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Czechoslovak Mathematical Journal, Vol. 61 (2011), No. 2, 389–400

Persistent URL: http://dml.cz/dmlcz/141542

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L^∞ ESTIMATES OF SOLUTION FOR m-LAPLACIAN PARABOLIC EQUATION WITH A NONLOCAL TERM

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(Received January 14, 2010)

Abstract. In this paper, we consider the global existence, uniqueness and L^∞ estimates of weak solutions to quasilinear parabolic equation of m-Laplacian type $u_t - \operatorname{div}(|\nabla u|^{m-2}\nabla u) = u|u|^{\beta-1}\int_\Omega |u|^\alpha \,\mathrm{d}x$ in $\Omega\times(0,\infty)$ with zero Dirichlet boundary condition in $\partial\Omega$. Further, we obtain the L^∞ estimate of the solution $u(t)$ and $\nabla u(t)$ for $t>0$ with the initial data $u_0\in L^q(\Omega)$ $(q>1)$, and the case $\alpha+\beta<m-1$.

Keywords: m-Laplacian parabolic equations, global existence, uniqueness, L^∞ estimates

MSC 2010: 35K65, 35K92

1. Introduction

In this paper we study the global existence, uniqueness, and L^∞ estimates of the solution for the initial boundary value problem for the parabolic equation of m-Laplacian type with a nonlocal term

$$(1.1)\qquad \begin{cases} u_t - \operatorname{div}(|\nabla u|^{m-2}\nabla u) = u|u|^{\beta-1}\int_\Omega |u|^\alpha \,\mathrm{d}x, & x\in\Omega,\ t>0,\\ u(x,0)=u_0(x), & x\in\Omega,\\ u(x,t)=0, & x\in\partial\Omega,\ t>0, \end{cases}$$

where $2<m<N$, $\alpha\geqslant 0$, $\beta\geqslant 1$ and Ω is a bounded domain in $\mathbb{R}^N$ $(N\geqslant 3)$ with the smooth boundary $\partial\Omega$. If $\alpha+\beta\geqslant m-1$ and $|\Omega|$ or $u_0(x)$ is properly large, we know the problem (1.1) need not have a global solution, see [9]. So we mainly consider the problem (1.1) with $\alpha+\beta<m-1$. Many results concerning global

The work was supported by the Fundamental Research Funds for the Central Universities (Grant No. 2010B17914) and the Science Funds of Hohai University (Grant No. 2008430211 and 2008408306).

existence, uniqueness, blow-up and asymptotic behavior of the solution for (1.1) have been established. In particular, it is well known that (1.1) admits a unique global solution if $\alpha = 0$ and $u_0 \in W_0^{1,m}(\Omega)$.

Many physical phenomena were formulated as non-local mathematical models and studied by many authors (cf. [1],[5], [9]). Li and Xie in [9] considered the problem

$$(1.2)\qquad \begin{cases} u_t - \operatorname{div}(|\nabla u|^{m-2}\nabla u) = \int_\Omega |u|^\alpha \,\mathrm{d}x, & x \in \Omega,\ t > 0,\\ u(x,0) = u_0(x) \geqslant 0, & x \in \Omega,\\ u(x,t) = 0, & x \in \partial\Omega,\ t > 0, \end{cases}$$

by making use of super-subsolution techniques with $2 < m < N, \alpha \geqslant 1$, $u_0 \in L^\infty(\Omega) \cap W_0^{1,m}(\Omega)$ and $\partial u_0/\partial\nu < 0$ on $\partial\Omega$, where ν denotes the unit outer normal vector on the boundary $\partial\Omega$. Under the appropriate hypotheses, they developed local theory of the solution and obtained that the solution either exists globally or blows up in finite time.

Rouchon in [14] proved the existence of a universal bound for all nonnegative global solutions of (1.2) with $m = 2$, where $\alpha > 1$ and $u_0 \in L^\infty(\Omega)$.

In [3] Chen considered the nonlocal problem (1.1) with $\alpha = 0$ and $u_0 \in L^q$ $(1 < q < 2)$, proved the global existence of $u(t)$ and gave an L^∞ estimates of $u(t)$ and $\nabla u(t)$ for $t \in (0,T]$. However, as far as we know, there are few results concerning the L^∞ estimates of $u(t)$ and $\nabla u(t)$ for $u_0 \in L^q(\Omega)$ $(q > 1)$ for the problem (1.1).

