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EXISTENCE OF ENTIRE SOLUTIONS OF NONLINEAR DIFFERENCE EQUATIONS

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Abstract. In this paper we obtain that there are no transcendental entire solutions with finite order of some nonlinear difference equations of different forms.

Keywords: entire functions, difference equations, finite order

MSC 2010: 30D35, 39B32

1. Introduction

A meromorphic function means meromorphic in the whole complex plane. Given a meromorphic function f, recall that α is a small function with respect to f if $T(r,\alpha) = S(r,f)$, where $S(r,f)$ is used to denote any quantity satisfying $S(r,f) = o(T(r,f))$, where $r \to \infty$ outside of a possible exceptional set of finite logarithmic measure. We use the notation $\sigma(f)$ to denote the order of growth of f. We assume that the reader is familiar with standard symbols and fundamental results of Nevanlinna theory [9], [17].

Recently, difference version of Nevanlinna theory has been established [2], [6], [7], [8], [10], including the lemma of difference analogue of logarithmic derivative, difference analogue of the Clunie lemma and the second main theorem in differences, which are good tools in dealing with the value distributions of difference polynomials [11], [12] and with the existence and growth of solutions of complex difference

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equations [10], [13], [16]. Yang and Laine [16] considered the existence of the nonlinear differential-difference equation of the form

$$f^n + L(z, f) = h, \tag{1.1}$$

where $L(z, f)$ is a finite sum of the product of f, derivatives of f and their shifts $f(z + c_j)$ with small meromorphic functions as the coefficients, c_j are nonzero constants, h is an entire function. They obtained some results in [16]. One of them can be stated as follows.

Theorem A [16, Theorem 2.4]**.** *Let $P(z)$, $Q(z)$ be polynomials. Then the nonlinear difference equation*

$$f(z)^2 + P(z)f(z + 1) = Q(z)$$

has no transcendental entire solutions of finite order.

Using a proof similar to [16, Theorem 2.4], Yang and Laine pointed out that if the degree of the differential-difference polynomial $L(z, f)$ is less than n and h is a polynomial, then equation (1.1) has no transcendental entire solutions of finite order. In this paper, we mainly investigate the existence of solutions of different types of nonlinear difference equations.

We first investigate solutions of the nonlinear difference equation of the form (1.2), where the special case can be seen as the Fermat type functional equation. We obtain some results that partly answer the following question.

Question. What can we know about the solutions of nonlinear difference equation

$$f(z)^n + q(z)f(z + c_1)f(z + c_2)\ldots f(z + c_n) = p(z), \tag{1.2}$$

where $p(z)$, $q(z)$ are polynomials, c_j, $j = 1, 2, \ldots, n$, are constants?

Remark 1. If $p(z) \equiv 0$, then the equation $f(z)^n + q(z)f(z + c_1)f(z + c_2)\ldots f(z+c_n) = 0$ may have entire solutions of infinite or finite order. For example, $f(z) = \sin z$ solves the equation $f(z)^n + f(z+\pi)f(z+2\pi)\ldots f(z+n\pi) = 0$, where $n = 4k+1$, $k \in \mathbb{N}$, and $f(z) = \mathrm{e}^{\mathrm{e}^z}$ solves the equation $f(z)^n - f(z+c_1)f(z+c_2)\ldots f(z+c_n) = 0$, where $\mathrm{e}^{c_1} + \mathrm{e}^{c_2} + \ldots + \mathrm{e}^{c_n} = n$.

For the existence of transcendental entire solutions of finite order of equation (1.2), we have the following theorem.

Theorem 1.1. *Equation* (1.2) *has no transcendental entire solutions of finite order provided that at least one of c_j satisfies $c_j = 0$, where $p(z)$, $q(z)$ are nonzero polynomials.*

Remark 2. If c_j are distinct nonzero constants and $n \leqslant 2$, then equation (1.2) may have entire solutions of finite order. If $n = 1$, then $f(z) = e^z + \frac{1}{2}$ solves the equation $f(z) + f(z + c_1) = 1$, where $c_1 = i\pi$. If $n = 2$, then $f(z) = \sin z$ is a solution of $f(z)^2 + f(z + c_1)f(z + c_2) = 1$, where $c_1 = \frac{1}{2}\pi$, $c_1 = \frac{5}{2}\pi$. But we cannot succeed in finding a transcendental solution of the equation (1.2) when $n \geqslant 3$.

