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SPECTRA OF WEIGHTED COMPOSITION OPERATORS ON ALGEBRAS OF ANALYTIC FUNCTIONS ON BANACH SPACES

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Abstract. Let E be a complex Banach space, with the unit ball B_E. We study the spectrum of a bounded weighted composition operator uC_φ on $H^\infty(B_E)$ determined by an analytic symbol φ with a fixed point in B_E such that $\varphi(B_E)$ is a relatively compact subset of E, where u is an analytic function on B_E.

Keywords: bounded analytic function spaces, weighted composition operators, essential norm, spectra

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1. Introduction

Let E denote a complex Banach space with the open unit ball B_E. The space $H^\infty(B_E)$ is the set $\{f\colon B_E \to \mathbb{C}\colon f$ is analytic and bounded$\}$ with the sup-norm $\|f\| = \sup\{|f(x)|\colon x \in B_E\}$. Let $\varphi\colon B_E \to B_E$ be an analytic map, then φ induces a composition operator $C_\varphi : H^\infty(B_E) \to H^\infty(B_E)$ given by $C_\varphi(f) = f \circ \varphi$. Let u be an analytic function on B_E. We consider weighted composition operators uC_φ defined by $uC_\varphi(f) = u \cdot (f \circ \varphi)$, acting on $H^\infty(B_E)$. Obviously, uC_φ is bounded if and only if $u \in H^\infty(B_E)$. We investigate the spectrum of uC_φ. We focus on the case in which φ has a fixed point $z_0 \in B_E$.

The paper is motivated by recent works [8], [6], [11], and [14]. When the symbol φ has an interior fixed point, the spectrum of C_φ on $H^\infty(B_H)$ is characterized by Galindo, Gamelin and Lindström in [8], where B_H is the unit ball of a Hilbert space. Similar results on $H^\infty(B_E)$ can be found in [6] and [11]. In [14], Yuan and Zhou

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characterized the spectrum of uC_φ on $H^\infty(B_N)$, where B_N is the N-dimensional complex ball. All these works can be traced back to [5] and [15].

We establish some notation before outlining the contents of the paper. Let $\|uC_\varphi\|_e$ and $\varrho_e(uC_\varphi)$ denote the essential norm and the essential spectral radius of uC_φ, respectively. The essential norm of an operator is the norm of its equivalence class in the Calkin algebra. We denote by φ_n the n-fold iterate of φ, so that $\varphi_n = \varphi\circ\varphi\circ\ldots\circ\varphi$ (n-times). The spectrum of an operator T is denoted by $\sigma(T)$.

Recently, the spectra of composition operators, both weighted and unweighted ($u \equiv 1$), have been well studied for several spaces of holomorphic functions on the disk or the ball; see [1], [3], [5], [8], [6], [9], [11], [12], [14], [15], and [16] for example. All the arguments follow an analogous pattern: For a disk centered at 0 with radius equal to the essential spectral radius, find a positive integer m, define an invariant subspace H_m of uC_φ, consider the restriction of uC_φ on H_m, denoted by C_m; then it is sufficient to show that $(C_m - \lambda I)^*$ is not bounded from below. In this paper, we characterize the spectra of uC_φ on $H^\infty(B_E)$ using the same strategy.

2. The main result

Recall that the essential norm of a bounded linear operator T is the distance from T to the compact operators, that is,

$$\|T\|_e = \inf\{\|T - K\|\colon\ K \text{ is compact}\}.$$

Clearly T is compact if and only if its essential norm is 0.

Recently, Pascal Lefèvre gave estimates of weighted composition operators on $H^\infty(B_E)$ in [10]. Suppose $u \in H^\infty(B_E)$ and $\varphi\colon\ B_E \to B_E$ is analytic with $\varphi(B_E)$ relatively compact. Let us define

$$n_\varphi(u) = \lim_{r\to 1^-} \sup\{|u(z)|;\ z \in B_E \text{ and } \|\varphi(z)\| \geqslant r\}.$$

The following lemma is quoted from [10].