In this paper we are interested in the global existence and the uniqueness of solution for (1.1) with $u_0 \in L^q(\Omega)$ $(q > 1)$, $\alpha + \beta < m - 1$, and give L^∞ estimates for $u(t)$ and $\nabla u(t)$ with $t > 0$. For L^∞ estimates, we use Moser's technique as in [2]–[4], [11]–[13]. To obtain an estimate of $\|\nabla u(t)\|_\infty$, we also make the assumption that the mean curvature $H(x)$ of $\partial\Omega$ at x is non-positive with respect to the outward normal; such assumption is made also in [2], [7]. We know that $H(x) \leqslant 0$ if Ω is convex.

This paper is organized as follows. In Section 2, we state the main results and present some lemmas which will be used below. In Sections 3 and 4, we use these lemmas to derive L^∞ estimates for $u(t)$ and $\nabla u(t)$, respectively. The proof of the main results will be given in Sections 3 and 4.

2. Preliminaries and results

Let $\|\cdot\|_p$ and $\|\cdot\|_{1,p}$ denote the $L^p(\Omega)$ and $W^{1,p}(\Omega)$ $(1 \leqslant p \leqslant \infty)$ norms respectively.

Definition 1. A measurable function $u(x,t)$ on $\Omega \times \mathbb{R}^+$ is said to be a weak solution of the problem (1.1) if $u = u(x,t) \in L^\infty((0,\infty), W_0^{1,m}) \cap L^{m-1}(\mathbb{R}^+, W_0^{1,m-1})$

and the equality

$$\int_0^t\int_\Omega \{-u\varphi_s + |\nabla u|^{m-2}\nabla u\nabla\varphi - \|u(s)\|_\alpha^\alpha |u|^{\beta-1}u\varphi\}\,\mathrm{d}x\,\mathrm{d}s = \int_\Omega \{u_0(x)\varphi(x,0) - u(x,t)\varphi(x,t)\}\,\mathrm{d}x \tag{2.1}$$

is valid for any $t > 0$ and $\varphi = \varphi(x,t) \in C^1(\mathbb{R}^+, C_0^1(\Omega))$, where $\mathbb{R}^+ = [0,+\infty)$.

We make the following assumptions.

(H_1) $u_0 \in L^q(\Omega)$, $q > 1$;

(H_2) $N > m > 2$, $\alpha \geqslant 0$, $\beta \geqslant 1$, and $\alpha + \beta < m - 1$;

(H_3) the mean curvature $H(x)$ of $\partial\Omega$ at x is non-positive with respect to the outward normal.

Remark 1. Since Ω is a bounded domain, we have $L^p(\Omega) \subset L^q(\Omega)$ for $p > q \geqslant 1$.

Our main results read as follows.

Theorem 1. *Assume* (H_1)–(H_2) *hold. Then* (1.1) *admits a unique global solution* $u(t)$ *which satisfies*

$$u(t) \in L^\infty(\mathbb{R}^+, L^q) \cap L^\infty_{\mathrm{loc}}((0,\infty), W_0^{1,m}) \cap L^{m-1}_{\mathrm{loc}}(\mathbb{R}^+, W_0^{1,m-1}), \quad u_t \in L^2_{\mathrm{loc}}((0,\infty), L^2), \tag{2.2}$$

and the estimates

$$\|u(t)\|_p \leqslant C_p(1 + t^{-1/(m-2)}), \quad t > 0,\ \forall p > q, \tag{2.3}$$

$$\|u(t)\|_\infty \leqslant C_0 t^{-\lambda}, \quad 0 < t \leqslant T, \tag{2.4}$$

$$\|\nabla u(t)\|_m \leqslant C_0 t^{-(1+2\lambda(\alpha+\beta))/m}, \quad 0 < t \leqslant T, \tag{2.5}$$

where $\lambda = N/((m-2)N + mq)$, $C_0 = C_0(|\Omega|, T, \|u_0\|_q) > 0$ *and* C_p *depends on* p.

Theorem 2. *Assume that* (H_1)–(H_3) *hold. Then the solution* $u(t)$ *of* (1.1) *has the gradient estimate*

$$\|\nabla u(t)\|_\infty \leqslant C_0 t^{-\mu}, \quad 0 < t \leqslant T. \tag{2.6}$$

Further, if $u_0 \in W_0^{1,m}(\Omega)$ *and* $2\beta < m^* = Nm/(N-m)$, *we have*

$$\|\nabla u(t)\|_m \leqslant \|\nabla u_0\|_m \mathrm{e}^{-\lambda_1 t} + C_0, \quad t \geqslant 0 \tag{2.7}$$

with some $\lambda_1 > 0$ *and* $\mu = (2(1 + 2\lambda(\alpha+\beta)) + N^2)/(2m + (m-2)N^2)$.

To obtain the above results, we will use the following lemmas.