Next, we will consider zeros of the difference polynomial of $f(z)^n - f(z + c_1) \times f(z + c_2) \ldots f(z + c_n)$. First, we give two examples.

Example 1. If $f(z) = e^z$, then the difference polynomial $f(z)^2 - f(z + i\pi) \times f(z + 2i\pi) = 2e^{2z}$ has no zeros.

Example 2. If $f(z) = e^z + 1$, then the difference polynomial $f(z)^2 - f(z + i\pi) \times f(z + 2i\pi) = 2e^{2z} + 2e^z$ has infinitely many zeros.

From the above examples, we know that if $f(z)$ is a transcendental entire function of finite order, we cannot definitely say that the difference polynomial $f(z)^n - f(z + c_1)f(z + c_2) \ldots f(z + c_n)$ must have infinitely many zeros. But if $\sigma(f) < 1$ and $n = 3$, we obtain an affirmative result, the main idea being from [3]. Using a method similar to the proof of Theorem 1.2 below, the result can be improved to the case of $n \geqslant 3$.

Theorem 1.2. *Let $f(z)$ be a transcendental entire function with $\sigma(f) < 1$. Then $f(z)^3 - f(z + c_1)f(z + c_2)f(z + c_3)$ where $c_1 + c_2 + c_3 \neq 0$ has infinitely many zeros.*

Corollary 1.3. *The equation $f(z)^3 - f(z + c_1)f(z + c_2)f(z + c_3) = p(z)$, where $c_1 + c_2 + c_3 \neq 0$ and $p(z)$ is a nonzero polynomial, has no transcendental entire solutions of order less than* 1.

The special case of equation (1.2) with $c_j = c$, $j = 1, 2, \ldots, n$ and $p(z) = q(z) \equiv 1$, can be viewed as the Fermat type functional equation

$$f(z)^n + f(z + c)^n = 1. \tag{1.3}$$

It is well known that equation (1.3) has no transcendental entire solutions when $n \geqslant 3$, which can be seen in [5]. We will investigate solutions of the difference equation

$$f(z)^n + f(z + c)^m = a(z), \tag{1.4}$$

where $a(z)$ is a rational function, which is a completion of [13, Proposition 5.1]. We get the following theorem.

Theorem 1.4. *If* $n(1-1/m) > 1$, *then equation* (1.4) *has no transcendental entire solutions, if* $m \neq n$, *then equation* (1.4) *has no transcendental entire solutions of finite order, where* $n, m \in \mathbb{N}$, $a(z)$ *is a nonzero rational function.*

Remark 3. The equation $f(z)^2 + f(z+c) = a(z)$ may admit an entire solution of infinite order, which can be seen by setting $a(z) = 2$, $f(z) = -1/\mathrm{e}^{\mathrm{e}^z} - \mathrm{e}^{\mathrm{e}^z}$ and $\mathrm{e}^c = 2$.

Denote $\Delta f = f(z+c) - f(z)$ and $\Delta^s f = \Delta(\Delta^{s-1} f)$, where $s \in \mathbb{N}$ and c is a nonzero constant. By using a method similar to the proof of Theorem 1.4, we also get the next result.

Theorem 1.5. *The equation*

$$f(z)^n + (\Delta^s f)^m = a(z) \tag{1.5}$$

has no transcendental entire solutions if $n(1-1/m) > 1$, *where* $n, m, s \in \mathbb{N}$, $a(z)$ *is a rational function.*

Finally, we consider zeros of the difference polynomial $f(z)^n - \sum_{j=1}^{m} a_j(z) f(z+c_j) - s(z)$ and obtain the following result, which improves [11, Theorem 1.1 & 1.2].

Theorem 1.6. *Let* f *be a transcendental entire function of finite order* σ, *let* $s(z)$ *be a small function of* $f(z)$, *and let* c_j *be complex constants. Then the difference polynomial* $f(z)^n - \sum_{j=1}^{m} a_j(z) f(z+c_j) - s(z)$ *has infinitely many zeros provided that* $n \geqslant 3$, $\sum_{j=1}^{m} a_j(z) f(z+c_j) + s(z) \not\equiv 0$ *or* $n \geqslant 2$, $s(z) \equiv 0$, $\sum_{j=1}^{m} a_j(z) f(z+c_j) \not\equiv 0$.

