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PROPERTIES OF DISTANCE FUNCTIONS ON CONVEX SURFACES AND APPLICATIONS

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Abstract. If X is a convex surface in a Euclidean space, then the squared intrinsic distance function $\operatorname{dist}^2(x,y)$ is DC (d.c., delta-convex) on $X\times X$ in the only natural extrinsic sense. An analogous result holds for the squared distance function $\operatorname{dist}^2(x,F)$ from a closed set $F\subset X$. Applications concerning r-boundaries (distance spheres) and ambiguous loci (exoskeletons) of closed subsets of a convex surface are given.

Keywords: distance function, convex surface, Alexandrov space, DC manifold, ambiguous locus, skeleton, r-boundary

MSC 2010: 53C45, 52A20

1. Introduction

The geometry of 2-dimensional convex surfaces in $\mathbb{R}^3$ was thoroughly studied by A. D. Alexandrov [1]. Important generalizations for n-dimensional convex surfaces in $\mathbb{R}^{n+1}$ are due to A. D. Milka (see, e.g., [12]). Many (but not all) results on geometry of convex surfaces are special cases of results of the theory of Alexandrov spaces with curvature bounded from below.

Let $X\subset\mathbb{R}^{n+1}$ be an n-dimensional (closed bounded) convex surface and $\emptyset\neq F\subset X$ a closed set. We will prove (Theorem 3.8) that

(A) *the intrinsic distance* $d_F(x):=\operatorname{dist}(x,F)$ *is locally DC on* $X\setminus F$ *in the natural extrinsic sense (with respect to natural local charts).*

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It is well known that, in a Euclidean space, d_F is not only locally DC but even locally semiconcave on the complement of F. This was generalized to smooth Riemannian manifolds in [11].

The result (A) can be applied to some problems from the geometry of convex surfaces that are formulated in the language of intrinsic distance functions. The reason of this is that DC functions (i.e., functions which are differences of two convex functions) have many nice properties which are close to those of C^2 functions. We present two applications.

The first (Theorem 4.1) concerns r-boundaries (distance spheres) of a closed set $F \subset X$ in the cases $\dim X = 2, 3$. It implies that, for almost all r, the r-boundary is a Lipschitz manifold, and so provides an analogue of well-known results proved (in Euclidean spaces) by Ferry [6] and Fu [7].

The second application (Theorem 4.4) concerns the ambiguous locus (exoskeleton) of a closed subset of an n-dimensional ($n \in \mathbb{N}$) convex surface. This result is essentially stronger than the corresponding result of T. Zamfirescu in Alexandrov spaces with curvature bounded from below.

It is not clear whether the results of these applications can be obtained as consequences of results in Alexandrov spaces (possibly with some additional properties). In any case, there are serious obstacles when trying to obtain such generalizations by our methods (see Remark 4.2).

To explain briefly what is the "natural extrinsic sense" from (A), consider for a while an unbounded convex surface $X \subset \mathbb{R}^{n+1}$ which is the graph of a convex function $f\colon \mathbb{R}^n \to \mathbb{R}$, and denote $x^* := (x, f(x))$ for $x \in \mathbb{R}^n$. Then (A) also holds (see Remark 3.9) and is equivalent to the statement

(B) *the function* $h(x) := \mathrm{dist}(x^*, F)$ *is locally DC on* $\{x \in \mathbb{R}^n\colon\ x^* \notin F\}$.

Moreover, it is true that

(C) $h^2(x) := \mathrm{dist}^2(x^*, F)$ *is DC on the whole* $\mathbb{R}^n$, and
(D) *the function* $g(x, y) := \mathrm{dist}^2(x^*, y^*)$ *is DC on* $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$.

For a natural formulation of the corresponding results (Theorems 3.8 and 3.4) for a closed bounded convex surface X, we will define in a canonical way the structure of a DC manifold on X and $X \times X$.

A weaker version of the result (C) (in the case $n = 2$) has been known for a long time to the second author, who used a method similar to that of Alexandrov's proof (for two-dimensional convex surfaces) of Alexandrov-Toponogov theorem, namely approximating a general convex surface by polyhedral convex surfaces and considering a developing of those polyhedral convex surfaces "along geodesics". However, he was not able to formalize the geometrically transparent method of developings in a rigorous way.

In the present article we use another method suggested by the first author. Namely, we use the well-known semiconcavity properties of distance functions on X and $X \times X$ in an intrinsic sense (i.e., in the sense of the theory of length spaces). When applying this method, it was not necessary to use developings. However, our proof still needs approximation by polyhedral surfaces.

Note that, in the case $n = 1$, the above statements (A)–(D) have straigthforward proofs. Moreover, the functions h, h^2 from (B) and (C) (even in the case $n = 1$) can happen to be neither locally semiconcave nor locally semiconvex on $\{x \in \mathbb{R}^n\colon\ x^* \notin F\}$. (To show this, it is sufficient to set $f(x) = \max(|x|, 1)$ and $F = \{(-2, 2)\}$. Then h is clearly positive and continuous on $(-2, \infty)$ and affine with the slopes $\sqrt{2}$, 1, $\sqrt{2}$ on the intervals $[-2, -1]$, $[-1, 1]$, $[1, \infty)$, respectively. Consequently, h is not semiconvex (resp. semiconcave) on any neighbourhood of -1 (resp. 1). The same is true also for the function $g := h^2$, since clearly $g'_-(-1) > g'_+(-1)$ and $g'_-(+1) < g'_+(+1)$.)

The organization of the paper is as follows. In Section 2 (Preliminaries) we recall some facts concerning length spaces, semiconcave functions, DC functions, DC manifolds, and DC surfaces. Further we prove (by standard methods) two needful technical lemmas on approximation of convex surfaces by polyhedral surfaces. In Section 3 we prove our main results on distance functions on closed bounded convex surfaces. Section 4 is devoted to applications which we already briefly described above. In the last short Section 5 we present several remarks and questions concerning DC structures on length spaces.

2. Preliminaries

In a metric space, $B(c, r)$ denotes the open ball with center c and radius r. The symbol $\mathscr{H}^k$ stands for the k-dimensional Hausdorff measure. If $a, b \in \mathbb{R}^n$, then $[a, b]$ denotes the segment joining a and b. If F is a Lipschitz mapping, then $\operatorname{Lip} F$ stands for the least Lipschitz constant of F.

If W is a unitary space and V is a subspace of W, then we denote by $V^\perp_W$ the orthogonal complement of V in W.

If f is a mapping from a normed space X to a normed space Y, then the symbol $df(a)$ stands for the (Fréchet) differential of f at $a \in X$. If $df(a)$ exists and

$$\lim_{x,y \to a,\, x \neq y} \frac{f(y) - f(x) - df(a)(y - x)}{\|y - x\|} = 0,$$

then we say that f is *strictly differentiable* at a (cf. [13, p. 19]).

For the sake of brevity, we introduce the following notation (we use the symbol Δ^2, though $\Delta^2 f(x, y)$ is one half of a second difference).

Definition 2.1. If f is a real function defined on a subset U of a vector space and $x, y, \frac{1}{2}(x+y) \in U$, we denote

$$\Delta^2 f(x,y) := \frac{f(x)+f(y)}{2} - f\Big(\frac{x+y}{2}\Big). \tag{1}$$

Note that, if $f(y) = \|y\|^2$, $y \in \mathbb{R}^n$, then

$$\Delta^2 f(x+h, x-h) = \frac{\|x+h\|^2 + \|x-h\|^2}{2} - \|x\|^2 = \|h\|^2. \tag{2}$$

We shall need the following easy lemma. Its first part is an obvious consequence of [22, Lemma 1.16] (which works with convex functions). The second part clearly follows from the first.

Lemma 2.2.

