Czechoslovak Mathematical Journal

Zong-Xuan Chen; Kwang Ho Shon

Properties of differences of meromorphic functions

Czechoslovak Mathematical Journal, Vol. 61 (2011), No. 1, 213–224

Persistent URL: `http://dml.cz/dmlcz/141529`

Terms of use:

© Institute of Mathematics AS CR, 2011

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.

This document has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* `http://dml.cz`

Czechoslovak Mathematical Journal, 61 (136) (2011), 213–224

PROPERTIES OF DIFFERENCES OF MEROMORPHIC FUNCTIONS

Zong-Xuan Chen, Guangzhou, and Kwang Ho Shon, Pusan

(Received November 9, 2009)

Abstract. Let f be a transcendental meromorphic function. We propose a number of results concerning zeros and fixed points of the difference $g(z)=f(z+c)-f(z)$ and the divided difference $g(z)/f(z)$.

Keywords: meromorphic function, difference, divided difference, zero, fixed point

MSC 2010: 30D35, 39A10

1. Introduction and results

Bergweiler and Langley [2] investigated the existence of zeros of the difference $f(z+c)-f(z)$ and the divided difference $(f(z+c)-f(z))/f(z)$. They obtained many profound and significant results. The results may be viewed as difference analogues of the following existing theorem on the zeros of f'.

Theorem A ([3], [8], [15]). *Let f be transcendental and meromorphic in the plane with*

$$\varliminf_{r\to\infty}\frac{T(r,f)}{r}=0. \tag{1.1}$$

Then f' has infinitely many zeros.

Theorem A is sharp, as shown by e^z, $\tan z$ and examples of arbitrary order greater than 1 constructed in [6].

This project was supported by the Brain Pool Program of Korean Federation of Science and Technology Societies (No: 072-1-3-0164) and by the National Natural Science Foundation of China (No: 10871076). The second author was supported by the Korea Research Foundation(KRF) grant funded by the Korea government(MEST) (No. 2009-0074210).

In this paper we assume the reader is familiar with the basic notions of Nevanlinna's value distribution theory (see e.g. [12], [17], [18]). In addition, we use the notations $\sigma(f)$ to denote the order of growth of the meromorphic function $f(z)$; $\lambda(f)$ and $\lambda(1/f)$ denote, respectively, the exponents of convergence of zeros and poles of $f(z)$. We also use the notation $\tau(f)$ to denote the exponent of convergence of fixed points of f that is defined as

$$\tau(f) = \overline{\lim_{r\to\infty}} \frac{\log N(r, 1/(f-z))}{\log r}.$$

For f as in the hypotheses of Theorem A it follows from Hurwitz's theorem that if z_1 is a zero of f' then $f(z+c) - f(z)$ has a zero near z_1 for all sufficiently small $c \in \mathbb{C} \setminus \{0\}$. This makes it natural to ask whether $f(z+c) - f(z)$, for such functions f, must always have infinitely many zeros or not. Bergeiler and Langley [2] answered this question, and obtained the following Theorems B–D.

Theorem B. *Let f be a function transcendental and meromorphic of lower order $\mu(f) < 1$ in the plane. Let $c \in \mathbb{C} \setminus \{0\}$ be such that at most finitely many poles z_j, z_k of f satisfy $z_j - z_k = c$.*

Then $g(z) = f(z+c) - f(z)$ has infinitely many zeros.

Theorem C. *Let $\varphi(r)$ be a positive non-decreasing function on $[1, \infty)$ which satisfies $\lim_{r\to\infty} \varphi(r) = \infty$. Then there exists a function f transcendental and meromorphic in the plane with*

$$\overline{\lim_{r\to\infty}} \frac{T(r,f)}{r} < \infty \quad \textit{and} \quad \underline{\lim_{r\to\infty}} \frac{T(r,f)}{\varphi(r)\log r} < \infty$$

such that $g(z) = f(z+1) - f(z)$ has only one zero. Moreover, the function g satisfies

$$\overline{\lim_{r\to\infty}} \frac{T(r,g)}{\varphi(r)\log r} < \infty.$$

Theorem D. *Let f be a function transcendental and meromorphic in the plane with*

$$T(r,f) = O(\log r)^2 \quad \textit{as} \quad r \to \infty,$$

and set

$$g(z) = f(z+1) - f(z) \quad \textit{and} \quad G_1(z) = \frac{g(z)}{f(z)} = \frac{f(z+1) - f(z)}{f(z)}.$$

Then at least one of $g(z)$ and $G_1(z)$ has infinitely many zeros.

