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STANLEY DECOMPOSITIONS AND POLARIZATION

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Abstract. We define nice partitions of the multicomplex associated with a Stanley ideal. As the main result we show that if the monomial ideal I is a CM Stanley ideal, then I^p is a Stanley ideal as well, where I^p is the polarization of I.

Keywords: monomial ideals, partitionable simplicial complexes, multicomplexes, Stanley ideals, polarization

MSC 2010: 13H10, 13C14, 13F20, 13F55

1. Introduction

Let K be a field and $S = K[x_1, \ldots, x_n]$ a polynomial ring in n variables. Let $I \subset S$ be a monomial ideal, $u \in S/I$ a monomial and $Z \subseteq \{x_1, \ldots, x_n\}$. We denote by $uK[Z]$ the K-subspace of S/I generated by all elements uv where v is a monomial in $K[Z]$. The K-subspace $uK[Z] \subset S/I$ is called a *Stanley space* of dimension $|Z|$, if $uK[Z]$ is a free $K[Z]$-module. A decomposition of S/I as a finite direct sum of Stanley spaces $\mathscr{P}\colon\ S/I = \bigoplus\limits_{i=1}^{r} u_i K[Z_i]$ is called a *Stanley decomposition*. Stanley [15] conjectured that there always exists such a decomposition with $|Z_i| \geqslant \operatorname{depth}(S/I)$. If Stanley conjecture holds for S/I then I is called a *Stanley ideal*. The conjecture is still open but true in some special cases [1], [2], [4], [5], [6], [7], [9], [11], [12], [13], [14].

Let Γ be a subset of $\mathbb{N}^n_\infty$. An element $m \in \Gamma$ is called maximal if there is no $a \in \Gamma$ with $a > m$. We denote by $\mathscr{M}(\Gamma)$ the set of maximal elements of Γ. If $a \in \Gamma$, we write $\operatorname{infpt}(a) = \{i\colon\ a(i) = \infty\}$. An element $a \in \Gamma$ is called a facet of Γ if for all $m \in \mathscr{M}(\Gamma)$ with $a \leqslant m$ one has $|\operatorname{infpt}(a)| = |\operatorname{infpt}(m)|$. Herzog and Popescu [8]

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modify Stanley's definition of multicomplexes [15]. Γ is called a multicomplex if for all $a \in \Gamma$ and for all $b \in \mathbb{N}_\infty^n$ with $b \leqslant a$ it follows that $b \in \Gamma$ and for all $a \in \Gamma$ there is a maximal element m in Γ such that $a \leqslant m$. We define an interval $\mathscr{I}$ of Γ as a subset of Γ for which there exists $a \leqslant b$ in Γ such that $\mathscr{I} = [a, b] = \{c \in \Gamma \colon a \leqslant c \leqslant b\}$. A partition $\mathscr{P} \colon \Gamma = \bigcup\limits_{i=1}^{t} [a_i, b_i]$ of Γ is a presentation of Γ as a finite disjoint union of intervals $[a_i, b_i]$.

Monomial ideals I in the polynomial ring $S = K[x_1, \ldots, x_n]$ and multicomplexes in $\mathbb{N}_\infty^n$ correspond to each other bijectively. The multicomplex associated with a monomial ideal I is denoted by $\Gamma(I)$ and similarly, $I(\Gamma)$ denotes the monomial ideal associated with the multicomplex Γ. We show that Stanley's conjecture holds for S/I if and only if there exists a partition of the multicomplex $\Gamma(I)$ such that $|\mathrm{infpt}(b_i)| \geqslant \operatorname{depth}(S/I)$ for all i. Any partition of a multicomplex satisfying this condition will be called nice.

Let $I \subset S = K[x_1, \ldots, x_n]$ be a monomial ideal and $\Gamma(I)$ the multicomplex associated with I. In Proposition 1.3 we show that a partition $\mathscr{P} \colon \Gamma(I) = \bigcup\limits_{i=1}^{t} [a_i, b_i]$ of $\Gamma(I)$ is nice if all b_i's are facets of $\Gamma(I)$. Also, when S/I is Cohen-Macaulay, we have this result in both directions (see Corollary 1.4).