(i) *Let $f\colon (a,b) \to \mathbb{R}$ be a continuous function. Suppose that for every $t \in (a,b)$ and $\delta > 0$ there exists $0 < d < \delta$ such that $\Delta^2 f(t+d, t-d) \leqslant 0$. Then f is concave on (a,b).*

(ii) *Let f be a continuous function on an open convex subset $C \subset \mathbb{R}^n$. Suppose that for every $x \in C$ there exists $\delta > 0$ such that $\Delta^2 f(x+h, x-h) \leqslant 0$ whenever $\|h\| < \delta$. Then f is concave on C.*

2.1. Length spaces and semiconcave functions.

A metric space (X,d) is called a *length (or inner or intrinsic) space* if, for each $x, y \in X$, $d(x,y)$ equals the infimum of the lengths of curves joining x and y (see [3, p. 38] or [17, p. 824]). If X is a length space, then a curve $\varphi : [a,b] \to X$ is called *minimal*, if it is a shortest curve joining its endpoints $x = \varphi(a)$ and $y = \varphi(b)$ parametrized by the arc-length. A length space X is called a *geodesic (or strictly intrinsic) space* if each pair of points in X can be joined by a minimal curve. Note that any complete, locally compact length space is geodesic (see [17, Theorem 8]).

Alexandrov spaces with curvature bounded from below are defined as length spaces which have a lower curvature bound in the sense of Alexandrov. The precise definition of these spaces can be found in [3] or [17]. (Frequently Alexandrov spaces are supposed to be complete and/or finite dimensional.)

If X is a length space and $\varphi\colon [a,b] \to X$ a minimal curve, then the point $s = \varphi(\frac{1}{2}(a+b))$ is called *the midpoint of the minimal curve φ*. A point t is called *a midpoint of x, y* if it is the midpoint of a minimal curve φ joining x and y. If φ as above can be chosen to lie in a set $G \subset X$, we will say that t is *a G-midpoint of x, y*.

One of several natural equivalent definitions (see [5, Definition 1.1.1 and Proposition 1.1.3]) of semiconcavity in $\mathbb{R}^n$ reads as follows.

Definition 2.3. A function u on an open set $A \subset \mathbb{R}^n$ is called *semiconcave* with a *semiconcavity constant* $c \geqslant 0$ if u is continuous on A and

$$\Delta^2 u(x+h, x-h) \leqslant \frac{c}{2}\|h\|^2 \tag{3}$$

whenever $x, h \in \mathbb{R}^n$ and $[x-h, x+h] \subset A$.

Remark 2.4. It is well known and easy to see (cf. [5, Proposition 1.1.3]) that u is semiconcave on A with semiconcavity constant c if and only if the function $g(x) = u(x) - \frac{1}{2}c\|x\|^2$ is locally concave on A.

The notion of semiconcavity extends naturally to length spaces X. The authors working in the theory of length spaces use mostly the following terminology (cf. [16, p. 5] or [17, p. 862]).

Definition 2.5. Let X be a geodesic space. Let $G \subset X$ be open, $c \geqslant 0$, and let $f\colon G \to \mathbb{R}$ be a locally Lipschitz function.

(i) We say that f is c-*concave* if for each minimal curve $\gamma\colon [a,b] \to G$, the function $g(t) = f \circ \gamma(t) - \frac{1}{2}ct^2$ is concave on $[a,b]$.

(ii) We say that f is *semiconcave* on G if for each $x \in G$ there exists $c \geqslant 0$ such that f is c-concave on an open neighbourhood of x.

Remark 2.6. If $X = \mathbb{R}^n$, then c-concavity coincides with semiconcavity with constant c.

We will need the following simple well-known characterization of c-concavity. Because of lack of the reference, we give the proof.

Lemma 2.7. *Let Y be a geodesic space. Let $M \subset Y$ be open, $c \geqslant 0$, and let $f\colon M \to \mathbb{R}$ be a locally Lipschitz function. Then the following statements are equivalent.*

(i) *f is c-concave on M.*

(ii) *If $x, y \in M$, and s is an M-midpoint of x, y, then*

$$\frac{f(x)+f(y)}{2} - f(s) \leqslant \frac{c}{2}d^2, \tag{4}$$

where $d := \frac{1}{2}\operatorname{dist}(x,y)$.

Proof. Suppose that (i) holds. To prove (ii), let x, y, s, d be as in (ii). Choose a minimal curve $\gamma\colon [a,b] \to M$ with $\gamma(a) = x$, $\gamma(b) = y$ and $\gamma(\frac{1}{2}(a+b)) = s$. By (i), the function $g(t) = f \circ \gamma(t) - \frac{1}{2}ct^2$ is concave on $[a,b]$. So $\tilde{f} := f \circ \gamma$ is semiconcave with semiconcavity constant c on (a,b) by Remark 2.4. Consequently, $\Delta^2 \tilde{f}(b-h, a+h) \leqslant \frac{1}{2}c|\frac{1}{2}(b-a)-h|^2$ for each $0 < h < \frac{1}{2}(b-a)$. By continuity of $\tilde{f}$ we clearly obtain (4), since $d = \frac{1}{2}(b-a)$.

To prove (ii) $\Rightarrow$ (i), consider a minimal curve $\gamma\colon [a,b] \to M$ and suppose that f satisfies (ii). It is easy to see that then $\tilde{f} := f \circ \gamma$ is semiconcave with semiconcavity constant c on (a,b). By Remark 2.4, $g(t) = f \circ \gamma(t) - \frac{1}{2}ct^2$ is concave on (a,b), and therefore (by continuity of g), also on $[a,b]$. $\square$

2.2. DC manifolds and DC surfaces.

Definition 2.8. Let C be a nonempty open convex set in a real normed linear space X. A function $f\colon C \to \mathbb{R}$ is called *DC* (or d.c., or delta-convex) if it can be represented as a difference of two continuous convex functions on C.

If Y is a finite-dimensional normed linear space, then a mapping $F\colon C \to Y$ is called *DC*, if $y^* \circ F$ is a DC function on C for each linear functional $y^* \in Y^*$.

Remark 2.9.

(i) To prove that F is DC, it is clearly sufficient to show that $y^* \circ F$ is DC for each y^* from a basis of Y^*.
(ii) Each DC mapping is clearly locally Lipschitz.
(iii) There are many works on optimization that deal with DC functions. A theory of DC (delta-convex) mappings in the case when Y is a general normed linear space was built in [22].

Some basic properties of DC functions and mappings are established in the following lemma.

Lemma 2.10. *Let X, Y, Z be finite-dimensional normed linear spaces, let $C \subset X$ be a nonempty open convex set, and $U \subset X$ and $V \subset Y$ open sets.*

(a) ([2]) *If the derivative of a function f on C is Lipchitz, then f is DC. In particular, each affine mapping is DC.*
(b) ([8]) *If a mapping $F\colon C \to Y$ is locally DC on C, then it is DC on C.*
(c) ([8]) *Let a mapping $F\colon U \to Y$ be locally DC, $F(U) \subset V$, and let $G\colon V \to Z$ be locally DC. Then $G \circ F$ is locally DC on U.*
(d) ([22]) *Let $F\colon U \to V$ be a bilipschitz bijection which is locally DC on U. Then F^{-1} is locally DC on V.*

Since locally DC mappings are stable with respect to compositions (Lemma 2.10 (c)), the notion of an n-dimensional DC manifold can be defined in an obvious way, see [10, § 2.6, § 2.7]. The importance of this notion was shown in Perelman's preprint [15], cf. Section 5.

Definition 2.11. Let X be a paracompact Hausdorff topological space and $n \in \mathbb{N}$.

(i) We say that (U, φ) is an *n-dimensional chart on X* if U is a nonempty open subset of X and $\varphi\colon U \to \mathbb{R}^n$ is a homeomorphism of U onto an open set $\varphi(U) \subset \mathbb{R}^n$.

(ii) We say that two n-dimensional charts (U_1, φ_1) and (U_2, φ_2) on X are *DC-compatible* if $U_1 \cap U_2 = \emptyset$ or $U_1 \cap U_2 \neq \emptyset$ and the *transition maps* $\varphi_2 \circ (\varphi_1)^{-1}$ and $\varphi_1 \circ (\varphi_2)^{-1}$ are locally DC (on their domains $\varphi_1(U_1 \cap U_2)$ and $\varphi_2(U_1 \cap U_2)$, respectively).

(iii) We say that a system $\mathscr{A}$ of n-dimensional charts on X is an *n-dimensional DC atlas on X*, if the domains of the charts from $\mathscr{A}$ cover X and any two charts from $\mathscr{A}$ are DC-compatible.