Lemma 1 (Theorem 2.5 in [10])**.** *Let $u \in H^\infty(B_E)$ and $\varphi\colon\ B_E \to B_E$ be analytic. We assume that $\varphi(B_E)$ is relatively compact. Then*

$$n_\varphi(u) \leqslant \|uC_\varphi\|_e \leqslant 2n_\varphi(u).$$

For uC_φ acting on $H^\infty(B_N)$, recall that its spectral radius is denoted by $\varrho(uC_\varphi)$. Then

$$\varrho(uC_\varphi) = \lim_{n\to\infty} \|(uC_\varphi)^n\|^{1/n}$$

and the essential spectral radius is given by

$$\varrho_e(uC_\varphi) = \lim_{n\to\infty} \|(uC_\varphi)^n\|_e^{1/n}. \tag{1}$$

For any $f \in H^\infty(B_E)$,

$$(uC_\varphi)^n(f(z)) = u(z)u(\varphi(z))\dots u(\varphi_{n-1}(z)) \cdot C_{\varphi_n} f(z). \tag{2}$$

So $(uC_\varphi)^n$ is a weighted composition operator with symbol φ_n and weight $u(z) \times u(\varphi(z))\dots u(\varphi_{n-1}(z))$. Using Lemma 1 and (1), (2) above, the essential spectral radius follows immediately.

Theorem 1. *Let $u \in H^\infty(B_E)$ and $\varphi\colon B_E \to B_E$ be analytic. We assume that $\varphi(B_E)$ is relatively compact. Then*

$$\varrho_e(uC_\varphi) = \lim_{n\to\infty} \Bigl(\lim_{r\to 1^-} \sup_{z\in E_{r_n}} |u(z)u(\varphi(z))\dots u(\varphi_{n-1}(z))|\Bigr)^{1/n}$$

where $E_{r_n} = \{z \in B_E\colon \|\varphi_n(z)\| > r\}$.

Corollary 1. *Suppose $\varphi(B_E)$ is a relatively compact subset of E. Then the essential spectral radius of C_φ on $H^\infty(B_E)$ is either 1 or 0. If C_{φ_n} $(= C_\varphi^n)$ is compact for some $n \geqslant 1$, then $\varrho_e(C_\varphi) = 0$, otherwise $\varrho_e(C_\varphi) = 1$.*

If $\varphi(B_E)$ is not a relatively compact subset of E, it can occur that $0 < \varrho_e(C_\varphi) < 1$. See [7] for more details. To characterize the spectra of uC_φ we first investigate the eigenvalues, which is inspired by [2].

Lemma 2. *Assume that $\varphi\colon B_E \to B_E$ is analytic with a unique interior fixed point a and $u \in H^\infty(B_E)$. If $\mu \neq 0$ is an eigenvalue of uC_φ, then $\mu \in \Bigl\{u(a)\prod_{i=1}^{n} \lambda_i\colon \lambda_i \in \sigma(\mathrm{d}\varphi(a)),\ i = 1,\dots,n$ and $n \in \mathbb{Z}^+\Bigr\} \cup \{u(a)\}$.*

P r o o f. The argument is essentially the same as in [2]. Without loss of generality we suppose $u(a) \neq 0$. Let $f \in H^\infty(B_E)$ be an eigenfunction corresponding to μ, so that

$$\mu f(z) = u(z) f(\varphi(z)).$$

Suppose that $\mu \neq u(a)$ and is not of the form $\Bigl\{u(a)\prod_{i=1}^{n} \lambda_i\colon \lambda_i \in \sigma(\mathrm{d}\varphi(a))\Bigr\}$. It is sufficient to show that $f \equiv 0$. In some neighborhood of a in B_E we have the uniformly convergent Taylor series of f around a:

$$f(z) = \sum_{m=0}^{\infty} \frac{\mathrm{d}^m f(a)}{m!}(z-a)^m.$$

If we show that $\mathrm{d}^m f(a) \equiv 0$ for $m = 0, 1, \ldots$, we are done. For $z = a$, we have $\mu f(a) = u(a)f(a)$, so $f(a) = 0$ as $\mu \neq u(a)$. Assume that $\mathrm{d}^m f(a) = 0$ for $m < n$. Consider