Lemma 1 ([11], [16]). *Let $\beta \geqslant 0$, $N > m \geqslant 1$, $\beta + 1 \leqslant q$ and $1 \leqslant r \leqslant p \leqslant (\beta+1)Nm/(N-m)$. Then for $|u|^\beta u \in W^{1,m}(\Omega)$ we have*

$$\|u\|_q \leqslant C_1^{1/(\beta+1)} \|u\|_r^{1-\theta} \||u|^\beta u\|_{1,m}^{\theta/(\beta+1)}$$

with $\theta = (\beta+1)(r^{-1} - p^{-1})/(N^{-1} - m^{-1} + (\beta+1)r^{-1})$, where C_1 is a constant independent of p, r, β and θ.

Lemma 2 ([13]). *Let $y(t)$ be a nonnegative differentiable function on $(0,T]$ satisfying*

$$y'(t) + At^{\lambda\theta-1}y^{1+\theta} \leqslant Bt^{-k}y(t) + Ct^{-\delta}$$

with $A, \theta > 0$, $\lambda\theta \geqslant 1$, $B, C \geqslant 0$, $k \leqslant 1$. Then we have

$$y(t) \leqslant A^{-1/\theta}(2\lambda + 2BT^{1-k})^{1/\theta}t^{-\lambda} + 2C(\lambda + BT^{1-k})^{-1}t^{1-\delta}, \quad 0 < t \leqslant T.$$

Lemma 3 ([15]). *Let $y(t)$ be a nonnegative differentiable function on $(0,\infty)$ satisfying*

$$y'(t) + Ay^{1+\mu}(t) \leqslant B, \quad t > 0,$$

with A, $\mu > 0$, $B \geqslant 0$. Then

$$y(t) \leqslant (BA^{-1})^{1/(1+\mu)} + (A\mu t)^{-1/\mu}, \quad t > 0.$$

Further, if $y(t)$ is continuous on $[0,+\infty)$ then

$$y(t) \leqslant (BA^{-1})^{1/(1+\mu)} + (y(0)^{-\mu} + A\mu t)^{-1/\mu}, \quad t > 0.$$

3. L^∞ estimate for $u(t)$

Let $u_{0,i} \in C_0^2(\Omega) \to u_0$ in $L^q(\Omega)$ $(q > 1)$ as $i \to \infty$. For $i = 1, 2, \ldots$, we consider the approximate problem of (1.1):

$$(3.1) \quad \begin{cases} u_t - \operatorname{div}((|\nabla u|^2 + i^{-1})^{(m-2)/2}\nabla u) = u|u|^{\beta-1}\int_\Omega |u|^\alpha \, \mathrm{d}x, & x \in \Omega,\ t > 0, \\ u(x,0) = u_{0,i}(x), & x \in \Omega, \\ u(x,t) = 0, & x \in \partial\Omega,\ t > 0. \end{cases}$$

Then the problem (3.1) has a unique smooth solution $u_i(x,t)$ (see [8]). For simplicity of notation, we write u instead of u_i and u^p for $|u|^{p-1}u$ when $p > 0$. Also, let C, C_j, μ_j be generic constants independent of i and p, and changeable from line to line.

Proposition 1. *Assume* (H_1)–(H_2) *hold and* $u(t)=u(x,t)$ *is the solution of* (3.1). *Then* $u(t)\in L^{\infty}(\mathbb{R}^+,L^q)$.

The proof of Proposition 1 is similar to that of Propsition 1 in [3] and is omitted here.

Proposition 2. *Under the assumptions of Proposition* 1 *and* $p\geqslant q>1$, *the solution* $u(t)$ *of* (3.1) *also satisfies*

$$\|u(t)\|_p\leqslant C_p(1+t^{-1/(m-2)}),\quad t>0,\ \forall p>q, \tag{3.2}$$

and for any $T>0$,

$$\|u(t)\|_\infty\leqslant C_1t^{-\lambda},\qquad 0<t\leqslant T, \tag{3.3}$$

$$\|\nabla u(t)\|_m\leqslant C_1t^{-(1+2\lambda(\alpha+\beta))/m},\quad 0<t\leqslant T, \tag{3.4}$$

$$\int_0^T s^{1+\gamma}\|u_s(s)\|_2^2\,\mathrm{d}s\leqslant C_1, \tag{3.5}$$

where C_p *depends on* p, $\lambda=N/((m-2)N+mq)$, $C_1=C_1(|\Omega|,T,\|u_0\|_q)$, $\gamma>\lambda(2-q)^+$, *and* $(2-q)^+=\max\{0,2-q\}$.