Obviously, each n-dimensional DC atlas $\mathscr{A}$ on X can be extended to a uniquely determined maximal n-dimensional DC atlas (which consists of all n-dimensional charts on X that are DC-compatible with all charts from A). We will say that *X is equipped with an (n-dimensional) DC structure* (or with a structure of an n-dimensional DC manifold), if a maximal n-dimensional DC atlas on X is determined (e.g., by a choice of an n-dimensional DC atlas).

Let X be equipped with a DC structure and let f be a function defined on an open set $G \subset X$. Then we say that f is *DC* if $f \circ \varphi^{-1}$ is locally DC on $\varphi(U \cap G)$ for each chart (U, φ) from the maximal DC atlas on X such that $U \cap G \neq \emptyset$. Clearly, it is sufficient to check this condition for each chart from an arbitrary fixed DC atlas.

Remark 2.12.

(i) If we consider, in the definition of the chart (U, φ), a mapping φ from U to an n-dimensional unitary space H_φ, the whole Definition 2.11 does not change sense. (Indeed, we can identify H_φ with $\mathbb{R}^n$ by an isometry because of Lemma 2.10 (a), (c).) In the sequel, it will be convenient for us to use such (formally more general) charts with range in an n-dimensional linear subspace of a Euclidean space.

(ii) If X, Y are nonempty spaces equipped with m, n-dimensional DC structures, respectively, then the Cartesian product $X \times Y$ is canonically equipped with an $(m+n)$-dimensional DC structure. Indeed, let $\mathscr{A}_X$, $\mathscr{A}_Y$ be m, n-dimensional DC atlases on X, Y, respectively. Then

$$\mathscr{A} = \{(U_X \times U_Y, \varphi_X \otimes \varphi_Y)\colon\ (U_X, \varphi_X) \in \mathscr{A}_X,\ (U_Y, \varphi_Y) \in \mathscr{A}_Y\}$$

is an $(m+n)$-dimensional DC atlas on $X \times Y$, if we define $(\varphi_X \otimes \varphi_Y)(x, y) = (\varphi_X(x), \varphi_Y(y))$.

(iii) If X, Y are equipped with m, n-dimensional DC structures, respectively, and $f\colon X \times Y \to \mathbb{R}$ is DC, then the section $x \mapsto f(x,y)$ is DC on X for any $y \in Y$, and the section $y \mapsto f(x,y)$ is DC on Y for any $x \in X$.

Definition 2.13. Let H be an $(n+k)$-dimensional unitary space ($n, k \in \mathbb{N}$). We say that a set $M \subset H$ is a *k-dimensional Lipschitz (or DC) surface*, if it is nonempty and for each $x \in M$ there exist a k-dimensional linear space $Q \subset H$, an open neighbourhood W of x, a set $G \subset Q$ open in Q and a Lipschitz (locally DC, respectively) mapping $h\colon G \to Q^{\perp}$ such that

$$M \cap W = \{u + h(u)\colon\ u \in G\}.$$

Remark 2.14.

(i) Lipschitz surfaces were considered e.g. by Whitehead [24, p. 165] or Walter [23], who called them strong Lipschitz submanifolds. Obviously, each DC surface is a Lipschitz surface. For some properties of DC surfaces see [27].

(ii) If we suppose, in the above definition of a DC surface, that G is convex and h is DC and Lipschitz, we obtain clearly the same notion.

(iii) Each Lipschitz (or DC) surface admits a natural structure of a Lipschitz (DC, respectively) manifold that is given by the charts of the form $(W \cap M, \psi^{-1})$, where $\psi(u) = u + h(u)$, $u \in G$ (cf. Remark 2.12 (i)).

Lemma 2.15. *Let H be an n-dimensional unitary space, $V \subset H$ an open convex set, and $f\colon V \to \mathbb{R}^m$ a DC mapping. Then there exists a sequence (T_i) of $(n-1)$-dimensional DC surfaces in H such that f is strictly differentiable at each point of $V \setminus \bigcup_{i=1}^{\infty} T_i$.*

P r o o f. Let $f = (f_1, \dots, f_m)$. By the definition of a DC mapping, $f_j = \alpha_j - \beta_j$, where α_j and β_j are convex functions. By [25], for each j we can find a sequence T_k^j, $k \in \mathbb{N}$, of $(n-1)$-dimensional DC surfaces in H such that both α_j and β_j are differentiable at each point of $D_j := H \setminus \bigcup_{k=1}^{\infty} T_k^j$. Since each convex function is strictly differentiable at each point at which it is (Fréchet) differentiable (see, e.g., [22, Proposition 3.8] for a proof of this well-known fact), we conclude that each f_j is strictly differentiable at each point of D_j. Since strict differentiablity of f clearly follows from strict differentiability of all f_j's, the proof is completed after ordering all sets T_k^j, $k \in \mathbb{N}$, $j = 1, \dots, m$, to a sequence (T_i). $\square$

2.3. Convex surfaces.

Definition 2.16. A *convex body* in $\mathbb{R}^n$ is a compact convex subset with nonempty interior. Under a *convex surface* in $\mathbb{R}^n$ we understand the boundary $X = \partial C$ of a convex body C. A convex surface X is said to be *polyhedral* if it can be covered by finitely many hyperplanes.

It is well known that a convex surface in $\mathbb{R}^n$ with its intrinsic metric is a complete geodesic space with nonnegative curvature (see [4] or [3, §10.2]).

Obviously, each convex surface X is a DC surface (cf. Remark 2.18 (iii)), and so has a canonical DC structure. In the sequel, we will work mainly with "standard" DC charts on X (which are considered in the generalized sense of Remark 2.12 (i)).

Definition 2.17. Let $X \subset \mathbb{R}^{n+1}$ be a convex surface and U a nonempty, relatively open subset of X. We say that (U, φ) is a *standard n-dimensional chart* on X, if there exist a unit vector $e \in \mathbb{R}^{n+1}$, a convex, relatively open subset V of the hyperplane $e^\perp$, and a Lipschitz convex function $f\colon V \to \mathbb{R}$ such that, setting $F(x) := x + f(x)e$, $x \in V$, we have $U = F(V)$ and $\varphi = F^{-1}$. In this case we will say that (U, φ) is an (e, V)*-standard chart* on X and f will be called *the convex function associated with the standard chart.*

Remark 2.18.

(i) Clearly, if (U, φ) is an (e, V)-standard chart on X and π denotes the orthogonal projection onto $e^\perp$, then $\varphi = \pi\restriction_U$.

(ii) Let (U_1, φ_1) and (U_2, φ_2) be standard charts as in the above definition. Then these charts are DC-compatible. Indeed, φ_1^{-1} is a DC mapping from V_1 to $\mathbb{R}^{n+1}$ and φ_2 is a restriction of a linear mapping π (see (i)). So $\varphi_2 \circ (\varphi_1)^{-1} = \pi \circ (\varphi_1)^{-1}$ is locally DC by Lemma 2.10 (a), (c).

(iii) Let $X \subset \mathbb{R}^{n+1}$ be a convex surface, $z \in X$, and let C be the convex body for which $X = \partial C$. Choose $a \in \operatorname{int} C$, set $e := (a - z)/\|a - z\|$ and $V := \pi(B(a, \delta))$, where $\delta > 0$ is sufficiently small and π is the orthogonal projection of $\mathbb{R}^{n+1}$ onto $e^\perp$. Then it is easy to see that there exists an (e, V)-standard chart (U, φ) on X with $z \in U$.

By (ii) and (iii) above, the following definition is correct.

Definition 2.19. Let $X \subset \mathbb{R}^{n+1}$ be a convex surface. Then the *standard DC structure* on X is determined by the atlas of all standard n-dimensional charts on X.