$$u(z) = \sum_{m=0}^{\infty} \frac{\mathrm{d}^m u(a)}{m!}(z-a)^m.$$

Similar to [2], since

$$\varphi(z) = a + \mathrm{d}\varphi(a)(z-a) + \sum_{m=2}^{\infty} \frac{\mathrm{d}^m \varphi(a)}{m!}(z-a)^m$$

converges uniformly in a neighborhood of a, it follows from $\mu f(z) = u(z)f(\varphi(z))$ by comparing the terms at $(z-a)^n$ that

$$\mu\overline{\mathrm{d}^n f(a)} = u(a)(\overline{\mathrm{d}^n f(a)} \circ (\mathrm{d}\varphi(a)\widehat{\otimes}_\pi \ldots \widehat{\otimes}_\pi \mathrm{d}\varphi(a))),$$

where we have used $\mathrm{d}^m f(a) = 0$ and the isomorphic isomorphism between $L^s({}^n E) \simeq (\widehat{\otimes}_{n,s,\pi} E)'$, which associates $A \in L^s({}^n E)$ with $\overline{A} \in (\widehat{\otimes}_{n,s,\pi} E)'$. Thus we have

$$\mu\overline{\mathrm{d}^n f(a)} = u(a)((\mathrm{d}\varphi(a)\widehat{\otimes}_\pi \ldots \widehat{\otimes}_\pi \mathrm{d}\varphi(a))^t \overline{\mathrm{d}^n f(a)}).$$

As is well known, $\sigma(\mathrm{d}\varphi(a)\widehat{\otimes}_\pi \ldots \widehat{\otimes}_\pi \mathrm{d}\varphi(a)) = \sigma((\mathrm{d}\varphi(a)\widehat{\otimes}_\pi \ldots \widehat{\otimes}_\pi \mathrm{d}\varphi(a))^t)$. If $\mathrm{d}^n f(a) \neq 0$, this means that $\mu \in \sigma(u(a)(\mathrm{d}\varphi(a)\widehat{\otimes}_\pi \ldots \widehat{\otimes}_\pi \mathrm{d}\varphi(a)))$. In view of a result of M. Schechter in [13] we have

Sublemma 1. *Let E_i, $I = 1, \ldots, n$ be complex Banach spaces and let $T_i \in L(E_i, E_i)$. Then $\sigma(T_1\widehat{\otimes}_\pi \ldots \widehat{\otimes}_\pi T_n) = \prod_{i=1}^{n} \sigma(T_i)$.*

This implies that $\mu = u(a) \prod_{i=1}^{n} \lambda_i$, where all $\lambda_i \in \sigma(\mathrm{d}\varphi(a))$. But this is a contradiction, so that $\mathrm{d}^n f(a) \equiv 0$ and hence $f = 0$. This completes the proof. □

Because of the Earle and Hamilton's fixed point theorem (see [2]), φ has a unique fixed point a in B_E if $\varphi(B_E)$ lies strictly inside B_E. We have the following lemma.

Lemma 3. *Assume that $\varphi\colon B_E \to B_E$ is analytic and $\varphi(B_E)$ is a relatively compact set of E which lies strictly inside B_E. Let $u \in H^\infty(B_E)$. Then $\left\{u(a) \prod_{i=1}^{n} \lambda_i \colon \lambda_i \in \sigma(\mathrm{d}\varphi(a)),\ i = 1, \ldots, n \text{ and } n \in \mathbb{Z}^+\right\} \cup \{u(a)\} \subset \sigma(uC_\varphi)$ where a is the unique interior fixed point of φ.*

P r o o f. Evidently, C_φ is compact on $H^\infty(B_E)$, so uC_φ is such as well. Trivially, $0 \in \sigma(uC_\varphi)$. Without loss of generality we may suppose $u(a) \neq 0$. Since no $f \in H^\infty(B_E)$ can satisfy the equation

$$u(a)f(z) - uC_\varphi(f) = 1,$$

it follows that $u(a) \in \sigma(uC_\varphi)$. Indeed, if $u(a)f(z) - u(z)f(\varphi(z)) = 1$, then $u(a)f(a) - u(a)f(\varphi(a)) = 1$. This is a contradiction since $\varphi(a) = a$. The rest of the proof can be done as in the proof of Lemma 4 in [2]. □

Indeed,we can give the spectrum of a compact weighted composition operator on $H^\infty(B_E)$.