P r o o f. Multiplying (3.1) by u^{p-1} $(p\geqslant q>1)$, we have

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_p^p+\mu_0p^{1-m}\|\nabla u^{(p+m-2)/m}(t)\|_m^m\leqslant\|u(t)\|_\alpha^\alpha\int_\Omega|u(t)|^{\beta+p-1}\,\mathrm{d}x\equiv A. \tag{3.6}$$

By the Sobolev inequality, we get

$$\|\nabla u^{(p+m-2)/m}\|_m^m\geqslant\mu_1\|u\|_{p+m-2}^{p+m-2}$$

where $\mu_0,\mu_1>0$ are independent of p.

Since $\alpha+\beta<m-1$, we use Young's inequality and get

$$\|u(t)\|_\alpha^\alpha\int_\Omega|u(t)|^{\beta+p-1}\,\mathrm{d}x=\|u(t)\|_\alpha^\alpha\|u(t)\|_{p+\beta-1}^{p+\beta-1}\leqslant\frac{\mu_0\mu_1}{2}\|u(t)\|_{p+m-2}^{p+m-2}+C_p. \tag{3.7}$$

Then (3.6) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_p^p+\frac{\mu_0\mu_1}{2}p^{2-m}\|u(t)\|_{p+m-2}^{p+m-2}\leqslant pC_p. \tag{3.8}$$

Note that

$$\|u(t)\|_{p+m-2}^{p+m-2}\geqslant C_0\|u(t)\|_p^{p+m-2} \tag{3.9}$$

with $C_0 = C_0(|\Omega|) > 0$. Then, the application of Lemma 3 to (3.8) gives

$$\|u(t)\|_p \leqslant C_p(1 + t^{-1/(m-2)}), \quad t > 0. \tag{3.10}$$

In order to derive (3.3), we must treat carefully the differential inequality (3.6). Since $0 \leqslant \alpha < m$, it follows from the Sobolev's inequality that

$$\|u(t)\|_\alpha \leqslant C_0 \|u^{(p+m-2)/m}(t)\|_m^{m/(p+m-2)} \leqslant C_0 \|\nabla u^{(p+m-2)/m}(t)\|_m^{m/(p+m-2)}$$

Further, by Lemma 1 and Proposition 1, we get

$$\begin{aligned} A = \|u(t)\|_\alpha^\alpha \|u(t)\|_{p+\beta-1}^{p+\beta-1} &\leqslant \|u(t)\|_\alpha^\alpha \|u(t)\|_p^{\theta_1} \|u(t)\|_q^{\theta_2} \|u(t)\|_{p^*}^{\theta_3} \\ &\leqslant C_0 \|\nabla u^{(p+m-2)/m}(t)\|_m^{m\alpha/(p+m-2)} \|u(t)\|_p^{\theta_1} \|u(t)\|_{p^*}^{\theta_3} \\ &\leqslant C_0 \|u\|_p^{\theta_1} \|\nabla u^{(p+m-2)/m}(t)\|_m^{m(\alpha+\theta_3)/(p+m-2)} \\ &\leqslant \frac{1}{2}\mu_0 p^{1-m} \|\nabla u^{(p+m-2)/m}(t)\|_m^m + C p^\sigma \|u(t)\|_p^p\ , \end{aligned}$$

in which $\sigma = \lambda\alpha$, $p^* = N(p+m-2)/(N-m)$ and $(\theta_1, \theta_2, \theta_3)$ is the positive solution of the following system

$$\begin{cases} \theta_1 + \theta_2 + \theta_3 = p + \beta - 1, \\ \dfrac{\theta_1}{p} + \dfrac{\theta_2}{q} + \dfrac{\theta_3(N-m)}{N(p+m-2)} = 1, \\ \dfrac{\theta_1}{p} + \dfrac{\theta_3 + \alpha}{p+m-2} = 1. \end{cases}$$

It is easy to obtain

$$\begin{aligned} \theta_1 &= \frac{p(p+m-2-\alpha)[N(m-2)+mq] - pN(p+m-2)(\beta-1) - pN\alpha(p-q)}{(p+m-2)[N(m-2)+mq]}, \\ \theta_2 &= q\alpha(p+m-2)^{-1} + mq[\beta - 1 + \alpha(p-q)(p+m-2)^{-1}][N(m-2)+mq]^{-1}, \\ \theta_3 &= [N(p+m-2)(\beta-1) + N\alpha(p-q)][N(m-2)+mq]^{-1}. \end{aligned}$$

Then (3.6) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_p^p + \frac{1}{2}C_0 p^{2-m} \|\nabla u^{(p+m-2)/m}(t)\|_m^m \leqslant C_2 p^{1+\sigma} \|u(t)\|_p^p\ . \tag{3.11}$$