Lemma 2.20. *Let $X \subset \mathbb{R}^{n+1}$ ($n \geqslant 2$) be a convex surface and let (U, φ) be an (e, V)-standard chart on X. Let $T \subset e^\perp$ be an $(n-1)$-dimensional DC surface in $e^\perp$ with $T \cap V \neq \emptyset$. Then $\varphi^{-1}(T \cap V)$ is an $(n-1)$-dimensional DC surface in $\mathbb{R}^{n+1}$.*

P r o o f. Let f be the convex function associated with (U,φ). Let z be an arbitrary point of $\varphi^{-1}(T\cap V)$. Denote $x:=\varphi(z)$. By Definition 2.13 there exist an $(n-1)$-dimensional linear space $Q\subset e^{\perp}$, a set $G\subset Q$ open in Q, an open neighbourhood W of x in $e^{\perp}$ and a locally DC mapping $h\colon G\to Q^{\perp}_{e^{\perp}}$ such that $T\cap W=\{u+h(u)\colon u\in G\}$. We can and will suppose that $W\subset V$. Observing that $z\in\varphi^{-1}(T\cap W)$ and $\varphi^{-1}(T\cap W)$ is an open set in $\varphi^{-1}(T\cap V)$,

$$\varphi^{-1}(T\cap W)=\{u+h(u)+f(u+h(u))e\colon u\in G\}$$

and $u\mapsto h(u)+f(u+h(u))e$ is a locally DC mapping $G\to Q^{\perp}_{\mathbb{R}^{n+1}}$, we complete the proof. $\square$

We shall need the following known facts. Because of lack of a reference, we supply proofs of (ii) and (iii).

Lemma 2.21.

(i) *Let X be a convex surface in $\mathbb{R}^m$. Then there exists a sequence (X_k) of polyhedral convex surfaces in $\mathbb{R}^m$ converging to X in the Hausdorff distance.*

(ii) *Let convex surfaces X_k converge in the Hausdorff distance to the convex surface X in $\mathbb{R}^m$ and let dist_X, $\operatorname{dist}_{X_k}$ denote the intrinsic distances on X, X_k, respectively. Assume that $a,b\in X$, $a_k,b_k\in X_k$, $a_k\to a$ and $b_k\to b$. Then* $\operatorname{dist}_{X_k}(a_k,b_k)\to\operatorname{dist}_X(a,b)$.

(iii) *If X_k, X are as in* (ii) *then* $\operatorname{diam}X_k\to\operatorname{diam}X$, *where* $\operatorname{diam}X_k$, $\operatorname{diam}X$ *are the intrinsic diameters of X_k, X, respectively.*

P r o o f. (i) is well-known, see e.g. [20, § 1.8.15].

(ii) can be proved as in [3, Lemma 10.2.7], where a slightly different assertion is shown. We present here the proof for completeness. Let C, C_k be convex bodies in $\mathbb{R}^m$ such that $X=\partial C$, $X_k=\partial C_k$, $k\in\mathbb{N}$, and assume, without loss of generality, that the origin lies in the interior of C. It is easy to show that, since the Hausdorff distances of X and X_k tend to zero, there exist $k_0\in\mathbb{N}$ and a sequence $\varepsilon_k\searrow 0$ such that

$$(1-\varepsilon_k)C\subset C_k\subset(1+\varepsilon_k)C,\quad k\geqslant k_0.$$

For a convex body D in $\mathbb{R}^m$ and the corresponding convex surface $Y=\partial D$, we shall denote by Π_Y the metric projection of $\mathbb{R}^m$ onto Y, defined outside of the interior of D. The symbol dist_Y denotes the intrinsic distance on the convex surface Y. Let a, b, a_k, b_k from the assumption be given, and (for $k\geqslant k_0$) denote $\tilde a_k=\Pi_{X_k}((1+\varepsilon_k)a)$, $\tilde b_k=\Pi_{X_k}((1+\varepsilon_k)b)$. Since Π_{X_k} is a contraction (see e.g. [20, Theorem 1.2.2]), we have

$$\operatorname{dist}_{X_k}(\tilde a_k,\tilde b_k)\leqslant\operatorname{dist}_{(1+\varepsilon_k)X}((1+\varepsilon_k)a,(1+\varepsilon_k)b)=(1+\varepsilon_k)\operatorname{dist}_X(a,b).$$

Further, clearly $\tilde{a}_k \to a$ and $\tilde{b}_k \to b$, which implies that $\operatorname{dist}_{X_k}(\tilde{a}_k, a_k) \to 0$ and $\operatorname{dist}_{X_k}(\tilde{b}_k, b_k) \to 0$. Consequently,

$$\limsup_{k\to\infty} \operatorname{dist}_{X_k}(a_k, b_k) \leqslant \operatorname{dist}_X(a, b).$$

The inequality $\liminf\limits_{k\to\infty} \operatorname{dist}_{X_k}(a_k, b_k) \geqslant \operatorname{dist}_X(a, b)$ is obtained in a similar way, considering the metric projections of a_k and b_k onto $(1-\varepsilon_k)X$.

(iii) is a straightforward consequence of (ii) and the compactness of X. $\square$

Lemma 2.22. *Let $X \subset \mathbb{R}^{n+1}$ be a convex surface, (U, φ) an (e, V)-standard chart on X, and let f be the associated convex function. Let (X_k) be a sequence of convex surfaces which tends in the Hausdorff metric to X, and let $W \subset V$ be an open convex set such that $\overline{W} \subset V$. Then there exists $k_0 \in \mathbb{N}$ such that, for each $k \geqslant k_0$, the surface X_k has an (e, W)-standard chart (U_k, φ_k), and the associated convex functions f_k satisfy*

$$(5) \qquad f_k(x) \to f(x), \;\; x \in W \quad \textit{and} \quad \limsup_{k\to\infty} \operatorname{Lip} f_k \leqslant \operatorname{Lip} f.$$

Proof. Denote by C and C_k the convex bodies for which $X = \partial C$ and $X_k = \partial C_k$, respectively. Clearly, the convex function f has the form

$$f(v) = \inf\{t \in \mathbb{R}\colon\; v + te \in C\}, \quad v \in V.$$

Let π be the orthogonal projection onto $e^\perp$ and denote

$$W_r := \{v \in e^\perp\colon\; \operatorname{dist}(v, W) < r\}, \quad r > 0.$$

Let $\varepsilon, \delta > 0$ be such that $W_{\varepsilon+\delta} \subset V$, and let $k_0 = k_0(\delta) \in \mathbb{N}$ be such that the Hausdorff distance of X and X_k (and, hence, also of C and C_k) is less than δ for all $k > k_0$. Fix a $k > k_0$. It is easy to show that

$$f_k^*(v) = \inf\{t \in \mathbb{R}\colon\; v + te \in C_k\}, \quad v \in W_\varepsilon$$

is a finite convex function. We shall show that

$$(6) \qquad |f_k^*(v) - f(v)| \leqslant (1 + \operatorname{Lip} f)\delta, \quad v \in W_\varepsilon.$$

Take a point $v \in W_\varepsilon$ and denote $x = v + f(v)e \in X$ and $y = v + f_k^*(v)e \in X_k$. The definition of the Hausdorff distance yields that there must be a point $c \in C$ with $\|c - y\| < \delta$. This implies that for $w := \pi(c)$ we have $f(w) \leqslant c \cdot e$ and

$$f_k^*(v) = y \cdot e \geqslant c \cdot e - \delta \geqslant f(w) - \delta \geqslant f(v) - \delta \operatorname{Lip} f - \delta.$$

For the other inequality, note that, since f_k^* is convex, there exists a unit vector $u \in \mathbb{R}^{n+1}$ with $u \cdot e =: -\eta < 0$ such that $(z - y) \cdot u \leqslant 0$ for all $z \in C_k$ (i.e., u is a unit outer normal vector to C_k at y). It is easy to see that $(z - y) \cdot u \leqslant \delta$ for all $z \in C$, since the Hausdorff distance of C and C_k is less than δ. Consider the point $z = w + f(w)e \in C$ with $w = v + \delta u^*$, where $u^* = \pi(u)/\|\pi(u)\|$ if $\pi(u) \neq 0$ and u^* is any unit vector in $e^\perp$ if $\pi(u) = 0$. Then

$$\begin{aligned}\delta \geqslant (z - y) \cdot u &= (w + f(w)e - v - f_k^*(v)e) \cdot u \\ &= (w - v) \cdot u + (f(w) - f_k^*(v))(e \cdot u) \\ &= \delta\sqrt{1 - \eta^2} + (f(w) - f_k^*(v))(-\eta) \\ &\geqslant \delta(1 - \eta) + (f_k^*(v) - f(w))\eta,\end{aligned}$$

which implies that

$$f_k^*(v) \leqslant f(w) + \delta \leqslant f(v) + \delta \operatorname{Lip} f + \delta$$

by the Lipschitz property of f, and (6) is verified.

