Naeem Alkoumi; Pedro J. Torres
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ON THE NUMBER OF LIMIT CYCLES OF A GENERALIZED ABEL EQUATION

Naeem Alkoumi, Pedro J. Torres, Granada

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Abstract. New results are proved on the maximum number of isolated T-periodic solutions (limit cycles) of a first order polynomial differential equation with periodic coefficients. The exponents of the polynomial may be negative. The results are compared with the available literature and applied to a class of polynomial systems on the cylinder.

Keywords: periodic solution, limit cycle, polynomial nonlinearity

MSC 2010: 34C25

1. Introduction and main results

This paper is motivated by some recent results on the number of isolated periodic solutions (limit cycles) of the first order differential equation with polynomial nonlinearity

$$u' = \sum_{i=0}^{n} a_i(t)u^i, \tag{1}$$

where the coefficients a_i are continuous and T-periodic functions for some $T > 0$. This is a classical problem. The first non-trivial situation is the Abel equation $n = 3$. If $a_3(t) > 0$, Pliss [13] proved that (1) has at most three limit cycles, but in the general case Lins-Neto [11] gave examples with an arbitrary number of limit cycles. Such examples can be easily extended to higher-order polynomial equations, even with a constant leading coefficient a_n. Sufficient conditions for $n = 3$ to have at most three limit cycles were proved in [8], [2].

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More recently, the equation with three terms,

$$u' = a_{n_1}(t)u^{n_1} + a_{n_2}(t)u^{n_2} + a_{n_3}(t)u^{n_3}, \tag{2}$$

has been considered in some related works. From now on, a continuous function $f\colon [0,T] \to \mathbb{R}$ is said to have a definite sign if it is not null and either $f(t) \geqslant 0$ or $f(t) \leqslant 0$, and we write $f \succ 0$ in the former case and $f \prec 0$ in the latter case. Gasull and Guillamon [7] proved that if $n_3 = 1$ and $a_{n_2}(t)$ or $a_{n_3}(t)$ have a definite sign, then (2) has at most two positive limit cycles. This gives a total maximum number of five limit cycles by the change $y = -u$, since $x = 0$ is always a solution. This remark leads us to focus our attention only on the positive limit cycles. In the same paper, if $n_3 > 1$ and only one of the coefficients has a definite sign, examples are given with an arbitrary number of limit cycles.

Therefore, for equations with 3 or more monomials, in order to obtain bounds on the number of limit cycles it is natural in some sense to assume that two coefficients have a definite sign. The first result following this idea was obtained by Alwash in [5], where it is proved that if $n \geqslant 3$ and $a_{n-3}(t) \leqslant 0$, the equation

$$u' = u^n + a_{n-1}(t)u^{n-2} + a_{n-3}(t)u^{n-3} \tag{3}$$

has at most one positive limit cycle. This result has been generalized very recently in the following way.

Theorem 1 ([1])**.** *Consider the differential equation*

$$u' = a_{n_1}(t)u^{n_1} + a_{n_2}(t)u^{n_2} + a_{n_3}(t)u^{n_3} + a_m(t)u^m, \tag{4}$$

where $n_1 > n_2 > n_3 > m = 1$. Suppose that $a_{n_1}(t)$ and $a_{n_2}(t)$, or $a_{n_2}(t)$ and $a_{n_3}(t)$ have the same definite sign, or that $a_{n_1}(t)$ and $a_{n_3}(t)$ have opposite definite signs. Then (4) has at most two positive limit cycles. If, moreover, $a_m(t)$ has null integral over $[0,T]$, then (4) has at most one positive limit cycle.

Our aim in this paper is to contribute to the literature by proving some related results which can be seen as a complement to the previous ones. Our main result is as follows.

