d_m)^T \in \mathbb{R}^m$, $A = (a_{ij})$ and $B = (b_{ij}) \in \mathbb{R}^{m\times n}$ are given matrices and vectors. We introduce the following:

$$\begin{aligned} y &= (f_1 \otimes x_1, f_2 \otimes x_2, \ldots, f_n \otimes x_n) \\ &= \operatorname{diag}(f) \otimes x, \end{aligned} \tag{7}$$

$\operatorname{diag}(f)$ means a diagonal matrix whose diagonal elements are $f_1, f_2, \ldots, f_n$ and off diagonal elements are $-\infty$. It therefore follows from (7) that

$$\begin{aligned} & f^T \otimes x = 0^T \otimes y \\ \Longleftrightarrow \ & x = (f_1^{-1} \otimes y_1, f_2^{-1} \otimes y_2, \ldots, f_n^{-1} \otimes y_n) \\ & \quad = (\operatorname{diag}(f))^{-1} \otimes y. \end{aligned} \tag{8}$$

Hence, by substituting (7) and (8) into (6) we have

$$\begin{aligned} & 0^T \otimes y \longrightarrow \min \\ & \text{subject to} \\ & A' \otimes y \oplus c = B' \otimes y \oplus d, \end{aligned} \tag{9}$$

where 0^T is transpose of the zero vector, $A' = A \otimes (\operatorname{diag}(f))^{-1}$ and $B' = B \otimes (\operatorname{diag}(f))^{-1}$.

Therefore we assume without loss of generality that $f = 0$ and hence (6) is equivalent to

$$\begin{aligned} & f(x) = \sum_{j=1,\ldots,n}^{\oplus} x_j \longrightarrow \min \\ & \text{subject to} \\ & A \otimes x \oplus c = B \otimes x \oplus d. \end{aligned} \tag{10}$$

The set of feasible solutions for (10) will be denoted by S and the set of optimal solutions by $S^{\min}$. A vector is called *constant* if all its components are equal. That is a vector $x \in \mathbb{R}^n$ is constant if $x_1 = x_2 = \cdots = x_n$. For any $x \in S$ we define the set $Q(x) = \{i \in M; (A \otimes x)_i > c_i\}$. We introduce the following notation of matrices. Let $A = (a_{ij}) \in R^{m\times n}$, $1 \le i_1 < i_2 < \cdots < i_q \le m$ and $1 \le j_1 < j_2 < \cdots < j_r \le n$. Then,

$$A\begin{pmatrix} i_1, i_2, \ldots, i_q \\ j_1, j_2, \ldots, j_r \end{pmatrix} = \begin{pmatrix} a_{i_1j_1} a_{i_1j_2} \ldots a_{i_1j_r} \\ a_{i_2j_1} a_{i_2j_2} \ldots a_{i_2j_r} \\ \cdots \\ a_{i_qj_1} a_{i_qj_2} \ldots a_{i_qj_r} \end{pmatrix} = A(Q, R),$$

where, $Q = \{i_1, \ldots, i_q\}$, $R = \{j_1, \ldots, j_r\}$. Similar notation is used for vectors $c(i_1, \ldots, i_r) = (c_{i_1} \ldots c_{i_r})^T = c(R)$. Given $\mathrm{MLP}^{\min}$ with $c \ge d$, we define the following sets

$$M^{=} = \{i \in M; c_i = d_i\} \text{ and}$$
$$M^{>} = \{i \in M; c_i > d_i\}.$$

We also define the following matrices:

$$\begin{gathered} A_{=} = A(M^{=}, N),\ A_{>} = A(M^{>}, N) \\ B_{=} = B(M^{=}, N),\ B_{>} = B(M^{>}, N) \\ c_{=} = c(M^{=}),\ c_{>} = c(M^{>}). \end{gathered} \tag{11}$$

An easily solvable case arises when there is a constant vector $x \in S$ such that the set $Q(x) = \emptyset$. This constant vector x satisfies the following equations and inequalities

$$\begin{aligned} A_{=} \otimes x &\le c_{=} \\ A_{>} \otimes x &\le c_{>} \\ B_{=} \otimes x &\le c_{=} \\ B_{>} \otimes x &= c_{>}, \end{aligned} \tag{12}$$

where $A_{=}, A_{>}, B_{=}, B_{>}$, $c_{=}$ and $c_{>}$ are defined in (11). The one-sided system of equation and inequalities (12) can be written as

$$\begin{aligned} G \otimes x &= p \\ H \otimes x &\le q, \end{aligned} \tag{13}$$

where,

$$\begin{gathered} G = (B_{>}),\ H = \begin{pmatrix} A_{=} \\ A_{>} \\ B_{=} \end{pmatrix}, \\ p = c_{>} \ \text{ and } \ q = \begin{pmatrix} c_{=} \\ c_{>} \\ c_{=} \end{pmatrix}. \end{gathered} \tag{14}$$

Recall that $S(G, H, p, q)$ is the set of solutions for (13).

Theorem 4.1. Let $Q(x) = \emptyset$ for some constant vector $x = (\alpha, \dots, \alpha)^T \in S$. If $z \in S^{\min}$ then $z \in S(G, H, p, q)$.

Proof. Let $x = (\alpha, \dots, \alpha)^T \in S$. Suppose $Q(z) = \emptyset$ and $z \in S^{\min}$. This implies that $f(z) \le f(x) = \alpha$. Therefore we have, $\forall j \in N,\ z \le \alpha$. Consequently, $z \le x$ and $(A \otimes z)_i \le (A \otimes x)_i$ for all $i \in M$. Since, $Q(z) = \emptyset$ and $z \in S(G, H, p, q)$. □

Corollary 4.2. If $Q(x) = \emptyset$ for some constant vector $x \in S$ then $S^{\min} \subseteq S^{\min}(G, H, p, q)$.

Proof. The statement follows from Theorem 4.1. □

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