We shall show now that for $k > k_0$, X_k has an (e, W)-standard chart with an associated convex function $f_k := f_k^* \restriction W$ (i.e., that f_k is Lipschitz) and that (5) holds. Given two different points $u, v \in W$, we define points $u^*, v^* \in W_\varepsilon$ as follows: we set $u^* = u - \varepsilon(v - u)/\|v - u\|$, $v^* = v$ if $f_k(u) \geqslant f_k(v)$, and $u^* = u$, $v^* = v + \varepsilon(v - u)/\|v - u\|$ if $f_k(u) \leqslant f_k(v)$. Then, using (6) and convexity of f_k^*, we obtain

$$\frac{|f_k(u) - f_k(v)|}{\|u - v\|} \leqslant \frac{|f_k^*(u^*) - f_k^*(v^*)|}{\|u^* - v^*\|} \leqslant \operatorname{Lip} f + \frac{(2 + 2 \operatorname{Lip} f)\delta}{\varepsilon}$$

whenever $k > k_0(\delta)$. Therefore, $\operatorname{Lip} f_k \leqslant \operatorname{Lip} f + \frac{1}{\varepsilon}(2 + 2 \operatorname{Lip} f)\delta$. Using this inequality, (6), and the fact that $\delta > 0$ can be arbitrarily small, we obtain (5). $\square$

3. Extrinsic properties of distance functions on convex surfaces

We will prove our results via the following result concerning intrinsic properties of distance functions on Alexandrov spaces, which is an easy consequence of well-known results.

Proposition 3.1. *Let X be a complete geodesic (Alexandrov) space with nonnegative curvature. Then the Cartesian product X^2 with the product metric*

$$\operatorname{dist}_{X \times X}((x_1, x_2), (y_1, y_2)) = \sqrt{\operatorname{dist}^2(x_1, y_1) + \operatorname{dist}^2(x_2, y_2)}$$

is a complete geodesic space with nonnegative curvature as well, and the squared distance $g(x_1, x_2) := \operatorname{dist}^2(x_1, x_2)$ is 4-concave on X^2.

P r o o f. The assertion on the properties of X^2 is well known, see e.g. [3, § 3.6.1, § 10.2.1]. In order to show the 4-concavity of g, we shall use the fact that

$$g(x_1, x_2) = 2\operatorname{dist}^2_{X\times X}((x_1, x_2), D), \quad x_1, x_2 \in X, \tag{7}$$

where D is the diagonal in $X \times X$. To see that (7) holds, note that

$$\begin{aligned}\operatorname{dist}^2_{X\times X}((x_1, x_2), D) &= \inf_{y\in X} \operatorname{dist}^2_{X\times X}((x_1, x_2), (y, y))\\ &= \inf_{y\in X} (\operatorname{dist}^2(x_1, y) + \operatorname{dist}^2(x_2, y)).\end{aligned}$$

Choosing a midpoint of x_1 and x_2 for y in the last expression, we see that $\operatorname{dist}^2_{X\times X}((x_1, x_2), D) \leqslant \frac{1}{2}\operatorname{dist}^2(x_1, x_2)$. On the other hand, if y is an arbitrary point of X, we get by the triangle inequality

$$\operatorname{dist}^2(x_1, x_2) \leqslant 2(\operatorname{dist}^2(x_1, y) + \operatorname{dist}^2(x_2, y)) = 2\operatorname{dist}^2_{X\times X}((x_1, x_2), (y, y)),$$

and thus we get the other inequality proving (7).

To finish the proof, we use the following fact: *If Y is a length space of nonnegative curvature and $\emptyset \neq F \subset Y$ a closed subset, then the squared distance function $d^2_F(\cdot) = \operatorname{dist}^2_Y(\cdot, F)$ is 2-concave on Y.* This is well known if F is a singleton (see e.g. [17, Proposition 116]) and follows easily for a general nonempty closed set F by the facts that $d^2_F(y) = \inf_{x\in F} d^2_{\{x\}}(y)$ and that the infimum of concave functions is concave. If we apply this for $Y = X \times X$ and $F = D$, (7) completes the proof. $\square$

Lemma 3.2. *Let X be a polyhedral convex surface in $\mathbb{R}^{n+1}$, $T \in X$, and (U, φ) be an (e, V)-standard chart on X such that $T \in U$. Let f be the associated convex function and $t := \varphi(T)$. Then there exists a $\delta > 0$ such that for all $x, y \in V$ with $t = \frac{1}{2}(x + y)$ and $\|x - t\| = \|y - t\| < \delta$ we have*

$$\operatorname{dist}(S, T) \leqslant 2\Delta^2 f(x, y),$$

whenever S is a midpoint of $\varphi^{-1}(x), \varphi^{-1}(y)$.

P r o o f. Denoting $F := \varphi^{-1}$, we have $F(u) = u + f(u)e$. Let L be the Lipschitz constant of f. It is easy to see that we can choose $\delta_0 > 0$ such that for any $x \in V$ with $\|x - t\| < \delta_0$, the function f is affine on the segment $[x, t]$. Then we take $\delta \leqslant \delta_0/L$, such that for any two points $x, y \in B(t, \delta)$, any minimal curve connecting $F(x)$ and $F(y)$ (and, hence, also any midpoint of $F(x)$, $F(y)$) lies in U. Let two points $x, y \in B(t, \delta)$ with $t = \frac{1}{2}(x + y)$ be given and denote $\Delta = \Delta^2 f(x, y)$. Let S be a midpoint of $F(x)$, $F(y)$ (lying necessarily in U) and set $s = \varphi(S)$. Note that $\Delta \leqslant L\delta$.

From the parallelogram law, we obtain

$$2\|F(x)-T\|^2+2\|F(y)-T\|^2=\|F(y)-F(x)\|^2+4\Delta^2,$$

since

$$\Delta=\Big\|\frac{F(x)+F(y)}{2}-T\Big\|. \tag{8}$$

Taking the square root and using the inequality $a+b\leqslant\sqrt{2a^2+2b^2}$, we obtain

$$\|F(x)-T\|+\|F(y)-T\|\leqslant\sqrt{\|F(y)-F(x)\|^2+4\Delta^2}.$$

It is clear that the geodesic distance of $F(x)$ and $F(y)$ is at most $\|F(x)-T\|+\|F(y)-T\|$ (which is the length of a curve in X connecting $F(x)$ and $F(y)$). Thus,

$$\|S-F(x)\|\leqslant\operatorname{dist}(S,F(x))=\frac{1}{2}\operatorname{dist}(F(x),F(y))\leqslant\sqrt{\Big(\frac{\|F(y)-F(x)\|}{2}\Big)^2+\Delta^2}$$

and the same upper bound applies to $\|S-F(y)\|$. Summing the squares of both the distances, we obtain

$$\|S-F(x)\|^2+\|S-F(y)\|^2\leqslant\frac{1}{2}\|F(y)-F(x)\|^2+2\Delta^2$$

and, since the left-hand side equals, again by the parallelogram law,

$$\frac{1}{2}(\|F(y)-F(x)\|^2+\|2S-(F(x)+F(y)\|^2),$$

we arrive at

$$\Big\|S-\frac{F(x)+F(y)}{2}\Big\|\leqslant\Delta. \tag{9}$$

Considering the orthogonal projections of S and $\frac{1}{2}(F(x)+F(y))$ onto $e^{\perp}$, we obtain

$$\|s-t\|\leqslant\Delta\leqslant L\delta\leqslant\delta_0$$

and, hence, we have

$$\operatorname{dist}(S,T)=\|S-T\|,$$

since f is affine on $[s,t]$. On the other hand, equations (8) and (9) imply $\|S-T\|\leqslant 2\Delta$, which completes the proof. $\square$