Thus, we can obtain the non-existence of solutions of certain difference equations.

Corollary 1.7. *The equation* $f(z)^n - \sum_{j=1}^{m} a_j(z) f(z+c_j) - s(z) = P(z)\mathrm{e}^{Q(z)}$ *has no transcendental entire solutions of finite order provided that* $n \geqslant 3$, *where* $a_j(z)$, $P(z), Q(z)$ *are nonzero polynomials, unless* $\sum_{j=1}^{m} a_j(z) f(z+c_j) + s(z) \equiv 0$.

2. Some lemmas

In order to prove Theorem 1.2, we need the following lemma.

Lemma 2.1 ([1], Lemma 3.5). *Let f be a transcendental meromorphic function of order $\sigma(f) < 1$, let $h > 0$ be a constant. Then there exists an ε-set E such that*

$$f(z+c) - f(z) = cf'(z)(1 + o(1)) \quad \text{as } z \to \infty \text{ in } \mathbb{C} \setminus E,$$

uniformly in c for $|c| \leqslant h$.

Lemma 2.2 ([9], Theorem 3.1). *Let $f(z)$ be an entire function of order σ, and let $\nu(r)$ be the central index of f. Then*

$$\sigma = \limsup_{r \to \infty} \frac{\log^+ \nu(r)}{\log r}.$$

The original Clunie lemma was given in [4], and the difference versions of Clunie lemma [6, Corollary 3.3] or [10, Theorem 2.2] will take an important part in proving Theorem 1.4. Here, we only give the following result.

Lemma 2.3 ([6], Corollary 3.3). *Let f be a transcendental meromorphic solution with finite order of*

$$f^n P(z,f) = Q(z,f),$$

where $P(z,f)$, $Q(z,f)$ are difference polynomials in f and its shifts with small meromorphic coefficients a_λ, $\lambda \in I$. If the total degree of $Q(z,f)$ as a polynomial in f and its shifts are $\leqslant n$, then

$$m(r, P(z,f)) = S(r,f)$$

for all r outside of a possible exceptional set with finite logarithmic measure.

3. Proofs of Theorems

P r o o f of Theorem 1.1. If $n = 1$, obviously, the equation (1.2) has no transcendental solutions. Now assume that $n \geqslant 2$ and f is a transcendental finite order entire solution of equation (1.2). Without loss of generality, let $c_1 = 0$. Then we have

$$q(z)f(z+c_2)f(z+c_3)\ldots f(z+c_n) = \frac{p(z) - f(z)^n}{f(z)}. \tag{3.1}$$

Thus, from [2, Theorem 2.1], we get

$$T(r, f(z+c_2)f(z+c_3)\ldots f(z+c_n)) \leqslant (n-1)T(r,f) + S(r,f).$$

From the Valiron-Mohon'ko theorem [14], we know that

$$T\Big(r, \frac{p(z) - f(z)^n}{f(z)}\Big) = nT(r,f) + S(r,f).$$

Hence, we get from (3.1) that

$$T(r,f) = S(r,f),$$

which is a contradiction. □

P r o o f of Theorem 1.2. Assume that $f(z)^3 - f(z+c_1)f(z+c_2)f(z+c_3)$ has finitely many zeros. From the Hadamard factorization theorem we get

$$f(z)^3 - f(z+c_1)f(z+c_2)f(z+c_3) = p(z). \tag{3.2}$$

By Lemma 2.1, we know that there exists an ε-set E, such that $z \to \infty$ in $\mathbb{C} \setminus E$,

$$f(z+c_1) - f(z) = c_1 f'(z)(1+o(1)), \tag{3.3}$$

$$f(z+c_2) - f(z) = c_2 f'(z)(1+o(1)), \tag{3.4}$$

$$f(z+c_3) - f(z) = c_3 f'(z)(1+o(1)). \tag{3.5}$$

By (3.2)–(3.5), we obtain

$$c_1c_2c_3\Big(\frac{f'(z)}{f(z)}\Big)^2(1+o(1)) + (c_1c_2 + c_1c_3 + c_2c_3)\frac{f'(z)}{f(z)}(1+o(1)) + (c_1+c_2+c_3) = \frac{-p(z)}{f'(z)f(z)^2}. \tag{3.6}$$