Chen and Shon [4] considered zeros and fixed points of the difference and the divided difference of entire functions with order of growth $\sigma(f) = 1$ and obtained the following theorem.

Theorem E. *Let $c \in \mathbb{C} \setminus \{0\}$ and let f be a transcendental entire function of order of growth $\sigma(f) = \sigma = 1$, that has infinitely many zeros with the exponent of convergence of zeros $\lambda(f) = \lambda < 1$. Then $g(z) = \Delta f(z) = f(z+c) - f(z)$ has infinitely many zeros and infinitely many fixed points.*

In particular, if a set $H = \{z_j\}$ consists of all different zeros of $f(z)$ satisfying any one of the following two conditions:
(i) *at most finitely many zeros z_j, z_k satisfy $z_j - z_k = c$;*
(ii) $\underline{\lim}_{j\to\infty} |z_{j+1}/z_j| = l > 1$, *then*

$$G(z) = \frac{\Delta f(z)}{f(z)} = \frac{f(z+c) - f(z)}{f(z)}$$

has infinitely many zeros and infinitely many fixed points.

From Theorem B we see that the condition "at most finitely many poles z_j, z_k of f satisfy $z_j - z_k = c$" guarantees that $g(z)$ has infinitely many zeros.

From Theorem C we see that Theorem B fails without the hypothesis on the value c, even for lower order 0.

Theorem C shows that for any given σ $(0 \leqslant \sigma \leqslant 1)$, there exists a transcendental meromorphic function of order of growth $\sigma(f) = \sigma$, such that $g(z)$ has only one zero.

Theorem D shows that even under the condition "$T(r,f) = O(\log r)^2$ as $r \to \infty$", we cannot prove that $g(z)$ has infinitely many zeros.

Theorem E shows that the fixed points of the difference and the divided difference have the same properties as their zeros.

In this paper, we consider the following three problems:

(i) What conditions will guarantee that the difference $f(z+c) - f(z)$ has infinitely many zeros without the hypothesis on c for a meromorphic function f?

(ii) What is the exponent of convergence of zeros of the difference $f(z+c) - f(z)$ if it has infinitely many zeros?

(iii) What can we say about the zeros of

$$f(z+c) - f(z) - p(z) \quad \text{and} \quad \frac{f(z+c) - f(z)}{f(z)} - p(z),$$

where $p(z)$ is a polynomial?

We prove the following three theorems concerning the above three problems.

Theorem 1. *Let $c \in \mathbb{C} \setminus \{0\}$ be a constant and f a meromorphic function of order of growth $\sigma(f) = \sigma \leqslant 1$. Suppose that f satisfies $\lambda(1/f) < \lambda(f) < 1$ or has infinitely many zeros (with $\lambda(f) = 0$) and finitely many poles. Then*

$$g(z) = f(z+c) - f(z) \tag{1.2}$$

has infinitely many zeros and satisfies $\lambda(g) = \lambda(f)$.

Theorem 2. *Let c and $f(z)$ satisfy the conditions of Theorem 1. Suppose that $p(z)$ is a polynomial. Then $g^*(z) = g(z) - p(z)$ has infinitely many zeros and satisfies $\lambda(g^*) = \sigma(f)$.*

Theorem 3. *Let $c \in \mathbb{C} \setminus \{0\}$ be a constant and f a transcendental meromorphic function of order of growth $\sigma(f) = \sigma < 1$ or of the form $f(z) = h(z)\mathrm{e}^{az}$ where $a \neq 0$ is a constant, $h(z)$ is a transcendental meromorphic function with $\sigma(h) < 1$. Suppose that $p(z)$ is a nonconstant polynomial. Then*

$$G(z) = \frac{f(z+c) - f(z)}{f(z)} - p(z) \tag{1.3}$$

has infinitely many zeros.

From Theorems 2 and 3 we easily obtain the following corollaries on fixed points of differences and divided differences.