Let $r > mq^{-1}$, $p_1 = q$, $p_n = rp_{n-1} - m + 2$, $\theta_n = rN(1 - p_n^{-1}p_{n-1})(m + Nr - N)^{-1}$, $\beta_n = (p_n + m - 2)\theta_n^{-1} - p_n$, $n = 2, 3, \ldots$. By Lemma 1 we know that

$$\|u(t)\|_{p_n} \leqslant C^{m/(p_n+m-2)} \|u(t)\|_{p_{n-1}}^{1-\theta_n} \|\nabla u^{(p_n+m-2)/m}(t)\|_m^{m\theta_n/(p_n+m-2)}.$$

Putting this into (3.11) ($p = p_n$) we find that for $0 < t \leqslant T$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_{p_n} + C_0 C^{-m/\theta_n} p_n^{2-m} \|u(t)\|_{p_{n-1}}^{m-2-\beta_n} \|u(t)\|_{p_n}^{1+\beta_n} \leqslant C_2 p_n^{1+\sigma} \|u(t)\|_{p_n}. \tag{3.12}$$

We claim that there exist a bounded sequence $\{\xi_n\}$ and a convergent sequence $\{\lambda_n\}$ such that

$$\|u(t)\|_{p_n} \leqslant \xi_n t^{-\lambda_n}, \quad 0 < t \leqslant T. \tag{3.13}$$

In fact, by Proposition 1, this holds for $n = 1$ if we take $\lambda_1 = 0$, $\xi_1 = \sup\limits_{t \geqslant 0}\{\|u(t)\|_q\}$. If (3.13) is true for $n - 1$, then we have from (3.12) that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_{p_n} &+ C_0 C^{-m/\theta_n} p_n^{2-m} \xi_{n-1}^{m-2-\beta_n} t^{\lambda_{n-1}(\beta_n - m + 2)} \|u(t)\|_{p_n}^{1+\beta_n} \\ &\leqslant C_2 p_n^{1+\sigma} \|u(t)\|_{p_n}.\end{aligned} \tag{3.14}$$

Applying Lemma 2 to (3.14), we conclude that (3.13) also holds for n with $\lambda_n = (1 + \lambda_{n-1}(\beta_n - m + 2))\beta_n^{-1}$ and $\xi_n = (C_0^{-1} C^{m/\theta_n} p_n^{2-m} \xi_{n-1}^{m-2})^{-1/\beta_n} (2\lambda_n + 2C_2 p_n^{1+\sigma})^{1/\beta_n} \xi_{n-1}$, $n = 2, 3, \ldots$

It is not difficult to show that $\lambda_n \to \lambda = N/((m-2)N + mq)$ as $n \to \infty$ and $\{\xi_n\}$ is bounded (cf. [11]). Then (3.3) follows from (3.13) as $n \to \infty$.

In order to obtain (3.4), we first choose $\gamma > \lambda(2-q)^+$. Without loss of generality, we suppose $q \in (1, 2)$. Further, let $\eta(t) \in C[0,\infty) \cap C^1(0,\infty)$ such that $\eta(t) = t^\gamma$ if $t \in [0,1]$; $\eta(t) = 2$ if $t \geqslant 2$ and $\eta(t)$, $\eta'(t) \geqslant 0$ in $[0,\infty)$.

Then, multiplying (3.1) by $\eta(t)u$ we arrive at

$$\begin{aligned}\int_0^t \int_\Omega \eta(s)|\nabla u(s)|^m \,\mathrm{d}x\,\mathrm{d}s &+ \frac{1}{2}\eta(t)\|u(t)\|_2^2 \\ &\leqslant \frac{1}{2}\int_0^t \eta'(s)\|u(s)\|_2^2 \,\mathrm{d}s + \int_0^t \int_\Omega \eta(s)|u(s)|^{\beta+1}\|u(s)\|_\alpha^\alpha \,\mathrm{d}x\,\mathrm{d}s.\end{aligned} \tag{3.15}$$

Noticing that for $t \in (0, T]$,

$$\int_0^t \eta'(s)\|u(s)\|_2^2 \,\mathrm{d}s \leqslant C_0 \int_0^t s^{\gamma-1}\|u(s)\|_q^q \|u(s)\|_\infty^{2-q} \,\mathrm{d}s \leqslant C t^{\gamma - \lambda(2-q)} \tag{3.16}$$

and

$$\|u(s)\|_\alpha^\alpha \int_\Omega |u(s)|^{\beta+1} \,\mathrm{d}x \leqslant C\|u(s)\|_m^{\alpha+\beta+1} \leqslant \frac{1}{2}\|\nabla u(s)\|_m^m + C, \tag{3.17}$$

where the fact $\alpha + \beta < m - 1$ has been used.