Proposition 3.3. *Let $X \subset \mathbb{R}^{n+1}$ be a convex surface and let (U_i, φ_i) be (e_i, V_i) standard charts, $i = 1, 2$. Let f_1, f_2 be the corresponding convex functions. Set*

$$g(x_1, x_2) = \operatorname{dist}^2(\varphi_1^{-1}(x_1), \varphi_2^{-1}(x_2)), \quad x_1 \in V_1, \; x_2 \in V_2,$$

where dist *is the intrinsic distance on X. Then the function $g - c - d$ is concave on $V_1 \times V_2$, where*

$$\begin{aligned} c(x_1, x_2) &= 4(1 + L^2)(\|x_1\|^2 + \|x_2\|^2), \\ d(x_1, x_2) &= 4M(f_1(x_1) + f_2(x_2)), \end{aligned}$$

$L = \max\{\operatorname{Lip} f_1, \operatorname{Lip} f_2\}$ *and M is the intrinsic diameter of X.*

P r o o f. Assume first that the convex surface X is polyhedral. We shall show that for any $t \in V_1 \times V_2$ there exists $\delta > 0$ such that

$$\Delta^2 g(x, y) \leqslant \Delta^2 c(x, y) + \Delta^2 d(x, y) \tag{10}$$

for all $x, y \in B(t, \delta) \subset V_1 \times V_2$ with $t = \frac{1}{2}(x + y)$, which implies the assertion, see Lemma 2.2. We have

$$\begin{aligned} \Delta^2 g(x, y) &= \frac{g(x) + g(y)}{2} - g(t) \\ &= \Big(\frac{g(x) + g(y)}{2} - g(s)\Big) + (g(s) - g(t)) \end{aligned}$$

whenever $s = (s_1, s_2) \in V_1 \times V_2$ is such that $(\varphi_1^{-1}(s_1), \varphi_2^{-1}(s_2))$ is a midpoint of $(\varphi_1^{-1}(x_1), \varphi_2^{-1}(x_2))$ and $(\varphi_1^{-1}(y_1), \varphi_2^{-1}(y_2))$ in X^2, where $x = (x_1, x_2)$ and $y = (y_1, y_2)$. By Proposition 3.1 and Lemma 2.7 (ii), the first summand is bounded from above by

$$2\,\frac{\operatorname{dist}^2(\varphi_1^{-1}(x_1), \varphi_1^{-1}(y_1)) + \operatorname{dist}^2(\varphi_2^{-1}(x_2), \varphi_2^{-1}(y_2))}{4}.$$

Since clearly

$$\operatorname{dist}(\varphi_i^{-1}(x_i), \varphi_i^{-1}(y_i)) \leqslant \sqrt{1 + (\operatorname{Lip} f_i)^2}\,\|x_i - y_i\|, \quad i = 1, 2,$$

we get

$$\begin{aligned} \frac{g(x) + g(y)}{2} - g(s) &\leqslant (2 + (\operatorname{Lip} f_1)^2 + (\operatorname{Lip} f_2)^2)\frac{\|x_1 - y_1\|^2 + \|x_2 - y_2\|^2}{2} \\ &\leqslant \Delta^2 c(x, y) \end{aligned}$$

(we use the fact that $\Delta^2 c(x,y) = 4(1+L^2)(\|x-y\|/2)^2$, see (2)). In order to verify (10), it remains thus to show that

$$|g(s) - g(t)| \leqslant \Delta^2 d(x,y). \tag{11}$$

Denote $t = (t_1, t_2)$, $s = (s_1, s_2)$, $T_i = \varphi_i^{-1}(t_i)$ and $S_i = \varphi_i^{-1}(s_i)$, $i = 1, 2$. We have

$$\begin{aligned} |g(s) - g(t)| &= |\operatorname{dist}^2(S_1, S_2) - \operatorname{dist}^2(T_1, T_2)| \\ &\leqslant 2M|\operatorname{dist}(S_1, S_2) - \operatorname{dist}(T_1, T_2)| \\ &\leqslant 2M(\operatorname{dist}(S_1, T_1) + \operatorname{dist}(S_2, T_2)), \end{aligned}$$

where the last inequality follows from the (iterated) triangle inequality. Applying Lemma 3.2 and the fact that S_i is a midpoint of $\varphi_i^{-1}(x_i)$, $\varphi_i^{-1}(y_i)$ (see [17, § 4.3]), we get $\operatorname{dist}(S_i, T_i) \leqslant 2\Delta^2 f_i(x_i, y_i)$, $i = 1, 2$, for δ sufficiently small. Since clearly

$$\Delta^2 d(x,y) = 4M(\Delta^2 f_1(x_1, y_1) + \Delta^2 f_2(x_2, y_2)),$$

(11) follows.

Let now X be an arbitrary convex surface. Let (X_k) be a sequence of polyhedral convex surfaces which tends in the Hausdorff metric to X. Consider arbitrary open convex sets $W_i \subset V_i$ with $\overline{W_i} \subset V_i$, $i = 1, 2$. Applying Lemma 2.22 (and considering a subsequence of X_k if necessary), we find (e_i, W_i)-standard charts $(U_{i,k}, \varphi_{i,k})$ of X_k such that the associated convex functions $f_{i,k}$ converge to $f_i\restriction_{W_i}$, $L_i^* := \lim\limits_{k\to\infty} \operatorname{Lip} f_{i,k}$ exists and $L_i^* \leqslant \operatorname{Lip} f_i$, $i = 1, 2$.

By the first part of the proof we know that the function

$$\psi_k(x_1, x_2) := g_k(x_1, x_2) - 4(1 + L_k^2)(\|x_1\|^2 + \|x_2\|^2) - 4M_k(f_{1,k}(x_1) + f_{2,k}(x_2)),$$

where M_k is the intrinsic diameter of X_k and $L_k = \max(\operatorname{Lip} f_{1,k}, \operatorname{Lip} f_{1,k})$, is concave on $W_1 \times W_2$. Obviously, $L_k \to L^* := \max(L_1^*, L_2^*) \leqslant L$ and Lemma 2.21 implies that $g_k \to g$ and $M_k \to M$. Consequently,

$$\lim_{k\to\infty} \psi_k(x_1, x_2) = g(x_1, x_2) - 4(1 + {L^*}^2)(\|x_1\|^2 + \|x_2\|^2) - 4M(f_1(x_1) + f_2(x_2))$$

is concave on $W_1 \times W_2$. Since $L^* \leqslant L$, we obtain that $g - c - d$ is concave on $W_1 \times W_2$. Thus $g - c - d$ is locally concave, and so concave, on $V_1 \times V_2$. □

Proposition 3.3 has the following immediate corollary (recall the definition of a DC function on a DC manifold, Definition 2.11, and the definition of the DC structure on X^2, Remark 2.12 (ii)).

Theorem 3.4. *Let X be a convex surface in $\mathbb{R}^{n+1}$. Then the squared distance function $(x,y) \mapsto \operatorname{dist}^2(x,y)$ is DC on X^2.*

Using Remark 2.12 (iii), we obtain

Corollary 3.5. *Let X be a convex surface in $\mathbb{R}^{n+1}$ and let $x_0 \in X$ be fixed. Then the squared distance from x_0, $x \mapsto \operatorname{dist}^2(x, x_0)$, is DC on X.*

Since the function $g(z) = \sqrt{z}$ is DC on $(0, \infty)$, Lemma 2.10 (c) easily implies

Corollary 3.6. *Let X be a convex surface in $\mathbb{R}^{n+1}$ and let $x_0 \in X$ be fixed. Then the distance from x_0, $x \mapsto \operatorname{dist}(x, x_0)$, is DC on $X \setminus \{x_0\}$.*

Remark 3.7. If $n = 1$, it is not difficult to show that the function $x \mapsto \operatorname{dist}(x, x_0)$ is DC on the whole X. On the other hand, we conjecture that this statement is not true in general for $n \geqslant 2$.