Theorem 2. *If uC_φ is compact on $H^\infty(B_E)$, $\varphi\colon B_E \to B_E$ is analytic and $\varphi(B_E)$ is a relatively compact set of E which lies strictly inside B_E, then $\sigma(uC_\varphi) = \Big\{u(a)\prod_{i=1}^{n}\lambda_i\colon \lambda_i \in \sigma(\mathrm{d}\varphi(a)),\ i = 1,\dots,n$ and $n \in \mathbb{Z}^+\Big\} \cup \{u(a)\}$ where a is the unique interior fixed point of φ.*

To give the spectra of non-compact weighted composition operators, we need several lemmas quoted from [11] and [6]. First, we need the following definitions.

Definition 1. An *interpolating sequence* $\{z_j\}$ in the ball B_E is a sequence for which, given any bounded sequence $\{c_j\}$ of complex numbers, there is a bounded analytic function f such that $f(z_j) = c_j$.

Definition 2. Let $\varphi\colon B_E \to B_E$ be an analytic map. A finite or infinite sequence $(x_k)_{k\geqslant 0} \subset B_E$ is said to be an *iteration sequence* for φ if $\varphi(x_k) = x_{k+1}$.

Definition 3. Let $\varphi\colon B_E \to B_E$ be an analytic map. We say that $\varphi\colon B_E \to B_E$ satisfies the *approaching condition* if $\varphi_n(B_E)$ is not strictly inside B_E for any $n \in \mathbb{N}$.

Lemma 4 (Lemma 4.2.9 in [11], Lemma 3.3 in [6])**.** *Let E be a complex Banach space and let $\varphi\colon B_E \to B_E$ be an analytic map such that $\varphi(0) = 0$ and $\|\mathrm{d}\varphi(0)\| < 1$. Suppose that there exist $\delta > 0$ and $\varepsilon > 0$ such that*

$$\frac{1 - \|\varphi(x)\|}{1 - \|x\|} \geqslant 1 + \varepsilon, \quad \text{for all } x \in \varphi(B_E) \text{ such that } \|x\| \geqslant \delta.$$

Then, there exists a constant $M \geqslant 1$ which depends only on ε, such that any finite iteration sequence $\{x_0, x_1, \dots, x_N\}$ satisfying $x_0 \in \varphi(B_E)$ and $\|x_N\| \geqslant \delta$ is an interpolating sequence for $H^\infty(B_E)$ with the constant of interpolation not greater than M.

Lemma 5 (Lemma 4.2.10 in [11], Lemma 3.4 in [6]). *Let E and F be Banach spaces. Let $C\colon E\oplus F\to E\oplus F$ be a linear operator which leaves F invariant and for which $C|_E\colon E\to E\oplus F$ is a compact operator. If the operator C has the matrix representation*

$$C=\begin{pmatrix} X & 0\\ Y & Z\end{pmatrix}$$

with respect to this decomposition, then $\sigma(C)=\sigma(X)\cup\sigma(Z)$.

Denote by P_nf the nth term of the Taylor series of the analytic function $f\in H^\infty(B_E)$ at 0. Set

$$H^\infty_m(B_E)=\{f\in H^\infty(B_E)\colon\ P_nf=0 \text{ for } n=0,1,\dots,m-1\}.$$

Denoting by $P(^{<m}E)$ the subspace of polynomials of degree less than m, it is clear that $H^\infty(B_E)$ is isomorphic to $H^\infty_m(B_E)\oplus P(^{<m}E)$.

Lemma 6 (Lemma 4.2.11 in [11], Lemma 3.5 in [6]). *Let $\varphi\colon B_E\to B_E$ be an analytic map such that $\varphi(0)=0$. Assume $u\in H^\infty(B_E)$. Then C_φ leaves the space $H^\infty_m(B_E)$ invariant for any $m\geqslant 1$. So does uC_φ.*

Now, we have the main result which describes the spectrum of uC_φ for the non compact case.