Let I^p be the polarization of the monomial ideal I and let Γ^p be the multicomplex associated with I^p. In Theorem 2.5 we prove that in the case of Cohen-Macaulay monomial ideals, if Γ has a nice partition then Γ^p has a nice partition. The converse of this theorem is still open. In [7] it is shown that Stanley's conjecture on Stanley decompositions of S/I holds provided it holds whenever S/I is Cohen-Macaulay. As a consequence, Theorem 2.5 is true even for all monomial ideals I (see Remark 2.6).

2. Partitions of Multicomplexes

Let Γ be a subset of $\mathbb{N}^n$. We define on $\mathbb{N}^n$ a partial order given by

$$(a(1), \ldots, a(n)) \leqslant (b(1), \ldots, b(n))$$

if $a(i) \leqslant b(i)$ for all i. According to Stanley [15] Γ is a multicomplex if for all $a \in \Gamma$ and all $b \in \mathbb{N}^n$ with $b \leqslant a$, it follows that $b \in \Gamma$. The elements of Γ are called faces.

Herzog and Popescu [8] modify Stanley's definition of multicomplexes. Before giving this definition we introduce some notation. We set $\mathbb{N}_\infty = \mathbb{N} \cup \{\infty\}$. As usual we set $a \leqslant \infty$ for all $a \in \mathbb{N}$, and extend the partial order on $\mathbb{N}^n$ naturally to $\mathbb{N}_\infty^n$. Thus now we take Γ as a subset of $\mathbb{N}_\infty^n$.

An element $m \in \Gamma$ is called *maximal* if there is no $a \in \Gamma$ with $a > m$. We denote by $\mathscr{M}(\Gamma)$ the set of maximal elements of Γ. If $a \in \Gamma$, we call

$$\operatorname{infpt}(a) = \{i\colon\ a(i) = \infty\}$$

the *infinite part* of a.

Definition 2.1. A subset $\Gamma \subset \mathbb{N}_\infty^n$ is called a multicomplex if

(1) for all $a \in \Gamma$ and for all $b \in \mathbb{N}_\infty^n$ with $b \leqslant a$ it follows that $b \in \Gamma$,
(2) for all $a \in \Gamma$ there exists an element $m \in \mathscr{M}(\Gamma)$ such that $a \leqslant m$.

An element $a \in \Gamma$ is called a *facet* of Γ if for all $m \in \mathscr{M}(\Gamma)$ with $a \leqslant m$ one has $\operatorname{infpt}(a) = \operatorname{infpt}(m)$. The set of all facets of Γ will be denoted by $\mathscr{F}(\Gamma)$. In [8] it is shown that each multicomplex has only a finite number of facets.

Monomial ideals I in the polynomial ring $S = K[x_1, \dots, x_n]$ and multicomplexes in $\mathbb{N}_\infty^n$ correspond to each other bijectively. The bijection is defined as follows: Let Γ be a multicomplex, and let $I(\Gamma)$ be the K-subspace in $S = K[x_1, \dots, x_n]$ spanned by all monomials x^a such that $a \notin \Gamma$. Note that if $a \in \mathbb{N}_\infty^n$ and $b \in \mathbb{N}_\infty^n \setminus \Gamma$, then $a + b \in \mathbb{N}_\infty^n \setminus \Gamma$, that is, if $x^b \in I(\Gamma)$ then $x^a x^b \in I(\Gamma)$ for all $x^a \in S$. In other words, $I(\Gamma)$ is a monomial ideal. In particular, the monomials x^a with $a \in \Gamma$ form a K-basis of $S/I(\Gamma)$.

Conversely, given an arbitrary monomial ideal $I \subset S$, there is a unique multicomplex Γ with $I = I(\Gamma)$, namely the smallest multicomplex (with respect to inclusion) which contains $A = \{a \in \mathbb{N}_\infty^n\colon\ x^a \notin I\}$. Such a multicomplex exists and is uniquely determined since an arbitrary intersection of multicomplexes is again a multicomplex.