Corollary 1. *Let c and $f(z)$ satisfy the conditions of Theorem 2. Then $g(z)$ has infinitely many fixed points and satisfies the exponent of convergence of fixed points $\tau(g) = \sigma(f)$.*

Corollary 2. *Let c and $f(z)$ satisfy the conditions of Theorem 3. Then $G_1(z) = (f(z+c) - f(z))/f(z)$ has infinitely many fixed points.*

Remark 1.1. The following examples show that the condition $\lambda(f) < 1$ of Theorem 1 and Corollary 1 cannot be replaced by $\lambda(f) \leqslant 1$.

For example, the function $f(z) = \mathrm{e}^z + 1$ satisfies $\lambda(f) = 1$, but

$$g(z) = f(z+1) - f(z) = (e-1)\mathrm{e}^z$$

has no zero. And for example, the function $f = \mathrm{e}^z + \frac{1}{2}z^2 - \frac{1}{2}z + 1$ satisfies $\lambda(f) = 1$ by Milloux's theorem (see [12], [18]), and $g(z) = f(z+1) - f(z) = (e-1)\mathrm{e}^z + z$ has no fixed point, but it has infinitely many zeros.

2. Proof of theorem 1

We need the following lemmas and notion to prove Theorem 1.

ε-set. Following Hayman [13, p. 75–76], we define an ε-set to be a countable union of open discs not containing the origin and subtending angles at the origin whose sum is finite. If E is an ε-set then the set of $r \geqslant 1$ for which the circle $S(0,r)$ meets E has finite logarithmic measure, and for almost all real θ the intersection of E with the ray $\arg z = \theta$ is bounded.

Lemma 2.1 ([2]). *Let f be a function transcendental and meromorphic in the plane of order < 1. Let $h > 0$. Then there exists an ε-set E such that*

$$f(z+c) - f(z) = cf'(z)(1+o(1)) \quad \text{as } z \to \infty \text{ in } \mathbb{C} \setminus E,$$

uniformly in c for $|c| \leqslant h$.

Lemma 2.2 ([2]). *Let g be a function transcendental and meromorphic in the plane of order < 1. Let $h > 0$. Then there exists an ε-set E such that*

$$\frac{g'(z+c)}{g(z+c)} \to 0, \qquad \frac{g(z+c)}{g(z)} \to 1 \text{ as } z \to \infty \text{ in } \mathbb{C} \setminus E,$$

uniformly in c for $|c| \leqslant h$. Further, E may be chosen such that for large z not in E the function g has no zeros or poles in $|\zeta - z| \leqslant h$.

Lemma 2.3 (Rouché's theorem ([7, p. 125]). *Suppose f and g are meromorphic in a neighborhood of $\{z\colon\ |z-a| \leqslant R\}$ with no zeros or poles on the circle $\gamma = \{z\colon\ |z-a| = R\}$. If*

$$|f(z) + g(z)| < |f(z)| + |g(z)|$$

on γ, then

$$n\Big(R, \frac{1}{f}\Big) - n(R, f) = n\Big(R, \frac{1}{g}\Big) - n(R, g).$$

Proof of Theorem 1. We divide this proof into two cases $\sigma(f) = \sigma < 1$ and $\sigma(f) = \sigma = 1$.

Case I. $\sigma(f) = \sigma < 1$. First, we suppose that f satisfies $\lambda(1/f) < \lambda(f)$. Suppose that $f(z) = u(z)/v(z)$, where $u(z)$ and $v(z)$ are canonical products ($v(z)$ may be a polynomial) formed by zeros and poles of $f(z)$, respectively, and

$$\sigma(u) = \lambda(u) = \lambda(f) > \sigma(v) = \lambda(v) = \lambda\Big(\frac{1}{f}\Big).$$

By Lemma 2.1, there exists an ε-set E such that

$$f(z+c) - f(z) = cf'(z)(1+o(1)) \quad \text{as } z \to \infty \text{ in } \mathbb{C} \setminus E. \tag{2.1}$$

Set

$$H = \{|z| = r \in (1,\infty)\colon\ z \in E \ \text{ or } \ g(z) = 0, \ \text{or } f'(z) = 0\}.$$