Theorem 3. *Suppose $u\in H^\infty(B_E)$ and φ is a holomorphic map of B_E into B_E satisfying $\varphi(0)=0$, $\|\mathrm{d}\varphi(0)\|<1$ such that $\varphi(B_E)$ is a relatively compact subset of E. Suppose that φ satisfies the approaching condition and the following Julia-type estimate: for any $0<\delta<1$, there exists $\varepsilon>0$ such that*

$$\frac{1-\|\varphi(x)\|}{1-\|z\|}\geqslant 1+\varepsilon,\quad \text{for all}\ \ x\in\varphi(B_E)\ \text{such that}\ x\geqslant\delta. \tag{3}$$

Then

$$\sigma(uC_\varphi)=\{\lambda\in\mathbb{C}\colon\ |\lambda|\leqslant\varrho_e(uC_\varphi)\}\cup\{u(0),u(0)\mu\}$$

where μ stands for all products of eigenvalues of $\mathrm{d}\varphi(0)$.

The proof is an adaptation of Theorem 8 in [1], Theorem 4.2.12 in [11] and Theorem 3.6 in [6].

Proof. By Lemma 3 we have that $\{u(0),u(0)\mu\}\subset\sigma(uC_\varphi)$. For $\lambda\in\sigma(uC_\varphi)$ with $|\lambda|>\varrho_e(uC_\varphi)$, it follows that λ is an eigenvalue (that is true for all bounded operators, see Proposition 2.2 in [4]). If $\lambda\neq 0$ is an eigenvalue, Lemma 3 gives that $\lambda\in\{u(0),u(0)\mu\}$, so it remains to show that

$$\{\lambda\in\mathbb{C}\colon\ |\lambda|\leqslant\varrho_e(uC_\varphi)\}\subset\sigma(uC_\varphi).$$

If $\varrho_e(uC_\varphi) = 0$, the result is proved since then 0 is in the (non-empty) essential spectrum, hence in the spectrum. Now assume that $\varrho_e(uC_\varphi) > 0$ and denote $\varrho_e(uC_\varphi)$ by ϱ. Fix a λ with $0 < |\lambda| < \varrho$.

As shown in [11],

$$C_\varphi|_{P(^{<m}E)} \colon P(^{<m}E) \to H^\infty(B_E)$$

is compact by Proposition 3 in [2], hence so is $uC_\varphi|_{P(^{<m}E)}$.

Since $H^\infty(B_E) = H^\infty_m(B_E) \oplus P(^{<m}E)$ and $H^\infty_m(B_E)$ is an invariant subspace of C_φ, it is sufficient to show by Lemma 5 that $\lambda \in \sigma(C_m)$ if we let C_m denote the restriction of uC_φ to $H^\infty_m(B_E)$.

We find a positive integer m such that $(C_m - \lambda I)^*$ is not bounded from below, which means $C_m - \lambda I$ is not invertible.

We use the argument in the proof of Theorem 4.2.12 in [11] (see also the proof of Theorem 3.6 in [6]). Since $u \in H^\infty(B_E)$ is continuous, $0 < C := \max\Big\{\sup\limits_{\|z\|\leqslant\delta} |u(z)|, |u(z_n)|\Big\} < \infty$. Choose m great enough so that

$$\frac{c^m C}{|\lambda|} < 1. \tag{4}$$

Next we will show $C_m^* - \bar{\lambda} I$ is not bounded below on $H^\infty_m(B_E)$.