One has the following obvious rules: let $\{\Gamma_j,\ j \in J\}$ be a family of multicomplexes. Then

(a) $I\Bigl(\bigcap\limits_{j\in J} \Gamma_j\Bigr) = \sum\limits_{j \in J} I(\Gamma_j)$,
(b) if J is finite, then $I\Bigl(\bigcup\limits_{j\in J} \Gamma_j\Bigr) = \bigcap\limits_{j\in J} I(\Gamma_j)$.

Let $\Gamma \subset \mathbb{N}_\infty^n$ be a multicomplex. We define an *interval* $\mathscr{I}$ of Γ as a subset of Γ for which there exists $a \leqslant b$ in Γ such that $\mathscr{I} = \{c \in \Gamma\colon\ a \leqslant c \leqslant b\}$. We denote an interval given by faces a and b by $[a, b]$. A *partition* $\mathscr{P}$ of Γ is a presentation of Γ as a finite disjoint union of intervals.

Lemma 2.2. *Let* $\mathscr{P}\colon\ \Gamma = \bigcup\limits_{i=1}^{t} [a_i, b_i]$ *be a partition of* Γ*. Then* $\operatorname{infpt}(a_i) = \emptyset$ *for all* i*.*

Proof. Assume that for some i, say for $i=1$, we have $\operatorname{infpt}(a_1) \neq \emptyset$. We may assume that $a_1(1)=\infty$. Set $a=a_1$ and let c be any integer. None of the faces $(c, a(2), \ldots, a(n))$ belong to $[a_1, b_1]$. Thus for each c there exists an $i \in \{2, \ldots, t\}$ such that $(c, a(2), \ldots, a(n)) \in [a_i, b_i]$. Hence for some $j>1$, infinitely many of the vectors $(c, a(2), \ldots, a(n))$ belong to $[a_j, b_j]$. This is only possible if $(\infty, a(2), \ldots, a(n))$ belongs to $[a_j, b_j]$. This is a contradiction, since $a_1=(\infty, a(2), \ldots, a(n)) \in [a_1, b_1]$. $\square$

Next we describe how Stanley decompositions and partitions are related to each other. Let $\Gamma \subset \mathbb{N}_\infty^n$ be a multicomplex, $[a, b] \subset \Gamma$ an interval and $U_{[a, b]}$ the K-subspace of S generated by all monomials $u=x_1^{c(1)} \ldots x_n^{c(n)}$ such that $c=(c(1), \ldots, c(n)) \in [a, b]$. Then obviously $U_{[a, b]}$ is a Stanley space if and only if

(i) $\operatorname{infpt}(a)=\emptyset$,
(ii) $i \notin \operatorname{infpt}(b) \Rightarrow a(i)=b(i)$.

Indeed, in this case $U_{[a, b]}=x^a K[Z_b]$, where $Z_b=\{x_i\colon\ b(i)=\infty\}$.

Let $I \subset S$ be a monomial ideal and $\Gamma(I)$ the multicomplex associated with I. Also let $S/I=\bigoplus_{i=1}^r x^{a_i} K[Z_i]$ be a Stanley decomposition of S/I. Set $b_i(j)=\infty$ if $x_j \in Z_i$ and $b_i(j)=a_i(j)$ if $x_j \notin Z_i$. Then $\bigcup_{i=1}^r [a_i, b_i]$ is a partition of $\Gamma(I)$. For instance, if $a \in [a_i, b_i] \cap [a_j, b_j] \cap \mathbb{N}^n$ for $i, j \in \{1, \ldots, r\}$ and $i \neq j$, then $x^a \in a_i K[Z_i] \cup a_j K[Z_j]$, a contradiction. Thus $\bigcup_{i=1}^r [a_i, b_i]$ is disjoint.