By $\sigma(f) < 1$ and the property of the ε-set, we see that H has finite logarithmic measure. Thus, for large $|z| = r \notin [0,1] \cup H$, $g(z)$ and $f'(z)$ have no zero on the circle $|z| = r$, and by (2.1),

$$|g(z) - cf'(z)| = |cf'(z)o(1)| < |cf'(z)| + |g(z)|. \tag{2.2}$$

Applying Lemma 2.3 (Rouché's theorem) to $g(z)$ and $cf'(z)$, by (2.2) we obtain that

$$n\Big(r, \frac{1}{g}\Big) - n(r,g) = n\Big(r, \frac{1}{f'}\Big) - n(r,f') \quad r \notin [0,1] \cup H. \tag{2.3}$$

Since $f'(z) = (u'(z)v(z) - u(z)v'(z))/v^2(z)$, $\sigma(f) = \sigma(f')$ and $\lambda(1/f) < \lambda(f) = \sigma(f) < 1$, we see that

$$\lambda\Big(\frac{1}{f'}\Big) = \lambda\Big(\frac{1}{f}\Big) < \lambda(f) = \sigma(f) = \sigma(f') = \lambda(f'). \tag{2.4}$$

By (1.2) and $\lambda\left(\frac{1}{f}\right) < \lambda(f) = \sigma(f)$, we see that

$$\lambda\Big(\frac{1}{g}\Big) \leqslant \lambda\Big(\frac{1}{f}\Big) < \lambda(f) = \lambda(f'). \tag{2.5}$$

Thus, (2.3)–(2.5) give

$$\lambda(g) = \lambda(f') = \lambda(f).$$

Secondly, we suppose that $f(z)$ has infinitely many zeros (with $\lambda(f) = 0$) and only finitely many poles. Using a method similar to the above, we can complete the proof of Case I.

Case II. $\sigma(f) = \sigma = 1$. First, we suppose that f satisfies $\sigma(f) = 1$ and $\lambda(1/f) < \lambda(f) < 1$. Then f can be rewritten as

$$f(z) = h(z)\mathrm{e}^{az} = \frac{u(z)}{v(z)}\,\mathrm{e}^{az}, \tag{2.6}$$

where $a \neq 0$ is a constant, $h(z)$ is a meromorphic function such that $h(z) = u(z)/v(z)$, $u(z)$ and $v(z)$ are canonical products ($v(z)$ may be polynomial) formed by zeros and poles of $f(z)$ respectively. Also,

$$\begin{aligned} 1 > \sigma(h) = \lambda(h) = \sigma(u) = \lambda(u) = \lambda(f) \\ > \lambda\Big(\frac{1}{h}\Big) = \sigma(v) = \lambda(v) = \lambda\Big(\frac{1}{f}\Big). \end{aligned} \tag{2.7}$$

Thus,

$$g(z) = [h(z+c)\mathrm{e}^{ac} - h(z)]\,\mathrm{e}^{az} = g_1(z)\mathrm{e}^{az},$$

where

$$g_1(z) = h(z+c)\mathrm{e}^{ac} - h(z).$$

Thus,

$$\sigma(g) = 1,\ \sigma(g_1) < 1,\ \lambda(g) = \lambda(g_1) \ \text{ and } \ \lambda\Big(\frac{1}{g}\Big) = \lambda\Big(\frac{1}{g_1}\Big).$$

If $\mathrm{e}^{ac} = 1$, then by Case I and (2.7), we see that the assertion holds in Case II.

Next, we suppose that $\mathrm{e}^{ac} \neq 1$. By Lemma 2.3, there exists an ε-set E such that

$$h(z+c) = h(z)(1+o(1)) \quad \text{as} \quad z \to \infty \ \text{ in } \ \mathbb{C} \setminus E. \tag{2.8}$$

Thus (2.8) yields

$$g_1(z) = \mathrm{e}^{ac}h(z)(1+o(1)) - h(z) = (\mathrm{e}^{ac} - 1)h(z)(1+o(1)). \tag{2.9}$$

So, since h is transcendental, we see that g_1 is transcendental. Set

$$H = \{|z| = r \in (1,\infty) \colon\ z \in E \ \text{ or } \ g_1(z) = 0, \ \text{ or } \ h(z) = 0\}.$$