Theorem 2. *Let us assume that a_{n_1} has a definite sign. Fix integers n_1, n_2, n_3, $m \in \mathbb{Z}$ such that $n_1 > n_2 > n_3$ verify the condition*

$$n_1 - 2n_2 + n_3 = 0. \tag{5}$$

If

$$\Delta = a_{n_2}^2(m-n_2)^2 - 4a_{n_1}a_{n_3}(m-n_1)(m-n_3) \leqslant 0, \tag{6}$$

then (4) *has at most one positive limit cycle.*

Here some comments are in order. First, if compared with Theorem 1, it is important to remark that the condition $m < n_3$ is not required. If compared with other results in the related literature, an original feature of Theorem 2 is that the exponents $n_1, n_2, n_3, m \in \mathbb{Z}$ can be negative. It is worthwhile to consider this case for applications to the study of the number of limit cycles in polynomial planar systems on the cylinder, as we will show in more detail in Section 4 with examples inspired by [3]. About condition (5), it is easy to realize that it is equivalent to impose that three of the terms of the equation have powers following an arithmetic sequence, that is, there exist $r \in \mathbb{N}$, $\beta \in \mathbb{Z}$, such that

$$n_1 = 2r + \beta, \quad n_2 = r + \beta, \quad n_3 = \beta.$$

If $r = 1$ we get consecutive numbers. On the other hand, m does not appear in condition (5), therefore the result is quite flexible and gives a whole family of new criteria.

The paper is divided into four sections. In Section 2 we will prove Theorem 2 and discuss some consequences for the comparison between this result and those previously published. The method of proof is based on the known result that the sign of the derivative up to order three of the nonlinearity on a given region gives a bound on the number of limit cycles (see for instance [7], [8], [14]), but we exploit the fact that this sign is not invariant under changes of variables. A similar idea can be found in [10]. In Section 3, we combine this technique with upper and lower solutions in order to get multiplicity results for the fourth-order differential equation. Finally, in the last section the main results are applied to some specific examples of polynomial planar systems in order to get information on the maximum number of limit cycles.

2. The equation with four monomials

For the proof of Theorem 2, we will need the following result, which can be found in [7], [8], [14].

Proposition 1. *Let us consider a general first order equation*

$$x' = g(t, x), \tag{7}$$

with g continuous and T-periodic in t. Fix $k \in \{1, 2, 3\}$. Let J be an open interval and let us assume that $g(t, x)$ has a continuous derivative $(\partial^k/\partial x^k)g(t, x)$ for all $(t, x) \in [0, T] \times J$. If $(\partial^k/\partial x^k)g(t, x) \geqslant 0$ for all $(t, x) \in [0, T] \times J$ (or $(\partial^k/\partial x^k)g(t, x) \leqslant 0$ for all $(t, x) \in [0, T] \times J$), then the equation (7) *has at most k limit cycles with range contained in J.*

Proof of Theorem 2. By means of the change in the independent variable $\tau = -t$, we can assume that $a_{n_1} \succ 0$ without loss of generality. Let us first consider the case $m = 1$. We write the equation as

$$u' = uF(t, u),$$

where

$$F(t, u) = a_{n_1}u^{n_1-1} + a_{n_2}u^{n_2-1} + a_{n_3}u^{n_3-1} + a_1.$$

By using the change of variable $u = \mathrm{e}^x$, we get

$$x' = F(t, \mathrm{e}^x) := g(t, x). \tag{8}$$

Now,

$$\begin{aligned} g_x(t, x) &= \mathrm{e}^x F_x(t, \mathrm{e}^x) \\ &= \mathrm{e}^{(n_3-1)x}[(n_1 - 1)a_{n_1}\mathrm{e}^{(n_1-n_3)x} + (n_2 - 1)a_{n_2}\mathrm{e}^{(n_1-n_2)x} + (n_3 - 1)a_{n_3}]. \end{aligned}$$

If we denote $S = \mathrm{e}^{(n_1-n_2)x}$, then $S^2 = \mathrm{e}^{(n_1-n_3)x}$ as a result of (5). Therefore, $g_x(t, x)$ can be written as

$$g_x(t, x) = \mathrm{e}^{(n_3-1)x}[(n_1 - 1)a_{n_1}S^2 + (n_2 - 1)a_{n_2}S + (n_3 - 1)a_{n_3}].$$

The last factor is a quadratic polynomial with negative discriminant by hypothesis (6). Hence by Proposition 1 there exists at most one limit cycle of equation (8), which corresponds to at most one positive limit cycle of (8).