Conversely, we observe that each interval $[a, b]$ with $\operatorname{infpt}(a)=\emptyset$ can be written as a disjoint union of intervals

$$[a, b]=\bigcup [c_i, b_i] \tag{2.1}$$

such that each $[c_i, b_i]$ corresponds to a Stanley space. Indeed, if, as we may assume, for some integer r we have that $b(k)<\infty$ for $k \leqslant r$ and $b(k)=\infty$ for $k>r$, then $[a, b]$ is the disjoint union of the intervals

$$[(c(1), \ldots, c(r), a(r+1), \ldots, a(n)),(c(1), \ldots, c(r), \infty, \ldots, \infty)]$$

with $a(k) \leqslant c(k) \leqslant b(k)$ for $k=1, \ldots, r$, and each of these intervals satisfies (i) and (ii). Therefore, due to (2.1) and Lemma 2.2, Stanley's conjecture holds for S/I if and only if there exists a partition $\mathscr{P}\colon\ \Gamma=\bigcup_{i=1}^t [a_i, b_i]$ of the multicomplex $\Gamma=\Gamma(I)$ such that

$$|\operatorname{infpt}(b_i)| \geqslant \operatorname{depth}(S/I(\Gamma)) \quad \text{for all } i. \tag{2.2}$$

Any partition of a multicomplex satisfying condition (2.2) will be called *nice*.

Proposition 2.3. *A partition* $\mathscr{P}\colon\ \Gamma = \bigcup_{i=1}^{t} [a_i, b_i]$ *of the multicomplex* Γ *is a nice partition if* $b_i \in \mathscr{F}(\Gamma)$ *for all* i.

Proof. Let $I(\Gamma) = \bigcap_{i=1}^{m} Q_i$ be the unique irredundant presentation of I as an intersection of irreducible monomial ideals, and let $P_i = \sqrt{Q_i}$ for $i = 1, \dots, m$. Then $\operatorname{Ass}(S/I) = \{P_1, \dots, P_m\}$.

By [8, Proposition 9.12] there is a bijection between Q_i and the set of $\mathscr{M}(\Gamma)$ of maximal faces of Γ. In fact, for each i there is a unique $m_i \in \mathscr{M}(\Gamma)$ such that $Q_i = I(\Gamma(m_i))$ where $\Gamma(m_i)$ denotes the smallest multicomplex containing m_i. The assignment $Q_i \mapsto m_i$ establishes this bijection. Moreover, $\dim S/P_i = \operatorname{infpt}(m_i)$ for all i. Therefore,

$$\begin{aligned}
\min\{|\operatorname{infpt}(b_i)|\colon\ b_i \in \mathscr{F}(\Gamma)\} &= \min\{|\operatorname{infpt}(m_j)|\colon\ m_j \in \mathscr{M}(\Gamma)\}\\
&= \min\{\dim(S/P_j)\colon\ P_j \in \operatorname{Ass}(S/I(\Gamma))\}\\
&\geqslant \operatorname{depth}(S/I(\Gamma)).
\end{aligned}$$

The first equation follows from the definition of the facets, while the last inequality is a basic fact of commutative algebra, see [3, Proposition 1.2.13]. These considerations show that the given partition is nice. $\square$

Corollary 2.4. *Let* $I \subset S$ *be a monomial ideal such that* S/I *is Cohen-Macaulay. Let* Γ *be the multicomplex associated with* I *and let* $\mathscr{P}\colon\ \Gamma = \bigcup_{i=1}^{t} [a_i, b_i]$ *be a partition of* Γ. *Then the following conditions are equivalent.*

(a) $\mathscr{P}$ *is nice.*
(b) $\{b_1, \dots, b_t\} \subseteq \mathscr{F}(\Gamma)$.
(c) $\mathscr{M}(\Gamma) \subseteq \{b_1, \dots, b_t\} \subseteq \mathscr{F}(\Gamma)$.

Proof. (a) $\Rightarrow$ (b): In case S/I is Cohen-Macaulay we have $|\operatorname{infpt}(b)| \leqslant \operatorname{depth}(S/I)$ for all faces of Γ, and equality holds for b if and only if b is a facet. Thus $\mathscr{P}$ can be nice only if $\{b_1, \dots, b_t\} \subseteq \mathscr{F}(\Gamma)$.