Theorem 3.8. *Let $X \subset \mathbb{R}^{n+1}$ be a convex surface and $\emptyset \neq F \subset X$ a closed set. Denoting $d_F := \operatorname{dist}(\cdot, F)$,*

(i) *the function $(d_F)^2$ is DC on X and*

(ii) *the function d_F is DC on $X \setminus F$.*

Proof. Since X is compact, we can choose a finite system (U_i, φ_i), $i \in I$, of (e_i, V_i)-standard charts which forms a DC atlas on X. Let f_i, $i \in I$, be the corresponding convex functions. Choose $L > 0$ such that $\operatorname{Lip} f_i \leqslant L$ for all $i \in I$ and let M be the intrinsic diameter of X. To prove (i), it is sufficient to show that, for all $i \in I$, $(d_F)^2 \circ (\varphi_i)^{-1}$ is DC on V_i. So fix $i \in I$ and consider an arbitrary $y \in F$. Choose $j \in I$ with $y \in U_j$. Set

$$\omega(x) := 4(1 + L^2)\|x\|^2 + 4M f_i(x), \quad x \in V_i.$$

Proposition 3.3 (used for $\varphi_1 = \varphi_i$ and $\varphi_2 = \varphi_j$) easily implies that the function $h_y(x) = \operatorname{dist}^2(\varphi_i^{-1}(x), y) - \omega(x)$ is concave on V_i. Consequently, the function

$$\psi(x) := (d_F)^2 \circ (\varphi_i)^{-1}(x) - \omega(x) = \inf_{y \in F} h_y(x)$$

is concave on V_i. So $(d_F)^2 \circ (\varphi_i)^{-1} = \psi + \omega = \omega - (-\psi)$ is DC on V_i. Thus (i) is proved. Since the function $g(z) = \sqrt{z}$ is DC on $(0, \infty)$, Lemma 2.10 (c) easily implies (ii). □

Remark 3.9. It is not difficult to show that Theorems 3.8 and 3.4 imply the corresponding results in n-dimensional closed unbounded convex surfaces $X \subset \mathbb{R}^{n+1}$; in particular that the statements (B), (C) and (D) from Introduction hold. To this end, it is sufficient to consider a bounded closed convex surface $\tilde{X}$ which contains a sufficiently large part of X.

4. Applications

Our results on distance functions can be applied to a number of problems from the geometry of convex surfaces that are formulated in the language of distance functions. We present below applications concerning r-boundaries (distance spheres), the multijoined locus, and the ambiguous locus (exoskeleton) of a closed subset of a convex surface. Recall that r-boundaries and ambiguous loci were studied (in Euclidean, Riemannian and Alexandrov spaces) in a number of articles (see, e.g., [6], [21], [28], [9]).

The first application (Theorem 4.1 below) concerning r-boundaries provides an analogue of well-known results proved (in Euclidean spaces) by Ferry [6] and Fu [7]. It is an easy consequence of Theorem 3.8 and the following general result on level sets of DC functions, which immediately follows from [18, Theorem 3.4]:

Theorem DC ([18]). *Let $n \in \{2,3\}$, let E be an n-dimensional unitary space, and let d be a locally DC function on an open set $G \subset E$. Suppose that d has no stationary point. Then there exists a set $N \subset \mathbb{R}$ with $\mathscr{H}^{(n-1)/2}(N) = 0$ such that for every $r \in d(G) \setminus N$, the set $d^{-1}(r)$ is an $(n-1)$-dimensional DC surface in E.*

Moreover, N can be chosen such that $N = d(C)$, where C is a closed set in G.

(Let us note that C can be chosen to be the set of all critical points of d, but we will not need this fact.)

Theorem 4.1. *Let $n \in \{2,3\}$ and let $X \subset \mathbb{R}^{n+1}$ be a convex surface and $\emptyset \neq K \subset X$ a closed set. For $r > 0$, consider the r-boundary (distance sphere) $K_r := \{x \in X\colon \operatorname{dist}(x,K) = r\}$. There exists a compact set $N \subset [0,\infty)$ with $\mathscr{H}^{(n-1)/2}(N) = 0$ such that for every $r \in (0,\infty) \setminus N$, the r-boundary K_r is either empty, or an $(n-1)$-dimensional DC surface in $\mathbb{R}^{n+1}$.*

P r o o f. Choose a system (U_i, φ_i), $i \in \mathbb{N}$, of (e_i, V_i)-standard charts on X such that $G := X \setminus K = \bigcup_{i=1}^{\infty} U_i$. By Theorem 3.8 we know that $d_i := d_K \circ \varphi_i^{-1}$ is locally DC on V_i, where $d_K := \operatorname{dist}(\cdot, K)$. Moreover, no $t \in \varphi_i(U_i)$ is a stationary point of d_i (i.e., the differential of d_i at t is nonzero). Indeed, otherwise there would exist $\delta > 0$ such that $|d_i(\tau) - d_i(t)| < \|\tau - t\|$ whenever $\|\tau - t\| < \delta$. Denote $x := \varphi^{-1}(t)$ and choose a minimal curve γ with endpoints x and $u \in K$ and length $s = \operatorname{dist}(x,K)$. Choosing a point x^* on the image of γ which is sufficiently close to x and putting $\tau := \varphi_i(x^*)$, we clearly have $\|\tau - t\| < \delta$ and $|d_i(\tau) - d_i(t)| = \operatorname{dist}(x, x^*) \geqslant \|\tau - t\|$, which is a contradiction.

Consequently, by Theorem DC we can find for each i a set $S_i \subset V_i$ closed in V_i such that, for $N_i := d_i(S_i)$, we know that $\mathscr{H}^{(n-1)/2}(N_i) = 0$ and, for each $r \in (0,\infty) \setminus N_i$, the set $d_i^{-1}(r)$ is either empty, or an $(n-1)$-dimensional DC surface in $e_i^{\perp}$.

Define S as the set of all points $x \in G$ such that $\varphi_i(x) \in S_i$ whenever $x \in U_i$. Obviously, S is closed in G. Set $N := d_K(S) \cup \{0\}$. Since clearly $N \subset \bigcup_{i=1}^{\infty} N_i \cup \{0\}$, we have $\mathscr{H}^{(n-1)/2}(N) = 0$. Since $K \cup S$ is compact, $N = d_K(K \cup S)$ and d_K is continuous, we obtain that N is compact.

Let now $r \in (0,\infty) \setminus N$ and $x \in K_r$. Let $x \in U_i$. Then clearly $K_r \cap U_i = \varphi_i^{-1}(d_i^{-1}(r))$. Since $d_i^{-1}(r)$ is an $(n-1)$-dimensional DC surface in $e_i^{\perp}$, Lemma 2.20 implies that $K_r \cap U_i$ is an $(n-1)$-dimensional DC surface in $\mathbb{R}^{n+1}$. Since $x \in K_r$ was arbitrary, we obtain that K_r is an $(n-1)$-dimensional DC surface in $\mathbb{R}^{n+1}$. $\square$

Remark 4.2. Let $n = 2$. Then the weaker version of Theorem 4.1 in which $\mathscr{H}^1(N) = 0$ (instead of $\mathscr{H}^{1/2}(N) = 0$) and K_r are $(n-1)$-dimensional Lipschitz manifolds follows from [21, Theorem B] proved in 2-dimensional Alexandrov spaces without boundary. In such Alexandrov spaces even the version in which $\mathscr{H}^{1/2}(N) = 0$ and K_r are $(n-1)$-dimensional Lipschitz manifolds holds; it is proved in [18] using Theorem DC and Perelman's DC structure (cf. Section 5). However, it seems to be impossible to deduce by this method Theorem 4.1 in its full strength; any proof that K_r are DC surfaces probably needs results of the present article.

If X is a 3-dimensional Alexandrov space without boundary, it is still possible that the version of Theorem 4.1 in which K_r are Lipschitz manifolds holds. Nevertheless, it cannot be proved using only Theorem DC and Perelman's DC structure even if X is a convex surface. The obstacle is that the set $X \setminus X^*$ of "Perelman's singular" points (cf. Section 5) can have positive 1-dimensional Hausdorff measure even if X is a convex surface in $\mathbb{R}^4$ (see [18, Example 6.5]).

Remark 4.3. Examples due to Ferry [6] show that Theorem 4.1 cannot be generalized to $n \geqslant 4$. For an arbitrary n-dimensional convex surface X we can, however, obtain (quite similarly to the way used in [18] for Riemannian manifolds or Alexandrov spaces without Perelman singular points) that for all $r > 0$ except a countable set, either K_r is empty or K_r contains an $(n-1)$-dimensional DC surface $A_r \subset X$ such that A_r is dense and open in K_r, and $\mathscr{H}^{n-1}(K_r \setminus A_r) = 0$.