By $\sigma(g_1) < 1$ and the property of the ε-set, we see that H has finite logarithmic measure. Thus, for large $|z| = r \notin [0,1] \cup H$, $g_1(z)$ and $(\mathrm{e}^{ac} - 1)h(z)$ have no zero on the circle $|z| = r$, and by (2.9),

$$|g_1(z) - (\mathrm{e}^{ac} - 1)h(z)| = |(\mathrm{e}^{ac} - 1)h(z)o(1)| < |(\mathrm{e}^{ac} - 1)h(z)| + |g_1(z)|. \tag{2.10}$$

Using a method similar to the proof of Case I, by (2.10) we get

$$\lambda(g_1) = \lambda(h) = \lambda(u) = \lambda(f).$$

Secondly, we suppose that $f(z)$ has infinitely many zeros (with $\lambda(f) = 0$) and only finitely many poles. Using a method similar to the above, we can complete the proof of Case II.

3. Proof of theorem 2

We need the following lemma to prove Theorem 2.

Lemma 3.1 ([19]). *Let $f_j(z)$ $(j=1,\dots,n)$ $(n\geqslant 2)$ be meromorphic functions, $g_j(z)$ $(j=1,\dots,n)$ entire functions, and let them satisfy*

(i) $\sum\limits_{j=1}^{n} f_j(z)\mathrm{e}^{g_j(z)}\equiv 0$;

(ii) *when* $1\leqslant j<k\leqslant n$, *then* $g_j(z)-g_k(z)$ *is not a constant;*

(iii) *when* $1\leqslant j\leqslant n$, $1\leqslant h<k\leqslant n$, *then*

$$T(r,f_j)=o\{T(r,\mathrm{e}^{g_h-g_k})\}\quad (r\to\infty, r\notin E),$$

where $E\subset(1,\infty)$ *is of finite linear measure or finite logarithmic measure.*

Then $f_j(z)\equiv 0$ $(j=1,\dots,n)$.

Proof of Theorem 2. We divide this proof into two cases $\sigma(f)=\sigma<1$ and $\sigma(f)=\sigma=1$.

Case I. $\sigma(f)=\sigma<1$. We suppose that f satisfies $\lambda(f)>\lambda(1/f)$. From Theorem 1 and its proof of Case I, we see that

$$\sigma(g)=\lambda(g)=\sigma(f)=\lambda(f),\quad \lambda\Big(\frac{1}{g}\Big)\leqslant\lambda\Big(\frac{1}{f}\Big)<\sigma(g).$$

Since $g^*(z)=g(z)-p(z)$ where $p(z)$ is a polynomial, we have

$$1>\sigma(g^*)=\sigma(g)=\lambda(g)>\lambda\Big(\frac{1}{g}\Big)=\lambda\Big(\frac{1}{g^*}\Big).$$

So, $\lambda(g^*)=\sigma(g^*)=\sigma(g)=\lambda(f)=\sigma(f)$.

For the case that f has infinitely many zeros (with $\lambda(f)=0$) and only finitely many poles, using a method similar to the above, we can complete the proof of Case I.

Case II. $\sigma=1$. We suppose that f satisfies $\lambda(1/f)<\lambda(f)<1$. From Theorem 1 and its proof of Case II, we see that

$$f(z)=h(z)\mathrm{e}^{az}\quad\text{and}\quad g(z)=[h(z+c)\mathrm{e}^{ac}-h(z)]\,\mathrm{e}^{az}$$

where $a\neq 0$ is a constant, $h(z)$ is a meromorphic function such that $\sigma(g)=1$ and

$$1>\lambda(h)=\lambda(f)>\lambda\Big(\frac{1}{f}\Big)=\lambda\Big(\frac{1}{h}\Big),\quad \lambda(g)=\lambda(f)>\lambda\Big(\frac{1}{f}\Big)\geqslant\lambda\Big(\frac{1}{g}\Big). \tag{3.1}$$