Therefore, we obtain from (3.15)–(3.17) that

$$\int_0^t\int_\Omega \eta(s)|\nabla u(s)|^m \,\mathrm{d}x\,\mathrm{d}s \leqslant Ct^{\gamma-\lambda(2-q)}, \quad 0<t\leqslant T. \tag{3.18}$$

Next, let $\varrho(t)=\int_0^t \eta(s)\,\mathrm{d}s$, $t\geqslant 0$. Similarly, multiplying (3.1) by $\varrho(t)u_t$, we get

$$\begin{aligned}\frac{1}{m}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \varrho(t)(|\nabla u(t)|^2+i^{-1})^{m/2}\,\mathrm{d}x+\varrho(t)\|u_t(t)\|_2^2 \\ \leqslant \frac{\varrho'(t)}{m}\int_\Omega(|\nabla u(t)|^2+i^{-1})^{m/2}\,\mathrm{d}x+\varrho(t)\|u(t)\|_\alpha^\alpha\int_\Omega |u(t)|^\beta|u_t(t)|\,\mathrm{d}x.\end{aligned} \tag{3.19}$$

Moreover, for $t\in(0,T]$ we have

$$\begin{aligned}\varrho(t)\|u(t)\|_\alpha^\alpha\int_\Omega |u(t)|^\beta|u_t(t)|\,\mathrm{d}x &\leqslant \varrho(t)\|u_t(t)\|_2\|u(t)\|_{2\beta}^\beta\|u(t)\|_\alpha^\alpha \\ &\leqslant \frac{1}{2}\varrho(t)\|u_t(t)\|_2^2+C\varrho(t)\|u(t)\|_{2\beta}^{2\beta}\|u(t)\|_\alpha^{2\alpha} \\ &\leqslant \frac{1}{2}\varrho(t)\|u_t(t)\|_2^2+Ct^{\gamma+1-2(\alpha+\beta)\lambda}.\end{aligned} \tag{3.20}$$

Now, the application of (3.18)–(3.20) and the integration of (3.19) on $[0,t]$ yield

$$\frac{1}{m}\varrho(t)\|\nabla u(t)\|_m^m+\frac{1}{2}\int_0^t \varrho(s)\|u_s(s)\|_2^2\,\mathrm{d}s\leqslant C(t^{\gamma-\lambda(2-q)}+t^{\gamma+2-2(\alpha+\beta)\lambda}). \tag{3.21}$$

Thus (3.21) gives

$$\frac{\varrho(t)}{m}\|\nabla u(t)\|_m^m\leqslant Ct^{\gamma-2\lambda(\alpha+\beta)}, \quad 0<t\leqslant T. \tag{3.22}$$

This implies (3.4). Similarly, we have the estimate (3.5) from (3.21) and (3.22). Then the proof of Proposition 2 is completed. □

P r o o f of Theorem 1. Noticing that the estimate constants C_1, C_p in (3.2)–(3.5) are independent of i, we can obtain the desired solution $u(t)$ as the limit of $\{u_i\}$(or a subsequence) by the standard compactness argument in [10,11]. The solution $u(t)$ of (1.1) also satisfies (3.2)–(3.5) and (2.3)–(2.5).

It remains to prove the uniqueness. First, for $n=1,2,\ldots$ we define $a_n^+(s)=1$ if $ns\geqslant 1$, and $a_n^+(s)=n^2s^2\mathrm{e}^{1-n^2s^2}$ if $0\leqslant ns<1$. Let $A_n(t)=\int_0^t a_n(s)\,\mathrm{d}s$, $t\in\mathbb{R}^1$, where $a_n(s)$ is an odd extension of $a_n^+(s)$ in $\mathbb{R}^1$.

Let $u_1(t)$, $u_2(t)$ be two solutions of (1.1) which satisfy (2.2) and (2.4). Denote $u(t)=u_1(t)-u_2(t)$. Then by Proposition 1 and Lemma 4.4 in [6, chap 1] we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega A_n(u(t))\,\mathrm{d}x+\gamma_0\int_\Omega |\nabla u|^m a_n'(u)\,\mathrm{d}x\leqslant I \tag{3.23}$$

for some $\gamma_0 > 0$, where

$$I = \int_\Omega (\|u_1\|_\alpha^\alpha |u_1|^{\beta-1}u_1 - \|u_2\|_\alpha^\alpha |u_2|^{\beta-1}u_2) a_n(u)\,\mathrm{d}x \tag{3.24}$$
$$\leqslant \|u_1\|_\alpha^\alpha \int_\Omega \big||u_1|^{\beta-1}u_1 - |u_2|^{\beta-1}u_2\big|\,\mathrm{d}x + \big|\|u_1\|_\alpha^\alpha - \|u_2\|_\alpha^\alpha\big| \int_\Omega |u_2|^{\beta-1}u_2\,\mathrm{d}x$$
$$\leqslant Ct^{-\lambda(\alpha+\beta-1)}\|u(t)\|_1.$$