(b) $\Rightarrow$ (c): Let $m \in \mathscr{M}(\Gamma)$; then $m \in [a_i, b_i]$ for some i. Since $m \leqslant b_i$ and since m is maximal it follows that $m = b_i$. Thus $\mathscr{M}(\Gamma) \subseteq \{b_1, \dots, b_t\}$.

(c) $\Rightarrow$ (a) follows from Proposition 2.3. $\square$

Remark 2.5. In the above corollary if $\mathscr{P}$ is nice then we can refine it in such a way that for the refinement

$$\mathscr{P}':\ \Gamma = \bigcup_{i=1}^{t'} [a'_i, b'_i]$$

we have $\{b'_1, \dots, b'_{t'}\} = \mathscr{F}(\Gamma)$. To prove this fact we first observe that $|\mathrm{infpt}(a_i)| = 0$ for all i, see Lemma 1.2. Since $\mathscr{F}(\Gamma) = \bigcup_{i=1}^{t} (\mathscr{F}(\Gamma) \cap [a_i, b_i])$, it is enough to write each interval $[a_i, b_i]$ as a disjoint union of intervals $\bigcup_{j=1}^{l} [c_j, e_j]$ where $\{e_1, e_2, \dots, e_l\} = \mathscr{F}(\Gamma) \cap [a_i, b_i]$.

For simplicity, we may assume that $b_i(k) < \infty$ for $k \leqslant r$ and $b_i(k) = \infty$ for $k > r$. Then $e \in [a_i, b_i]$ is a facet of Γ if and only if $a_i(k) \leqslant e(k) \leqslant b_i(k)$ for $k \leqslant r$ and $e(k) = \infty$ for $k > r$. Thus if we set $c_j(k) = e_j(k)$ for $k \leqslant r$ and $c_j(k) = a_i(k)$ for $k > r$, then $[a_i, b_i] = \bigcup_{j=1}^{l} [c_j, e_j]$ is the desired refinement of $[a_i, b_i]$.

3. Partitions and polarization

Let $S = K[x_1, \dots, x_n]$ be the polynomial ring in n variables over the field K, and let $u = \prod_{i=1}^{n} x_i^{a_i}$ be a monomial in S. Then

$$u^p = \prod_{i=1}^{n} \prod_{j=1}^{a_i} x_{ij} \in K[x_{11}, \dots, x_{1a_1}, \dots, x_{n1}, \dots, x_{na_n}]$$

is called the *polarization* of u.

Let I be a monomial ideal in S with monomial generators $u_1, \dots, u_r$. Then $(u_1^p, \dots, u_r^p)$ is called a *polarization* of I and is denoted by I^p. It is known that I is Cohen-Macaulay if and only if I^p is Cohen-Macaulay. Indeed, the elements $x_{ij} - x_{i1}$, $i = 1, \dots, n$ and $j = 1, 2, \dots$ form a regular sequence on T/I^p, and T/I^p modulo this regular sequence is isomorphic to S/I.

Let $I = (u_1, \dots, u_s) \subset S$ be a monomial ideal. We may assume that for each $i \in [n]$ there exists j such that x_i divides u_j. Let $u_j = x_1^{a_{j1}} \dots x_n^{a_{jn}}$ for $j = 1, \dots, s$ and set $r_i = \max a_{ji} \colon j = 1, \dots, s$ for $i = 1, \dots, n$. Moreover, set $r = \sum_{i=1}^{n} r_i$.

Let $I = \bigcap_{i=1}^{t} Q_i$ be the unique irredundant presentation of I as an intersection of irreducible monomial ideals. In particular, each Q_i is generated by pure powers of some of the variables. Then $I^p = \bigcap_{i=1}^{t_1} Q_i^p$ is an ideal in the polynomial ring

$$T = K[x_{11}, \dots, x_{1r_1}, x_{21}, \dots, x_{n1}, \dots, x_{nr_n}]$$

in r variables.

We denote by Γ, Γ^p, Γ_i and Γ_i^p the multicomplexes associated with I, I^p, Q_i and Q_i^p, respectively, and by $\mathscr{F}$, $\mathscr{F}^p$, $\mathscr{F}_i$ and $\mathscr{F}_i^p$ the set of facets of Γ, Γ^p, Γ_i and Γ_i^p, respectively.