Suppose that $\lambda(g^*) < 1$. Then by $\sigma(g^*) = \sigma(g - p) = 1$, $g^*(z)$ can be rewritten as

$$g^*(z) = g(z) - p(z) = h^*(z)\mathrm{e}^{dz} \tag{3.2}$$

where $h^*(z)$ is a meromorphic function such that

$$\lambda(h^*) = \lambda(g^*),\ \lambda\Big(\frac{1}{h^*}\Big) = \lambda\Big(\frac{1}{g^*}\Big),\ \sigma(h^*) = \max\Big\{\lambda(h^*),\ \lambda\Big(\frac{1}{h^*}\Big)\Big\} < 1.$$

By (3.1), we see that $h^*(z) \not\equiv 0$ and

$$\lambda\Big(\frac{1}{g^*}\Big) = \lambda\Big(\frac{1}{g}\Big) = \lambda\Big(\frac{1}{h^*}\Big) \leqslant \lambda\Big(\frac{1}{f}\Big). \tag{3.3}$$

Thus (3.2) gives

$$[h(z+c)\mathrm{e}^{ac} - h(z)]\,\mathrm{e}^{az} - h^*(z)\mathrm{e}^{dz} - p(z)\mathrm{e}^{0z} = 0. \tag{3.4}$$

If $a \neq d$, then by Lemma 3.1 we see that

$$h(z+c)\mathrm{e}^{ac} - h(z) \equiv h^*(z) \equiv p(z) \equiv 0.$$

This is a contradiction. So, $a = d$. By (3.4), we see that

$$[h(z+c)\mathrm{e}^{ac} - h(z) - h^*(z)]\,\mathrm{e}^{az} - p(z)\mathrm{e}^{0z} = 0. \tag{3.5}$$

Again applying Lemma 3.1, we obtain that

$$p(z) \equiv 0, \quad h(z+c)\mathrm{e}^{ac} - h(z) - h^*(z) \equiv 0.$$

This is also a contradiction. Hence $\lambda(g-p) = 1$. Case II of Theorem 2 is thus proved.

4. Proof of theorem 3

We need the following lemmas to prove Theorem 3.

Lemma 4.1 ([2]). *Let $c \in \mathbb{C} \setminus \{0\}$ be a constant and f a function transcendental and meromorphic in the plane which satisfies* (1.1). *Then both $f(z+c) - f(z)$ and $(f(z+c) - f(z))/f(z)$ are transcendental.*

Lemma 4.2 ([9]). *Let f be a transcendental meromorphic function with $\sigma(f) = \sigma < \infty$, let $H = \{(k_1, j_1), (k_2, j_2), \ldots, (k_q, j_q)\}$ be a finite set of distinct pairs of integers that satisfy $k_i > j_i \geqslant 0$ for $i = 1, \ldots, q$. Let $\varepsilon > 0$ be a given constant. Then there exists a set $E \subset (1, \infty)$ with finite logarithmic measure such that for all z satisfying $|z| \notin E \cup [0, 1]$ and for all $(k, j) \in H$, we have*

$$\Bigl|\frac{f^{(k)}(z)}{f^{(j)}(z)}\Bigr| \leqslant |z|^{(k-j)(\sigma-1+\varepsilon)}.$$

The following Lemma 4.3 can be got by using a method similar to the proof of Lemma 4.1 (see [2]).

Lemma 4.3. *Let a and $c \in \mathbb{C} \setminus \{0\}$ be constants and h a function transcendental and meromorphic in the plane which satisfies (1.1). Then $(h(z + c)\mathrm{e}^{ac} - h(z))/h(z)$ is transcendental.*

Proof of Theorem 3. We divide this proof into two cases $\sigma(f) = \sigma < 1$, and $f(z)$ is of the form $f(z) = h(z)\mathrm{e}^{az}$ where $a \neq 0$ is a constant and $h(z)$ is a transcendental meromorphic function with $\sigma(h) < 1$.

Case I. $\sigma(f) = \sigma < 1$. By $\sigma(f) < 1$, we see that f satisfies (1.1). By Lemma 4.1, we see that $(f(z + c) - f(z))/f(z)$ is transcendental, and so is $G(z)$.