If $(z_k)_{k=0}^\infty$ is an iteration sequence for φ with n defined as above, let us define a linear functional $L_{\lambda,u}$ on $H^\infty_m(B_E)$ by

$$L_{\lambda,u}(f) = \sum_{k=0}^\infty \lambda^{-k} u(z_0)\dots u(z_{k-1}) f(z_k), \quad f \in H^\infty_m(B_E)$$

where we agree that $u(z_0)u(z_{-1}) = 1$ in the first term of the sum. Apparently,

$$\begin{aligned}((\lambda I - C_m)^*(L_{\lambda,u}))(f) &= \lambda L_{\lambda,u}(f) - L_{\lambda,u}(C_m(f)) \\ &= \lambda L_{\lambda,u}(f) - L_{\lambda,u}(u(f\circ\varphi)) \\ &= \lambda \sum_{k=0}^\infty \lambda^{-k} u(z_0)\dots u(z_{k-1}) f(z_k) \\ &\quad - \sum_{k=0}^\infty \lambda^{-k} u(z_0)\dots u(z_{k-1}) u(z_k) f(\varphi(z_k)) \\ &= f(z_0).\end{aligned}$$

Notice that for $f \in H_m^\infty(B_E)$, the maximum principle implies that $|f(z)| \leqslant \|f\|_\infty \|z\|^m$ for all $x \in B_E$. Now we obtain

$$\begin{aligned}
&\sum_{k=n+1}^{\infty} \frac{|u(z_0)| \dots |u(z_{k-1})||f(z_k)|}{|\lambda|^k} \\
&\quad \leqslant \sum_{k=n+1}^{\infty} \frac{|u(z_0)| \dots |u(z_{k-1})|\|f\|_\infty \|z_k\|^m}{|\lambda|^k} \\
&\quad \leqslant \frac{|u(z_0)| \dots |u(z_{n-1})|}{|\lambda|^n} \sum_{k=n+1}^{\infty} \frac{|u(z_n)| \dots |u(z_{k-1})|\|f\|_\infty \|z_k\|^m}{|\lambda|^{k-n}} \\
&\quad \leqslant \frac{|u(z_0)| \dots |u(z_{n-1})|\|z_n\|^m}{|\lambda|^n} \sum_{k=n+1}^{\infty} \frac{C^{k-n}\|f\|_\infty c^{(k-n)m}}{|\lambda|^{k-n}}.
\end{aligned}$$

Thus

$$\left|\sum_{k=n+1}^{\infty} \frac{|u(z_0)| \dots |u(z_{k-1})||f(z_k)|}{|\lambda|^k}\right| \leqslant \frac{|u(z_0)| \dots |u(z_{n-1})|\|z_n\|^m \|f\|_\infty}{|\lambda|^n} \sum_{k=1}^{\infty} \frac{(c^m C)^k}{|\lambda|^k}$$

for $f \in H_m^\infty(B_E)$.

Now choose an m-homogeneous polynomial P satisfying $\|P\| = 1$ and $|P(z_n)| = \|z_n\|^m$. Lemma 4 gives that there exist an interpolation constant $M = M(c)$ and $g \in H^\infty(B_E)$ such that $\|g\| \leqslant M$, $g(z_k) = 0$ for $0 \leqslant k < n$ and $|g(z_n)| = 1$ with $g(z_k)u(z_0) \dots u(z_{n-1}) = |u(z_0)| \dots |u(z_{n-1})|$. Then $P \cdot g \in H_m^\infty(B_E)$ satisfies $\|P \cdot g\| \leqslant M$, hence for $f = P \cdot g$ we have

$$\begin{aligned}
&|L_{\lambda,u}(P \cdot g)| \\
&\quad = \left|\sum_{k=0}^{\infty} \frac{u(z_0) \dots u(z_{k-1})(P \cdot g)(z_k)}{\lambda^k}\right| \\
&\quad = \left|\frac{u(z_0) \dots u(z_{n-1})(P \cdot g)(z_n)}{\lambda^n} + \sum_{k=n+1}^{\infty} \frac{u(z_0) \dots u(z_{k-1})(P \cdot g)(z_k)}{\lambda^k}\right| \\
&\quad \geqslant \left|\frac{u(z_0) \dots u(z_{n-1})(P \cdot g)(z_n)}{\lambda^n}\right| - \left|\sum_{k=n+1}^{\infty} \frac{u(z_0) \dots u(z_{k-1})(P \cdot g)(z_k)}{\lambda^k}\right| \\
&\quad \geqslant \frac{|u(z_0)| \dots |u(z_{n-1})|\|z_n\|^m}{|\lambda|^n} - \frac{|u(z_0)| \dots |u(z_{n-1})|\|z_n\|^m M}{|\lambda|^n} \sum_{k=1}^{\infty} \frac{(c^m C)^k}{|\lambda|^k} \\
&\quad = \frac{|u(z_0)| \dots |u(z_{n-1})|\|z_n\|^m}{|\lambda|^n} \left(1 - M \sum_{k=1}^{\infty} \frac{(c^m C)^k}{|\lambda|^k}\right).
\end{aligned}$$