Then combining (3.23) with (3.24), we obtain for $t \in (0, T]$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega A_n(u(t))\,\mathrm{d}x \leqslant Ct^{-\lambda(\alpha+\beta-1)}\|u(t)\|_1, \tag{3.25}$$

where $C > 0$ is independent of i and n. Integrating (3.25) on $[r, t]$ and letting $n \to \infty$, we have

$$\|u(t)\|_1 \leqslant \|u(r)\|_1 + \int_r^t s^{-\lambda(\alpha+\beta-1)}\|u(s)\|_1\,\mathrm{d}s, \quad 0 < r < t \leqslant T. \tag{3.26}$$

Since $u(t) \in L^q([0, T], L^q(\Omega))$ $(q > 1)$ and $u(0) = 0$, we let $r \to 0^+$ and find that

$$\|u(t)\|_1 \leqslant \int_0^t s^{-\lambda(\alpha+\beta-1)}\|u(s)\|_1\,\mathrm{d}s, \quad 0 < t \leqslant T.$$

Since $0 < \lambda(\alpha+\beta-1) < 1$, the application of the Gronwall's Lemma brings $\|u(t)\|_1 = 0$ on $[0, T]$. Thus $u_1(t) = u_2(t)$ on $[0, T]$. This completes the proof of Theorem 1. □

4. L^∞ estimate for $\nabla u(t)$

In this section we give the proof of Theorem 2. We also use an argument similar to that in [2], [7], [13], but we must treat carefully the nonlinear nonlocal term in the L^∞ estimate of $\nabla u(t)$. As above, we only consider the estimate of $\|\nabla u(t)\|_\infty$ for the smooth solution $u(t)$ of (3.1). As above, let C,C_j be generic constants independent of p and i changeable from line to line. Denote

$$|D^2u|^2 = \sum_{i,j=1}^N u_{i,j}^2, \quad u_{i,j} = \frac{\partial^2 u}{\partial x_i \partial x_j}\ .$$

Multiplying (3.1) by $-\operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $p \geqslant 2$ and integrating on Ω by parts, we have

$$(4.1) \qquad \frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u(t)\|_p^p + \int_\Omega |\nabla u(t)|^{p+m-4}|D^2u(t)|^2\,\mathrm{d}x + C_p\int_\Omega |\nabla u(t)|^{p+m-6}|\nabla(|\nabla u(t)|^2)|^2\,\mathrm{d}x - (N-1)\int_{\partial\Omega} H(x)|\nabla u(t)|^{p+m-2}\,\mathrm{d}S \leqslant -\|u(t)\|_\alpha^\alpha \int_\Omega |u(t)|^{\beta-1}u(t)\operatorname{div}(|\nabla u(t)|^{p-2}\nabla u(t))\,\mathrm{d}x \equiv B$$

with $C_p = \frac{1}{4}(p-2)$. It follows from (2.4) that

$$(4.2) \qquad B = \beta\|u(t)\|_\alpha^\alpha \int_\Omega |u(t)|^{\beta-1}|\nabla u(t)|^p\,\mathrm{d}x \leqslant C\|u(t)\|_\infty^{\alpha+\beta-1}\|\nabla u(t)\|_p^p \leqslant Ct^{-\lambda(\alpha+\beta-1)}\|\nabla u(t)\|_p^p\,.$$

If $H(x) \leqslant 0$ on $\partial\Omega$ and $N > 1$, then by the argument for the elliptic eigenvalue problem (cf. [7]) there exists $\lambda_0 > 0$ such that

$$(4.3) \qquad \|\nabla v\|_2^2 - (N-1)\int_{\partial\Omega} v^2 H(x)\,\mathrm{d}S \geqslant \lambda_0\|v\|_{1,2}^2\,, \qquad \forall v \in W^{1,2}(\Omega).$$

From (4.1)–(4.3) we see that there exist C_1 and C_2, independent of p, such that

$$(4.4) \qquad \frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u(t)\|_p^p + C_1\||\nabla u(t)|^{(p+m-2)/2}\|_{1,2}^2 \leqslant C_2pt^{-\lambda(\alpha+\beta-1)}\|\nabla u(t)\|_p^p\,.$$

Let $p_1 = m$, $p_n = Np_{n-1} - m + 2$, $\theta_n = N(1 - p_{n-1}p_n^{-1})(N - 1 + 2/N)^{-1}$, $n = 2, 3, \ldots$ Then it follows from Lemma 1 that

$$(4.5) \qquad \|\nabla u(t)\|_{p_n} \leqslant C^{2/(p_n+m-2)}\|\nabla u(t)\|_{p_{n-1}}^{1-\theta_n}\||\nabla u(t)|^{(p_n+m-2)/2}\|_{1,2}^{2\theta_n/(p_n+m-2)}.$$