If K is a closed subset of a length space X, the *multijoined locus* $M(K)$ of K is the set of all points $x \in X$ such that the distance from x to K is realized by at least two different minimal curves in X. If two such minimal curves exist that connect x with two different points of K, x is said to belong to the *ambiguous locus* $A(K)$ of K. The ambiguous locus of K is also called the skeleton of $X \setminus K$ (or the exoskeleton of K, [9]).

Zamfirescu [28] studies the multijoined locus in a complete geodesic (Alexandrov) space of curvature bounded from below and shows that it is σ-porous. An application of Theorem 3.8 yields a stronger result for convex surfaces:

Theorem 4.4. *Let K be a closed subset of a convex surface $X \subset \mathbb{R}^{n+1}$ ($n \geqslant 2$). Then $M(K)$ (and, hence, also $A(K)$) can be covered by countably many $(n-1)$-dimensional DC surfaces lying in X.*

Proof. Let (U, φ) be an (e, V)-standard chart on X. It is clearly sufficient to prove that $M(K) \cap U$ can be covered by countably many $(n-1)$-dimensional DC surfaces. Set $F := \varphi^{-1}$ and denote by $d_K(z)$ the intrinsic distance of $z \in X$ from K. Since both the mapping F and the function $d_K \circ F$ are DC on V (see Theorem 3.8 and Lemma 2.10), they are by Lemma 2.15 strictly differentiable at all points of $V \setminus N$, where N is a countable union of $(n-1)$-dimensional DC surfaces in $e^{\perp}$. By Lemma 2.20, $F(N \cap V)$ is a countable union of $(n-1)$-dimensional DC surfaces in $\mathbb{R}^{n+1}$. So it is sufficient to prove that $M(K) \cap U \subset F(N)$. To prove this inclusion, suppose to the contrary that there exists a point $x \in M(K) \cap U$ such that both F and $d_K \circ F$ are strictly differentiable at x.

We can assume without loss of generality that $x = 0$. Let $T := (dF(0))(e^{\perp})$ be the vector tangent space to X at 0. Let P be the projection of $\mathbb{R}^{n+1}$ onto T in the direction of e and define $Q := (P\restriction_U)^{-1}$. It is easy to see that $Q = F \circ (dF(0))^{-1}$ and therefore $dQ(0) = (dF(0)) \circ (dF(0))^{-1} = \mathrm{id}_T$.

Since $0 \in M(K)$, there exist two different minimal curves $\beta, \gamma\colon [0, r] \to X$ such that $r = d_K(0)$, $\beta(0) = \gamma(0) = 0$, $\beta(r) \in K$, and $\gamma(r) \in K$. As any minimal curves on a convex surface, β and γ have right semitangents at 0 (see [4, Corollary 2]); let $u, v \in \mathbb{R}^{n+1}$ be unit vectors from these semitangents. Further, [12, Theorem 2] easily implies that $u \neq v$.

Clearly $d_K \circ \beta(t) = r - t$, $t \in [0, r]$, and $(P \circ \beta)'_+(0) = P(\beta'_+(0)) = u$. Further observe that $d_K \circ Q$ is differentiable at 0, since $d_K \circ F$ is differentiable at $0 = (dF(0))^{-1}(0)$. Using the above facts, we obtain

$$\begin{aligned}(d(d_K \circ Q)(0))(u) &= (d(d_K \circ Q)(0))((P \circ \beta)'_+(0)) \\ &= (d_K \circ Q \circ P \circ \beta)'_+(0) \\ &= (d_K \circ \beta)'_+(0) = -1.\end{aligned}$$

In the same way we obtain $(d(d_K \circ Q)(0))(v) = -1$.

Thus, $u + v \neq 0$ and, by linearity of the differential,

$$(d(d_K \circ Q)(0))\Big(\frac{u+v}{\|u+v\|}\Big) = \frac{-2}{\|u+v\|} < -1.$$

Thus there exists $\varepsilon > 0$ such that

$$\|d(d_K \circ Q)(0)\| > 1 + \varepsilon. \tag{12}$$

Since $dQ(0) = \mathrm{id}_T$ and $Q = F \circ (dF(0))^{-1}$ is clearly strictly differentiable at 0, there exists $\delta > 0$ such that

$$\|Q(p) - Q(q) - (p - q)\| \leqslant \varepsilon \|p - q\|, \quad p, q \in B(0, \delta) \cap T,$$

and consequently Q is Lipschitz on $B(0, \delta) \cap T$ with constant $1 + \varepsilon$. Let $p, q \in B(0, \delta) \cap T$ and consider the curve $\omega\colon [0, 1] \to X$, $\omega(t) = Q(tp + (1 - t)q)$. Then clearly

$$\operatorname{dist}(Q(p), Q(q)) \leqslant \operatorname{length} \omega \leqslant (1 + \varepsilon)\|p - q\|.$$

Consequently,

$$\|d_K \circ Q(p) - d_K \circ Q(q)\| \leqslant \operatorname{dist}(Q(p), Q(q)) \leqslant (1 + \varepsilon)\|p - q\|.$$

Thus the function $d_K \circ Q$ is Lipschitz on $B(0, \delta) \cap T$ with constant $1 + \varepsilon$, which contradicts (12). $\square$

Remark 4.5. An analoguous result on ambiguous loci in a Hilbert space was proved in [26].

5. Remarks and questions

The results of [15] and Corollary 3.6 suggest that the following definition is natural.

Definition 5.1. Let X be a length space and let an open set $G \subset X$ be equipped with an n-dimensional DC structure. We will say that this DC structure is *compatible* with the intrinsic metric on X, if the following statements hold.

(i) For each DC chart (U, φ), the map $\varphi\colon U \to \mathbb{R}^n$ is locally bilipschitz.
(ii) For each $x_0 \in X$, the distance function $\operatorname{dist}(x_0, \cdot)$ is DC (with respect to the DC structure) on $G \setminus \{x_0\}$.

If M is an n-dimensional Alexandrov space with curvature bounded from below and without boundary, the results of [15] (cf. [10, § 2.7]) give that there exist an open dense set $M^* \subset M$ with $\dim_H(M \setminus M^*) \leqslant n - 2$ and an n-dimensional DC structure on M^* compatible with the intrinsic metric on M (cf. [15, p. 6, line 9 from below]). Since the components of each chart of this DC structure are formed by distance functions, Lemma 2.10 (d) easily implies that *no other DC structure on M^* compatible with the intrinsic metric exists.*

Let $X \subset \mathbb{R}^{n+1}$ be a convex surface. Then Corollary 3.6 gives that the standard DC structure on X is compatible with the intrinsic metric on X. By the above observations, there is no other compatible DC structure on the (open dense) "Perelman's set" X^*. *We conjecture that this uniqueness is true also on the whole X.* Further note that the standard DC structure on X has an atlas such that all the corresponding transition maps are C^∞. Indeed, let C be the convex body for which $X = \partial C$. We can suppose $0 \in \operatorname{int} C$ and find $r > 0$ such that $B(0, r) \subset \operatorname{int} C$. Now "identify" X with the C^∞ manifold $\partial B(0, r)$ via the radial projection of X on $\partial B(0, r)$. Then, this bijection transfers the C^∞ structure of $\partial B(0, r)$ on X.

We conclude with the following problem.

Problem. Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a semiconcave (or a DC) function. Consider the "semiconcave surface" (the DC surface, respectively) $X := \operatorname{graph} f$ equipped with the intrinsic metric. Let $x_0 \in X$. Is it true that the distance function $\operatorname{dist}(x_0, \cdot)$ is DC on $X \setminus \{x_0\}$ with respect to the natural DC structure (given by the projection onto $\mathbb{R}^n$)? In other words, is the natural DC structure on X compatible with the intrinsic metric on X?

If f is convex, then the answer is affirmative, see Remark 3.9. If f is semiconcave, then each minimal curve φ on X has bounded turn in $\mathbb{R}^{n+1}$ by [19]. Thus an interesting result concerning the intrinsic distance extends from convex surfaces to the case of semiconcave surfaces. So, there is a chance that the above problem has the affirmative answer in this case. However, we have not been able to extend our proof to this case.

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