For $m \neq 1$, the equation is written as

$$u' = u^m F(t, u).$$

Now the adequate change is $u = x^\alpha$, satisfying $(m-1)\alpha + 1 = 0$. This change is well defined for positive solutions and keeps the number of positive limit cycles. It leads to

$$x' = \frac{1}{\alpha} F(t, x^\alpha) := g(t, x).$$

The derivative is

$$\begin{aligned} g_x(t,x) &= x^{\alpha-1} F_x(t, x^\alpha) \\ &= \alpha x^{(n_3-m+1)\alpha-2}[a_{n_1}(n_1-m)S^2 + a_{n_2}(n_2-m)S + a_{n_3}(n_3-m)], \end{aligned}$$

where $S = x^{(n_1-n_2)\alpha}$. The conclusion is analogous. □

After this proof, we will compare our result with the related literature through some corollaries. The first generalizes the result by Alwash already mentioned in Introduction.

Corollary 1. *If $n_1 > n_2 > n_3$, the condition (5) holds and a_{n_1}, a_{n_3} have opposite definite signs, then the equation (2) has at most two nontrivial limit cycles, at most one positive and at most one negative.*

Proof. Take $m = n_2$ and apply Theorem 2, then (2) has at most one positive limit cycle. For the negative one, make the change $y = -x$. □

For comparison with Theorem 1, note that it does not cover the case of a_{n_1} and $a_{n_3}(t)$ with the same definite sign. In fact, in [1] the authors provide examples under this assumption with at least three limit cycles. Now we get the following complementary result.

Corollary 2. *Fix $n_1 > n_2 > n_3 > m = 1$ verifying (5) and assume that a_{n_1} and a_{n_3} have the same definite sign. If*

$$a_{n_1}(t)a_{n_3}(t) \geqslant \frac{(n_2-1)^2}{4(n_1-1)(n_3-1)} a_{n_2}(t)^2$$

for all t, then (4) has at most one positive limit cycle.

The proof is direct. Other variant is the following one.

Corollary 3. *Fix $n_1 > n_2 > n_3$ verifying* (5) *and assume that a_{n_1} and a_{n_3} have the same definite sign. If*

$$4a_{n_1}(t)a_{n_3}(t) > a_{n_2}^2(t)$$

for all t, then there exists $m_0 > 0$ such that if $|m| > m_0$ then (4) *has at most one positive limit cycle.*

The number m_0 is explicitly computable, for the proof follows easily from passing to the limit in condition (6).

We close the section by pointing out that Theorem 2 and its corollaries can be complemented with stability and exact multiplicity information by using the explicit behavior near the origin, as is done for instance in [2], [7].

3. The complete fourth-order equation

The aim of this section is to provide some sufficient conditions for limiting the number of limit cycles of the (4,3,2,1,0)-polynomial equation

$$u' = a_4(t)u^4 + a_3(t)u^3 + a_2(t)u^2 + a_1(t)u + a_0(t). \tag{9}$$

In [7, Theorem 5] it is proved that (9) with $a_4(t) \equiv 1$ may have an arbitrary number of T-periodic solutions. On the other hand, when $a_0 \equiv 0$, the main result of [1] implies that (9) has at most two positive T-periodic solutions if $a_4, a_3 \succ 0$, or $a_3, a_2 \succ 0$, or $a_4 \succ 0 \succ a_2$. Our results can be seen as a partial counterpart.

Our first result is very similar to some results in [4] for the fifth-order homogeneous equation.

Theorem 3. *If $a_2, a_4 \succ 0$ and $a_3^2 - \frac{8}{3}a_4a_2 \leqslant 0$, equation* (9) *has at most two limit cycles.*

P r o o f. The second derivative of the right-hand side of equation (9) is

$$12a_4(t)u^2 + 6a_3(t)u + 2a_2(t).$$

Viewing this as a second-order polynomial, the discriminant is $36a_3^2 - 96a_4a_2$. By hypothesis, this is negative, hence by Proposition 1 there exist at most two limit cycles. □

On the other hand, the next results are of a different nature.