By Lemma 2.1, there is an ε-set E, such that

$$f(z + c) - f(z) = cf'(z)(1 + o(1)) \quad \text{as} \quad z \to \infty \text{ in } \mathbb{C} \setminus E. \tag{4.1}$$

By Lemma 4.2, for a given $\varepsilon > 0$ there exists a set $H_1 \subset (1, \infty)$ with finite logarithmic measure such that for all z satisfying $|z| \notin [0, 1] \cup H_1$ we have

$$\Bigl|\frac{f'(z)}{f(z)}\Bigr| \leqslant |z|^{\sigma-1+\varepsilon} \tag{4.2}$$

where $\sigma(f) = \sigma < 1$. Set

$$H_2 = \{|z| = r \in (1, \infty)\colon\ z \in E, \quad \text{or} \quad G(z) = 0, \quad \text{or} \quad p(z) = 0\}.$$

Using the inequality $\sigma(f) < 1$ and the property of an ε-set, we see that H_2 has finite logarithmic measure. Thus for large $|z| = r \notin [0, 1] \cup H_1 \cup H_2$, $G(z)$ and $p(z)$ have no zero on the circle $|z| = r$. By (4.1) and (4.2), we obtain that

$$\begin{aligned} |G(z) + p(z)| &= \Bigl|\frac{cf'(z)}{f(z)}(1 + o(1))\Bigr| \\ &\leqslant |c(1 + o(1))||z|^{\sigma-1+\varepsilon} < |G(z)| + |p(z)|. \end{aligned} \tag{4.3}$$

Applying Lemma 2.3 (Rouché's theorem) to $G(z)$ and $p(z)$, by (4.3) we obtain that

$$n\Big(r, \frac{1}{G}\Big) - n(r, G) = n\Big(r, \frac{1}{p}\Big) - n(r, p) = \deg p, \quad r \notin [0, 1] \cup H_1 \cup H_2. \tag{4.4}$$

Since G is transcendental and $\sigma(G) < 1$, we see that at least one of $n\,(r, 1/G) \to \infty$ and $n(r, G) \to \infty$ $(r \to \infty)$ is true. So, by (4.4), we see that both $n\,(r, 1/G) \to \infty$ and $n(r, G) \to \infty$ $(r \to \infty)$ hold. Hence $G(z)$ must have infinitely many zeros. Thus, Case I of Theorem 3 is proved.

Case II. $f(z)$ is of the form $f(z) = h(z)\mathrm{e}^{az}$ where $a \neq 0$ is a constant and $h(z)$ is a transcendental meromorphic function with $\sigma(h) < 1$. Substituting $f(z) = h(z)\mathrm{e}^{az}$ into $G(z)$, we get that

$$G(z) = \frac{h(z+c)\mathrm{e}^{ac} - h(z)}{h(z)} - p(z), \tag{4.5}$$

where $h(z)$ is transcendental and $\sigma(h) < 1$.

If $\mathrm{e}^{ac} = 1$, then by Case I and (4.5) we see that $G(z)$ has infinitely many zeros.

Assume henceforth that $\mathrm{e}^{ac} \neq 1$. We use a method similar to the proof of Case I. By Lemmas 2.1 and 4.2, for a given $\varepsilon > 0$ there exist an ε-set E and a set $H_1 \subset (1, \infty)$ having finite logarithmic measure, such that for all z satisfying $z \in \mathbb{C} \setminus E$ and $|z| \notin [0, 1] \cup H_1$ we have

$$\begin{aligned} \Big|\frac{h(z+c)\mathrm{e}^{ac} - h(z)}{h(z)}\Big| &= \Big|\frac{ch'(z)}{h(z)}\mathrm{e}^{ac} + (\mathrm{e}^{ac} - 1)\Big| \\ &\leqslant |c\mathrm{e}^{ac}||z|^{\sigma-1+\varepsilon} + |\mathrm{e}^{ac} - 1|, \end{aligned} \tag{4.6}$$

where $\sigma(h) = \sigma < 1$. Set

$$H_2 = \{|z| = r \in (1, \infty)\colon\ z \in E, \quad \text{or} \quad G(z) = 0, \quad \text{or} \quad p(z) = 0\}.$$

So, H_2 has finite logarithmic measure. Thus for large $|z| = r \notin [0, 1] \cup H_1 \cup H_2$, $G(z)$ and $p(z)$ have no zero on the circle $|z| = r$. By (4.5) and (4.6), we obtain that

$$|G(z) + p(z)| \leqslant |c\mathrm{e}^{ac}||z|^{\sigma-1+\varepsilon} + |\mathrm{e}^{ac} - 1| < |G(z)| + |p(z)|. \tag{4.7}$$

By Lemma 2.3 (Rouché's theorem) and (4.7), we obtain (4.4). By the same argument as in the proof of Case I and noting that $G(z)$ is transcendental, by Lemma 4.3 we obtain $n\,(r, 1/G) \to \infty$ $(r \to \infty)$. Case II of Theorem 3 is thus proved.