Each Γ_i has only one maximal facet, say m_i, and $m_i(k) \leqslant r_k - 1$ for all k with $m_i(k) \neq \infty$. Moreover, $\mathscr{M}(\Gamma) = \{m_1, \ldots, m_t\}$. It follows that the set of facets of Γ is a subset of the set

$$\mathscr{B} = \{b \in \mathbb{N}_\infty^n \colon\ b(i) < r_i \text{ if } b(i) \neq \infty\}.$$

We define the map

$$\beta\colon\ \mathscr{B} \to \{0, \infty\}^r, \quad b \mapsto b',$$

where the components of the vectors b' are indexed by pairs of numbers ij, where for each $i = 1, \ldots, n$ the second index j runs in the range $j = 1, \ldots, r_i$. The map β is defined as follows:

$$b'(ij) = \begin{cases} 0, & \text{if } b(i) < \infty \text{ and } j = b(i) + 1, \\ \infty, & \text{otherwise.} \end{cases}$$

We quote the following result by Soleyman Jahan [10, Proposition 3.8].

Proposition 3.1. *With the above assumptions and notation the restriction of the map β to $\mathscr{F}$ induces a bijection $\mathscr{F} \to \mathscr{F}^p$.*

The following example demonstrates this bijection: let $I = (x_1^2, x_1x_2, x_3^2) = (x_1, x_3^2) \cap (x_1^2, x_2, x_3^2) \subset K[x_1, x_2, x_3]$. Then the multicomplex Γ associated with I has the facets

$$(0, \infty, 0),\ (0, \infty, 1),\ (1, 0, 0),\ (1, 0, 1),$$

while the multicomplex of the polarized ideal

$$I^p = (x_{11}x_{12}, x_{11}x_{21}, x_{31}x_{32}) \subset K[x_{11}, x_{12}, x_{21}, x_{31}, x_{32}]$$

has the facets

$$(0, \infty, \infty, 0, \infty),\ (0, \infty, \infty, \infty, 0),\ (\infty, 0, 0, 0, \infty),\ (\infty, 0, 0, \infty, 0).$$

Let $\Gamma = \bigcup\limits_{i=1}^{t} [a_i, b_i]$ be a nice partition of Γ with $\mathscr{F}(\Gamma) = \{b_1, \ldots, b_t\}$. With the notation introduced above we have

Lemma 3.2. $a_i(j) \leqslant r_j$ *for all i and j.*

P r o o f. Suppose without loss of generality that $a_1(1) > r_1$. Then $b_1(1) = \infty$, because if $b_1(1) < \infty$ then it follows that $a_1(1) \leqslant b_1(1) < r_1$, a contradiction. Now since $\Gamma = \bigcup_{i=1}^{t} [a_i, b_i]$ and since $a = (r_1, a_1(2), \dots, a_1(n)) \in \Gamma \backslash [a_1, b_1]$, there exists $i > 1$ such that $a \in [a_i, b_i]$. As above, $b_i(1) = \infty$ because if $b_i(a) < \infty$ then $r_1 \leqslant b_i(1) < r_1$, which is not possible. Hence we conclude that $a_i \leqslant a < a_1 < b_i \Rightarrow a_1 \in [a_i, b_i]$, a contradiction. $\square$

We want to "polarize" the nice partition $\Gamma = \bigcup_{i=1}^{t} [a_i, b_i]$. For this purpose we consider the set $\mathscr{A} = \{a \in \mathbb{N}\colon\ a(i) \leqslant r_i\}$ and the map $\gamma\colon\ \mathscr{A} \to \{0,1\}^r$ with

$$\gamma(a)(ij) = \begin{cases} 0, & \text{if } j > a(i), \\ 1, & \text{otherwise.} \end{cases}$$

We observe that γ is injective. Indeed, for $a \neq a'$ there exists i such that $a(i) \neq a'(i)$, say, $a(i) < a'(i)$. Then $a(ij) = 0$ for $j = a(i) + 1$, while $a'(ij) = 1$ for $j = a(i) + 1$.