Theorem 4. *Let us assume that* $a_0(t)a_4(t) > 0$ *for all* t. *If* $4\sqrt[4]{a_0a_4^3} + a_3 \geqslant 0$, *equation* (9) *has at most two positive limit cycles.*

P r o o f. We can assume without loss of generality that a_0, a_4 are both strictly positive functions. After the change $x = 1/u$, the equation is

$$x' = -xF\Big(t, \frac{1}{x}\Big),$$

where

$$F(t,x) = a_4(t)x^3 + a_3(t)x^2 + a_2(t)x + a_1(t) + \frac{a_0(t)}{x}.$$

By defining $g(t,x) := -xF(t,1/x)$, the second derivative is

$$g_{xx}(t,x) = \frac{-1}{x^3}F_{xx}\Big(t, \frac{1}{x}\Big).$$

Therefore, the proof is reduced to showing that $F_{xx}(t,x)$ is positive for $x > 0$. It turns out that

$$F_{xx}(t,x) = 6a_4(t)x + 2a_3(t) + \frac{2a_0(t)}{x^3}.$$

Since a_0, a_4 are strictly positive, the function $6a_4(t)x + 2a_0(t)/x^3$ attains its global minimum at $a_4(t)^{-1/4}a_0(t)^{1/4}$. Hence, for any $x > 0$,

$$F_{xx}(t,x) \geqslant 8a_0(t)^{1/4}a_4(t)^{3/4} + 2a_3(t) \geqslant 0$$

and the proof is done by a direct application of Proposition 1 with $J =]0, +\infty[$. □

Theorem 5. *Let us assume that* $a_4(t) > 0$ *for all* t. *Then equation* (9) *has at most three limit cycles verifying the condition*

$$u(t) > \frac{-a_3(t)}{4a_4(t)} \quad \textit{for all } t. \tag{10}$$

Analogously, equation (9) *has at most three limit cycles verifying the condition*

$$u(t) < \frac{-a_3(t)}{4a_4(t)} \quad \textit{for all } t. \tag{11}$$

P r o o f. First, we consider the case that the function $\varphi(t) := a_3(t)/4a_4(t)$ has a continuous derivative. By introducing the change $x = u + \varphi$ in eq. (9), the resulting equation is

$$x' = a_4(t)(x-\varphi)^4 + a_3(t)(x-\varphi)^3 + a_2(t)(x-\varphi)^2 + a_1(t)(x-\varphi) + a_0(t) + \varphi'(t). \tag{12}$$

The third derivative of the right-hand side of the equation is

$$g_{xxx}(t, x) = 24a_4(t)x.$$

Then $g_{xxx}(t, x) > 0$ if $x > 0$. By Proposition 1 with $J =]0, +\infty[$, there are at most three positive limit cycles of eq. (12). Going back to the original equation, it gives at most three limit cycles of eq. (9) verifying (10).

Now, let us prove the general case of a continuous function $\varphi(t)$ by a limiting argument. The set C_T^1 of T-periodic functions with continuous derivatives is dense in the set C_T of T-periodic and continuous functions. Hence, it is easy to prove that there exists a sequence $\{\varphi_n(t)\} \subset C_T^1$ converging uniformly to $\varphi(t)$ and such that $\varphi_n(t) > \varphi(t)$ for all n, t. Using the previous reasoning for each $\varphi_n(t)$ and passing to the limit we get the desired result.

In the same way it is proved that there are at most three limit cycles verifying (10). □

Of course, in this latter result additional T-periodic solutions crossing $-a_3(t) \times (4a_4(t))^{-1}$ may appear. This possibility is excluded by an additional assumption. At this moment we will need some basic facts about the concept of upper and lower solutions. See for instance [12] for more details.

Definition 1. A T-periodic function φ is called a strict lower (upper) solution of equation (4) if

$$\varphi'(t) < g(t, \varphi(t)) \quad (\varphi'(t) > g(t, \varphi(t)))$$

for all t.