Acknowledgements. The authors are grateful to the referee for a number of helpful suggestions improving the paper.

References

[1] *M. Ablowitz, R. G. Halburd and B. Herbst*: On the extension of Painlevé property to difference equations. Nonlinearity *13* (2000), 889–905.

[2] *W. Bergweiler and J. K. Langley*: Zeros of differences of meromorphic functions. Math. Proc. Camb. Phil. Soc. *142* (2007), 133–147.

[3] *W. Bergweiler and A. Eremenko*: On the singularities of the inverse to a meromorphic function of finite order. Rev. Mat. Iberoamericana *11* (1995), 355–373.

[4] *Z. X. Chen and K. H. Shon*: On zeros and fixed points of differences of meromorphic functions. J. Math. Anal. Appl. *344-1* (2008), 373–383.

[5] *Y. M. Chiang and S. J. Feng*: On the Nevanlinna characteristic of $f(z+\eta)$ and difference equations in the complex plane. Ramanujan J. *16* (2008), 105–129.

[6] *J. Clunie, A. Eremenko and J. Rossi*: On equilibrium points of logarithmic and Newtonian potentials. J. London Math. Soc. *47-2* (1993), 309–320.

[7] *J. B. Conway*: Functions of One Complex Variable. New York, Spring-Verlag.

[8] *A. Eremenko, J. K. Langley and J. Rossi*: On the zeros of meromorphic functions of the form $\sum_{k=1}^{\infty} a_k/(z-z_k)$. J. Anal. Math. *62* (1994), 271–286.

[9] *G. Gundersen*: Estimates for the logarithmic derivative of a meromorphic function, plus similar estimates. J. London Math. Soc. *37-2* (1988), 88–104.

[10] *R. G. Halburd and R. Korhonen*: Difference analogue of the lemma on the logarithmic derivative with applications to difference equations. J. Math. Anal. Appl. *314* (2006), 477–487.

[11] *R. G. Halburd and R. Korhonen*: Nevanlinna theory for the difference operator. Ann. Acad. Sci. Fenn. Math. *31* (2006), 463–478.

[12] *W. K. Hayman*: Meromorphic Functions. Oxford, Clarendon Press, 1964.

[13] *W. K. Hayman*: Slowly growing integral and subharmonic functions. Comment. Math. Helv. *34* (1960), 75–84.

[14] *J. Heittokangas, R. Korhonen, I. Laine, J. Rieppo and K. Tohge*: Complex difference equations of Malmquist type. Comput. Methods Funct. Theory *1* (2001), 27–39.

[15] *J. D. Hinchliffe*: The Bergweiler-Eremenko theorem for finite lower order. Results Math. *43* (2003), 121–128.

[16] *K. Ishizaki and N. Yanagihara*: Wiman-Valiron method for difference equations. Nagoya Math. J. *175* (2004), 75–102.

[17] *I. Laine*: Nevanlinna Theory and Complex Differential Equations. Berlin, W. de Gruyter, 1993.

[18] *L. Yang*: Value Distribution Theory. Beijing, Science Press, 1993.

[19] *C. C. Yang and H. X. Yi*: Uniqueness Theory of Meromorphic Functions. Dordrecht, Kluwer Academic Publishers Group, 2003.

Authors' addresses: Zong-Xuan Chen, School of Mathematical Sciences, South China Normal University, Guangzhou, 510631, P. R. China, e-mail: chzx@vip.sina.com; Kwang Ho Shon, Department of Mathematics, College of Natural Sciences, Pusan National University, Pusan 609-735, Korea, e-mail: khshon@pusan.ac.kr.