Putting this into (4.4)$(p = p_n)$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u(t)\|_{p_n}^{p_n} + C_1C^{-2/\theta_n}\|\nabla u(t)\|_{p_{n-1}}^{(p_n+m-2)(1-1/\theta_n)}\|\nabla u(t)\|_{p_n}^{(p_n+m-2)/\theta_n} \leqslant C_2p_nt^{-\lambda(\alpha+\beta-1)}\|\nabla u(t)\|_{p_n}^{p_n}\,.$$

Then we take $y_1 = \max\{1, C_0\}$, $z_1 = (1 + 2\lambda(\alpha+\beta))/m$, where C_0 is the constant in the estimate (2.5).

As the proof of Proposition 2, we can show that there exist a bounded sequence $\{y_n\}$ and a convergent sequence $\{z_n\}$ such that

$$\|\nabla u(t)\|_{p_n} \leqslant y_n t^{-z_n}, \qquad 0 < t \leqslant T, \tag{4.6}$$

for which $z_n \to \mu = (2(1+2\lambda(\alpha+\beta)) + N^2)/(2m + (m-2)N^2)$, see [11]. Then the estimate (2.6) is obtained from (4.6) as $n \to \infty$.

Now we consider the estimate (2.7). Let

$$F(t) = \frac{1}{m}\int_\Omega |\nabla u(t)|^m \,\mathrm{d}x.$$

Multiplying (1.1) by u_t and integrating on Ω by parts, we obtain

$$\|u_t(t)\|_2^2 + F'(t) = \|u(t)\|_\alpha^\alpha \int_\Omega u_t(t)|u(t)|^{\beta-1}u(t)\,\mathrm{d}x \leqslant \frac{1}{2}\|u_t(t)\|_2^2 + \frac{1}{2}\|u(t)\|_\alpha^{2\alpha}\|u(t)\|_{2\beta}^{2\beta}.$$

Hence,

$$\frac{1}{2}\|u(t)\|_\alpha^{2\alpha}\|u(t)\|_{2\beta}^{2\beta} - F'(t) \geqslant \frac{1}{2}\|u_t(t)\|_2^2 . \tag{4.7}$$

Similarly, multiplying (1.1) by $u(t)$ gives

(4.8)
$$\|\nabla u(t)\|_m^m = \|u(t)\|_{\beta+1}^{\beta+1}\|u(t)\|_\alpha^\alpha - \int_\Omega u(t)u_t(t)\,\mathrm{d}x \leqslant \frac{1}{2}\|u_t(t)\|_2^2 + \frac{1}{2}\|\nabla u(t)\|_m^m + C_1.$$

This shows that

$$\frac{1}{2}\|u(t)\|_\alpha^{2\alpha}\|u(t)\|_{2\beta}^{2\beta} - F'(t) \geqslant \lambda_1 F(t) - C_1 \tag{4.9}$$

for some $\lambda_1 \in (0,1)$ and $C_1 > 0$. We now estimate the first term of (4.9).

By the assumption $u_0 \in W_0^{1,m}(\Omega)$, we obtain $u_0 \in L^{m^*}(\Omega)$ by the Sobolev embedding theorem, where $m^* = mN/(N-m)$. Then, by Proposition 1, the solution satisfies $u(t) \in L^\infty(\mathbb{R}^+, L^{m^*}(\Omega))$. Since $\alpha+\beta < m-1$ and $2\beta \leqslant m^*$, we get

$$\|u(t)\|_\alpha^{2\alpha} \leqslant C_0, \quad \|u(t)\|_{2\beta}^{2\beta} \leqslant C_0, \quad \forall t \geqslant 0, \tag{4.10}$$

where C_0 depends only on the initial data u_0. Then we have from (4.9) that

$$F'(t) + \lambda_1 F(t) \leqslant C_0. \tag{4.11}$$

This implies that

$$F(t) \leqslant F(0)\mathrm{e}^{-\lambda_1 t} + C_0, \quad t \geqslant 0, \tag{4.12}$$

or

$$\|\nabla u(t)\|_m \leqslant \|\nabla u_0\|_m \mathrm{e}^{-\lambda_1 t} + C_0, \quad t \geqslant 0. \tag{4.13}$$

This is the estimate (2.7). We have completed the proof of Theorem 2. □

Acknowledgement. The authors express their sincere gratitude to the anonymous referees for a number of valuable comments and suggestions.

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