From the Wiman-Valiron theory, we see that there exists a subset $F \subset (1,\infty)$ of finite logarithmic measure such that for large $r \notin [0,1] \cup F \cup E$, for all z satisfying $|z| = r$ and $|f(z)| = M(r,f)$ we have

$$\frac{f'(z)}{f(z)} = \frac{\nu(r)}{z}(1+o(1)).$$

Thus,

$$c_1c_2c_3\Big(\frac{\nu(r)}{z}\Big)^2(1+o(1)) + (c_1c_2+c_1c_3+c_2c_3)\frac{\nu(r)}{z}(1+o(1)) + \frac{p(z)z}{\nu(r)f(z)^3} = -(c_1+c_2+c_3). \tag{3.7}$$

Since $\sigma(f) < 1$, and f is a transcendental entire function, then we get

$$\Big|\frac{\nu(r)}{z}\Big| \to 0, \quad z \to \infty, \tag{3.8}$$

and

$$\Big|\frac{p(z)z}{\nu(r)f(z)^3}\Big| = \Big|\frac{p(z)r}{\nu(r)M(r,f)^3}\Big| \to 0, \quad z \to \infty. \tag{3.9}$$

Thus, from (3.7)–(3.9) and $c_1+c_2+c_3 \neq 0$ we get a contradiction. This completes the proof of Theorem 1.2. □

P r o o f of Theorem 1.4. Assume that f is a transcendental entire function of equation (1.4), then we get

$$nT(r,f(z)) = mT(r,f(z+c)) + S(r,f).$$

Combining the above with the second main theorem yields

$$\begin{aligned} nT(r,f(z)) &\leqslant \overline{N}(r,f(z)) + \overline{N}\Big(r,\frac{1}{f(z)}\Big) + \overline{N}\Big(r,\frac{1}{f(z)^n - a(z)}\Big) + S(r,f) \\ &\leqslant \overline{N}\Big(r,\frac{1}{f(z)}\Big) + \overline{N}\Big(r,\frac{1}{f(z+c)}\Big) + S(r,f) \\ &\leqslant \Big(1+\frac{n}{m}\Big)T(r,f(z)) + S(r,f). \end{aligned} \tag{3.10}$$

If $n(1-1/m) > 1$, we get a contradiction from (3.10). Thus, if $n(1-1/m) \leqslant 1$, then for any positive integers m, n one of the following four cases should be satisfied:

Case 1. $m = n = 1$. It is easy to give an entire solution of this case, such that $f(z) = z + \sin z$ is a solution of the equation $f(z) + f(z+\pi) = 2z + \pi$.

Case 2. $n \in \mathbb{N}\backslash\{1\}, m = 1$. We get that $f(z)^n + f(z+c) = a(z)$ has no transcendental entire solutions with finite order. Otherwise, using Lemma 2.3, $m(r, f) = S(r, f)$ follows, which contradicts the fact that f is an entire function.

Case 3. $n = 1$, $m \in \mathbb{N} \setminus \{1\}$. We obtain the same conclusion as that derived from Case 2. Actually, we need to consider the equation $f(z)^m + f(z-c) = a(z-c)$.

Case 4. $m = n = 2$. In fact, we had investigated this case in [13, Proposition 5.1] and obtained that if the equation

$$f(z)^2 + f(z+c)^2 = b(z)^2 \tag{3.11}$$

has a transcendental entire solution with finite order, then $f(z) = \frac{1}{2}(h_1(z) + h_2(z))$, where $h_1(z+c)/h_1(z) = \mathrm{i}$, $h_2(z+c)/h_2(z) = -\mathrm{i}$ and $h_1(z)h_2(z) = b(z)^2$, and $b(z)$ is a non-vanishing small entire function of $f(z)$ with period c.

Based on the above result, we will give more details about the properties of the solutions of (3.11). We first consider the solutions of the first order linear difference equation

$$h(z+c) = A(z)h(z), \tag{3.12}$$

where $A(z)$ is a nonzero rational function. It is well known that the Gamma function is a solution of equation (3.12) provided that $c = 1$ and $A(z) \equiv z$. If $A(z) \equiv b$ (a constant), then the solutions of equation (3.12) can be written in the form

$$h(z) = b^{z/c}\Pi(z),$$

where $\Pi(z)$ is a periodic function with period c.