Let $\mathscr{I} = [a, b] \subset \Gamma \subset \mathbb{N}_\infty^n$ be an interval such that $a = (a(1), a(2), \dots, a(n))$ and $b = (b(1), b(2), \dots, b(n))$. We define an *$i$-subinterval* as

$$\{c \in \mathbb{N}_\infty\colon\ a(i) \leqslant c \leqslant b(i)\}$$

and denote it by $\mathscr{I}(i) = [a(i), b(i)]$.

Example 3.3. Let $a, b \in \Gamma \subset \mathbb{N}_\infty^2$, $a = (2, 5)$, $b = (4, \infty)$. Then

$$\mathscr{I}(1) = [a(1), b(1)] = [2, 4] \quad \text{i.e. } \mathscr{I}(1) = \{2, 3, 4\},$$
$$\mathscr{I}(2) = [a(2), b(2)] = [5, \infty] \quad \text{i.e. } \mathscr{I}(2) = \{5, 6, \dots\}.$$

Now we need the following elementary lemma.

Lemma 3.4. *Let $\mathscr{I}_1, \mathscr{I}_2$ be two intervals of a multicomplex $\Gamma \subset \mathbb{N}_\infty^n$ such that $\mathscr{I}_1 = [a, b]$ and $\mathscr{I}_2 = [c, d]$. Suppose $\mathscr{I}_1 \cap \mathscr{I}_2 = \emptyset$. Then there exists i such that $\mathscr{I}_1(i) \cap \mathscr{I}_2(i) = \emptyset$.*

Let $I \subset S$ be a monomial ideal and let $S/I = \bigoplus_{i=1}^{r} u_i K[Z_i]$ be its Stanley decomposition, where $u_i = x^{a_i}$ for $i = 1, \dots, r$. Then the Hilbert series is given by $H(S/I) = \sum_{i=1}^{r} t^{|a_i|}/(1-t)^{|Z_i|}$, where $|a_i|$ denotes the sum of the components of a_i and $|Z_i|$ the cardinality of Z_i. Thus if Γ is the multicomplex associated with I and

$\Gamma = \bigcup_{i=1}^{t} [a_i, b_i]$ is the corresponding partition (with $b_i(j) = a_i(j)$ for $x_j \notin Z_i$ and $b_i(j) = \infty$ for $x_j \in Z_i$), then $H(S/I) = \sum_{i=1}^{r} t^{|a_i|}/(1-t)^{|b_i|_\infty}$, where $|b_i|_\infty = |\operatorname{infpt} b_i|$.

Theorem 3.5. *Let $I \subset S$ be a monomial ideal such that S/I is Cohen-Macaulay, and let I^p be the polarization of I. Suppose I satisfies the Stanley Conjecture. Then I^p satisfies it too.*

Proof. Let Γ be the multicomplex associated with I. Since I satisfies the Stanley Conjecture, Γ has a nice partition. Let Γ^p be the multicomplex associated with I^p. Then we show that Γ^p has a nice partition.

Let $\Gamma = \bigcup_{i=1}^{\hat{t}} [\hat{a}_i, \hat{b}_i]$ be a nice partition of Γ. Then by Corollary 1.4, $\hat{b}_i \in \mathscr{F}(\Gamma)$ for all i. Again by Remark 1.5, we can refine this partition to another nice partition, say $\mathscr{P}\colon\ \Gamma = \bigcup_{i=1}^{t} [a_i, b_i]$, such that $\{b_1, \ldots, b_t\} = \mathscr{F}(\Gamma)$.

Let β and γ be the functions defined above and set $\beta(b_i) = \bar{b}_i$ and $\gamma(a_i) = \bar{a}_i$ for all $i = 1, \ldots, t'$. We will show that $\mathscr{P}^p\colon\ \Gamma^p = \bigcup_{i=1}^{t} [\bar{a}_i, \bar{b}_i]$ is a nice partition of Γ^p.