It is easy to check that

$$\sum_{k=1}^{\infty} \frac{(c^m C)^k}{|\lambda|^k} \to 0 \quad \text{as } m \to 0.$$

Choose m so large that, in addition to (4) to hold, we have

$$M \sum_{k=1}^{\infty} \frac{(c^m C)^k}{|\lambda|^k} < \frac{1}{2}.$$

Then, since $|L_{\lambda,u}(P \cdot f)| \leqslant M\|L_{\lambda,u}\|$ and $\|z_n\| > \delta > 1/4$, we get

$$|L_{\lambda,u}(P \cdot g)| \geqslant \frac{|u(z_0)| \ldots |u(z_{n-1})| \|z_n\|^m}{2|\lambda|^n} \geqslant \frac{|u(z_0)| \ldots |u(z_{n-1})|}{2 \cdot 4^m |\lambda|^n}.$$

Recall that

$$((\lambda I - C_m)^*(L_{\lambda,u}))(f) = f(z_0).$$

Hence

$$\|(\lambda I - C_m)^*(L_{\lambda,u})\| \leqslant 1.$$

Since $|\lambda| < \varrho$, we can pick μ such that $|\lambda| < \mu < \varrho$. So there exists n_0 such that for all $s \geqslant n_0$,

$$\|(uC_\varphi)^s\|_e > \mu^s.$$

Hence for any $n \geqslant n_0$ we can find a $w \in B_E$ such that $|u(w)||u(\varphi(w))| \ldots \times |u(\varphi_{n-1}(w))| \geqslant \mu^n/2 > 0$ and $\|\varphi_n(w)\| \geqslant \delta$.

This defines an iteration sequence $(x_k)_{k=0}^{\infty}$ by letting $x_0 = w$ and $x_{k+1} = \varphi(x_k)$ for $n \geqslant 0$. Then $\|x_n\| = \|\varphi_n(w)\| \geqslant \delta$ and $|u(x_0)||u(x_1)| \ldots |u(x_{n-1})| \geqslant \mu^n/2 > 0$, and

$$\frac{\|(C_m - \lambda I)^* L_{\lambda,u}\|}{\|L_{\lambda,u}\|} \leqslant \frac{2 \cdot 4^m |\lambda|^n}{|u(x_0)| \ldots |u(x_{n-1})|} \leqslant 4^{m+1} \frac{|\lambda|^n}{\mu^n}. \tag{5}$$

Thus, we can form iteration sequences for which n is arbitrary. Hence $(C_m - \lambda I)^*$ is not bounded below as desired. This completes the proof since the spectrum is a closed set. □

In Theorem 3 we assume that the Julia estimate (3) is satisfied for E to describe the spectrum of uC_φ. It is shown that the estimate exists when E is a Hilbert space ([8]) and $E = C_0(X)$, the continuous $\mathbb{C}$-valued functions vanishing at infinity on a locally compact space X ([6] and [11]). Thus we have the following corollary.

Corollary 2. *Let E be a Hilbert space or a $C_0(X)$ space. Suppose $u \in H^\infty(B_E)$ and φ is a holomorphic map of B_E into B_E satisfying $\varphi(0) = 0$, $\|\mathrm{d}\varphi(0)\| < 1$ such that $\varphi(B_E)$ is a relatively compact subset of E. Suppose that φ satisfies the approaching condition. Then*

$$\sigma(uC_\varphi) = \{\lambda \in \mathbb{C} \colon |\lambda| \leqslant \varrho_e(uC_\varphi)\} \cup \{0, u(0), u(0)\mu\}$$

where μ stands for all products of eigenvalues of $\mathrm{d}\varphi(0)$.

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