In addition, about the growth of solutions of equation (3.12), Whittaker [15] showed that equation (3.12) admits a meromorphic solution f such that $\sigma(A) \leqslant \sigma(f) \leqslant \sigma(A) + 1$, provided that $A(z)$ is a finite order meromorphic function. Furthermore, Chiang and Feng [2, Corollary 9.3] showed that equation (3.12) admits a meromorphic solution of order $\sigma(f) = \sigma(A) + 1$, provided that $A(z)$ is an entire function of finite order.

From the above discussion, since f is an entire solution of equation (3.11), its general solutions can be given by

$$f(z) = \frac{1}{2}\big((-\mathrm{i})^{z/c}\Pi_1(z) + \mathrm{i}^{z/c}\Pi_2(z)\big) \tag{3.13}$$

for c-periodic functions $\Pi_1(z)$ and $\Pi_2(z)$ with

$$\Pi_1(z)\Pi_2(z) = b(z)^2. \tag{3.14}$$

Next, we prove that

$$\{\Pi_1(z), \Pi_2(z)\} = \pm b(z) \times \{\mathrm{e}^{2m\pi \mathrm{i} z/c+k}, \mathrm{e}^{-2m\pi \mathrm{i} z/c-k}\}$$

for an integer m and a constant k.

From equation (3.14), a zero of $\Pi_j(z)$ should be a zero of $b(z)$. We note that the finite-order zero-free c-periodic entire function can be written in the form $\mathrm{e}^{2m\pi \mathrm{i} z/c+k}$, so that there are two small c-periodic entire functions $b_j(z)$ with $b_1(z)b_2(z) = b(z)^2$, where

$$\Pi_j(z) = b_j(z)\mathrm{e}^{(-1)^j(2m\pi \mathrm{i} z/c+k)} \quad (j = 1, 2)$$

holds for an integer m and a constant k. This implies that $f(z)$ must be of order at most one, thus $b(z)$, as well as $b_1(z)$ and $b_2(z)$, is a constant. $\square$

P r o o f of Theorem 1.6. If $\sum\limits_{j=1}^{m} a_j(z)f(z+c_j) \equiv 0$ and $s(z) \not\equiv 0$, then the result follows by using the second main theorem. In fact, let $\psi(z) = f(z)^n - s(z)$. Then

$$\begin{aligned} (3.15)\qquad nT(r,f) &+ S(r,f) \\ &= T(r,\psi) \leqslant \overline{N}(r,\psi) + \overline{N}\Big(r,\frac{1}{\psi}\Big) + \overline{N}\Big(r,\frac{1}{\psi+s(z)}\Big) + S(r,\psi) \\ &\leqslant \overline{N}\Big(r,\frac{1}{\psi}\Big) + \overline{N}\Big(r,\frac{1}{f}\Big) + S(r,f). \end{aligned}$$

Since $n \geqslant 3$, we get that $\psi(z)$ has infinitely many zeros.

Consider now the case $\sum\limits_{j=1}^{m} a_j(z)f(z+c_j) \not\equiv 0$, $s(z) \not\equiv 0$ and $\sum\limits_{j=1}^{m} a_j(z)f(z+c_j) + s(z) \not\equiv 0$. Assume that $f(z)^n - \sum\limits_{j=1}^{m} a_j(z)f(z+c_j) - s(z)$ has finitely many zeros. Then from the Hadamard factorization theorem we obtain

$$f(z)^n - \sum_{j=1}^{m} a_j(z)f(z+c_j) - s(z) = P(z)\mathrm{e}^{Q(z)}, \tag{3.16}$$

where $P(z)$ is a nonzero polynomial, $Q(z)$ is a nonconstant polynomial. Otherwise, if $Q(z) \equiv A$, then by Lemma 2.3 we have $m(r,f) = S(r,f)$, which is a contradiction. Differentiating (3.16) and eliminating $\mathrm{e}^{Q(z)}$, we obtain

$$\begin{aligned} (3.17)\qquad f(z)^{n-1}&\big[nf'(z) - f(z)\varphi(z)\big] \\ &= \sum_{j=1}^{m} a_j'(z)f(z+c_j) + \sum_{j=1}^{m} a_j(z)f'(z+c_j) \\ &\quad - \varphi(z)\Big[\sum_{j=1}^{m} a_j(z)f(z+c_j) + s(z)\Big] + s'(z), \end{aligned}$$