$\mathscr{P}^p$ is a partition if the intervals $[\bar{a}_i, \bar{b}_i]$ are disjoint for all $i = 1, \ldots, t$ and $\mathscr{P}^p$ covers all faces of Γ^p.

Suppose that the intervals are not disjoint and, say, there exists a face $a \in [\bar{a}_i, \bar{b}_i] \cap [\bar{a}_j, \bar{b}_j]$ for some $i \neq j$, $i, j \in \{1, \ldots, t\}$. Since $a_i \neq a_j$ we get $\bar{a}_i \neq \bar{a}_j$, γ being injective.

The intervals $[a_i, b_i]$ and $[a_j, b_j]$ are disjoint and so by Lemma 2.4 there exists at least one pair of i_1-subintervals, say $[a_i(i_1), b_i(i_1)]$ and $[a_j(i_1), b_j(i_1)]$ for $i_1 \in \{1, \ldots, n\}$, such that $[a_i(i_1), b_i(i_1)] \cap [a_j(i_1), b_j(i_1)] = \emptyset$.

So at least one of $b_i(i_1), b_j(i_1)$ is finite, say $b_i(i_1) \neq \infty$, thus $i_1 \notin \operatorname{infpt}(b_i)$, so by condition (ii) of being Stanley space, $b_i(i_1) = a_i(i_1)$. Also we can assume that $a_i(i_1) < a_j(i_1)$. If not and $b_j(i_1) = \infty$ then $[a_i(i_1), b_i(i_1)] \subset [a_j(i_1), b_j(i_1)]$, which is not possible; if $b_j(i_1) < \infty$ then change i by j.

Let $a_i(i_1) = b_i(i_1) = k$ and $a_j(i_1) = m > k$. Then by definition of γ and β we have $\bar{a}_i(i_1\overline{k+1}) = 0 = \bar{b}_i(i_1\overline{k+1})$ and $\bar{a}_j(i_1 l) = 1$ for $l \leqslant m$, thus $\bar{a}_j(i_1\overline{k+1}) = 1$.

It follows that $a(i_1\overline{k+1}) = 0$. On the other hand, since $a \geqslant \bar{a}_j$ we get $a(i_1\overline{k+1}) \geqslant \bar{a}_j(i_1\overline{k+1}) = 1$, a contradiction.

Now for the second part of the proof, we will use the Hilbert series. We have $H(S/I) = \sum_{i=1}^{t} s^{|a_i|}/(1-s)^{|b_i|_\infty}$. The definition of the function γ implies that $|a_i| = |\bar{a}_i|$ for all $i = \{1, \ldots, t\}$. Now for each polarization step, the depth of S/I increases by 1. Also by the definition of β for each polarization step the number of infinite points increases by 1. Thus after n_1 polarization steps we have $|\operatorname{infpt}(\bar{b}_i)| = |\operatorname{infpt}(b_i)| + n_1$.

So

$$H\left(\bigcup_{i=1}^{t}[\bar{a}_i,\bar{b}_i]\right)=\sum_{i=1}^{t}\frac{s^{|a_i|}}{(1-s)^{|b_i|_\infty+n_1}}=\frac{1}{(1-s)^{n_1}}H(S/I)$$

is in fact the Hilbert series of $H(S^p/I^p)$. Hence $S^p/I^p=\bigcup_{i=1}^{t}[\bar{a}_i,\bar{b}_i]$.

Note that $\mathscr{P}^p$ is a nice partition because $|\bar{b}_i|_\infty=|b_i|_\infty+n_1\geqslant \operatorname{depth}_S(S/I)+n_1=\operatorname{depth}_{S^p}(S^p/I^p)$ for all i. □

Remark 3.6. In the above theorem, the condition for S/I to be Cohen-Macaulay is even not necessary. As in [7], in Corollary 2.2 it is shown that for each monomial ideal I, Stanley's conjecture holds for S/I provided it holds whenever S/I is Cohen-Macaulay.

The converse of Theorem 2.5 is still open. If one can prove the converse, then Stanley's Conjecture will reduce to the case of squarefree monomial ideals I.

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