where

$$\varphi(z) = \frac{P'(z)}{P(z)} + Q'(z). \tag{3.18}$$

We infer that $nf'(z) - f(z)\varphi(z) \not\equiv 0$. Otherwise,

$$n\frac{f'(z)}{f(z)} = \frac{P'(z)}{P(z)} + Q'(z). \tag{3.19}$$

Hence,

$$f(z)^n = P(z)\mathrm{e}^{Q(z)+B} \tag{3.20}$$

for a constant B. If $\mathrm{e}^B \neq 1$, from (3.16) we get

$$(1 - \mathrm{e}^{-B})f(z)^n - \sum_{j=1}^{m} a_j(z)f(z + c_j) - s(z) = 0. \tag{3.21}$$

Combining the above with Lemma 2.3 and $n \geqslant 3$, we obtain

$$m(r, f) = S(r, f), \tag{3.22}$$

which is a contradiction. Hence, $\mathrm{e}^B = 1$. From (3.21) we get $\sum\limits_{j=1}^{m} a_j(z)f(z + c_j) + s(z) \equiv 0$, which is also a contradiction. Hence, $nf'(z) - f(z)\varphi(z) \not\equiv 0$.

Since $n \geqslant 3$, by using Lemma 2.3 we obtain from (3.17) that

$$m(r, nf'(z) - f(z)\varphi(z)) = S(r, f), \tag{3.23}$$

and

$$m(r, nf(z)f'(z) - f(z)^2\varphi(z)) = S(r, f). \tag{3.24}$$

Since f is an entire function, combining (3.23) and (3.24) we get $T(r, f) = S(r, f)$, which is a contradiction.

Suppose now that $n = 2$ and $s(z) \equiv 0$. Then (3.17) assumes the form

$$\begin{aligned} f(z)\bigl(2f'(z) - \varphi(z)f(z)\bigr) &\\ = \sum_{j=1}^{m} a_j'(z)f(z + c_j) + \sum_{j=1}^{m} a_j(z)f'(z + c_j) - \varphi(z)\sum_{j=1}^{m} a_j(z)f(z + c_j).& \end{aligned} \tag{3.25}$$

Similarly to the case $n \geqslant 3$, we can conclude that $\psi := 2f' - \varphi f \neq 0$. From Lemma 2.3 we have

$$T(r, \psi(z)) = S(r, f). \tag{3.26}$$

Moreover,

$$m\Big(r, \frac{\psi}{f}\Big) = m\Big(r, 2\frac{f'}{f} - \varphi\Big) = S(r, f). \tag{3.27}$$

Differentiating $\psi(z)$, we obtain

$$2f'' - \varphi f' - \varphi' f = \psi' = \frac{\psi'}{\psi}\psi = \frac{\psi'}{\psi}(2f' - \varphi f),$$

and so

$$2f'' - \Big(\varphi + 2\frac{\psi'}{\psi}\Big)f' - \Big(\varphi' - \varphi\frac{\psi'}{\psi}\Big)f = 0.$$

This can be written as

$$2\frac{f''}{f} - \Big(\varphi + 2\frac{\psi'}{\psi}\Big)\frac{f'}{f} - \Big(\varphi' - \varphi\frac{\psi'}{\psi}\Big) = 0. \tag{3.28}$$

Suppose z_0 is a zero of f with multiplicity at least 2. Then, by the expression of ψ, z_0 is also a zero of ψ, thus the contribution to $N(r, 1/f)$ is $S(r, f)$. If z_0 is a simple zero of f and $\psi(z_0) \neq 0$, then by virtue of (3.28) and the Taylor expansion of $f(z) = a_1(z - z_0) + a_2(z - z_0)^2 + a_3(z - z_0)^3 + \ldots$, z_0 must be a zero of $\varphi + 2\psi'/\psi - 4a_2/a_1$, hence z_0 makes a contribution of $S(r, f)$ to $N(r, 1/f)$. We get

$$N\Big(r, \frac{1}{f}\Big) = S(r, f). \tag{3.29}$$

Hence, combining (3.26), (3.27) and (3.29), we get

$$T\Big(r, \frac{\psi}{f}\Big) = S(r, f). \tag{3.30}$$

From the first main theorem, (3.26), (3.30), we know that $T(r, f) = S(r, f)$, which is a contradiction. We have completed the proof.

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