Lemma 1. *A T-periodic solution does not intersect any eventual strict upper or lower solution.*

By using this notion, the following result is proved.

Corollary 4. *Let us assume that $-a_3(t)/4a_4(t)$ is an upper (lower) solution of eq. (9). Then there are at most 6 limit cycles.*

Proof. If $-a_3(t)/4a_4(t)$ is an upper (or lower) solution, by Lemma 1 a T-periodic solution can not cross it, so there are at most 3 of them above and at most 3 below. □

4. Applications to polynomial systems in the cylinder

In this section we study the maximum number of limit cycles of some polynomial vector fields in $\mathbb{R}^2$, the so-called Hilbert number. The first example is known in literature as a *rigid system* (see for instance [9], [7]).

The planar system

$$x' = -y + xP(x,y), \qquad y' = x + yP(x,y) \tag{13}$$

where $P(x,y)$ is a polynomial, is known in the related literature as a *rigid system* (see for instance [7], [8], [9] and their references). In polar coordinates, the system is rewritten as

$$r' = rP(r\cos\theta, r\sin\theta), \quad \theta' = 1.$$

If r is considered as a function of θ, we get the first order differential equation

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = rP(r\cos\theta, r\sin\theta), \tag{14}$$

and now it is easy to give applications of the results of Section 2 for suitable choices of the polynomial P.

In the recent paper [3], the authors study the number of non-contractible limit cycles of a family of systems in the cylinder $\mathbb{R} \times \mathbb{R}/[0, 2\pi]$ of the form

$$\begin{cases} \dfrac{\mathrm{d}\varrho}{\mathrm{d}t} = \tilde{\alpha}(\theta)\varrho + \tilde{\beta}(\theta)\varrho^{k+1} + \tilde{\gamma}(\theta)\varrho^{2k+1}, \\ \dfrac{\mathrm{d}\theta}{\mathrm{d}t} = b(\theta) + c(\theta)\varrho^k, \end{cases} \tag{15}$$

where $k \in \mathbb{Z}^+$ and all the above functions are continuous and 2π-periodic in θ. A contractible limit cycle is an isolated periodic orbit which can be deformed continuously to a point, otherwise it is called non-contractible. This type of systems arises as the polar expression of several types of planar polynomial systems. Of course, when $b(\theta) \equiv 1$ and $c(\theta) \equiv 0$ we have a rigid system. In general, if $b(\theta)$ does not vanish, a widely used change of variables due to Cherkas [6] transforms the system into a common Abel equation. We will consider the reciprocal case $b(\theta) \equiv 0$, $c(\theta) \equiv 1$. Let us consider the system

$$\begin{cases} \dfrac{\mathrm{d}\varrho}{\mathrm{d}t} = \tilde{\alpha}(\theta)\varrho + \tilde{\beta}(\theta)\varrho^{N_3} + \tilde{\gamma}(\theta)\varrho^{N_2} + \tilde{\delta}(\theta)\varrho^{N_1}, \\ \dfrac{\mathrm{d}\theta}{\mathrm{d}t} = \varrho^k, \end{cases} \tag{16}$$

where $N_1 > N_2 > N_3 > 0$ and $k > 0$. A limit cycle of this system is always non-contractible and as a function of θ it is a limit cycle of the first order equation

$$r' = \tilde{\beta}(\theta)r^{n_1} + \tilde{\gamma}(\theta)r^{n_2} + \tilde{\delta}(\theta)r^{n_3} + \tilde{\alpha}(\theta)r^{m},$$

where $n_i = N_i - k$ for $i = 1, 2, 3$ and $m = 1 - k$. Now, a direct application of Theorem 2 gives the following result.

Corollary 5. *Take N_1, N_2, N_3 such that $N_1 - 2N_2 + N_3 = 0$ and assume*

$$\tilde{\gamma}(\theta)^2(N_2 - 1)^2 - 4\tilde{\beta}(\theta)\tilde{\delta}(\theta)(N_1 - 1)(N_3 - 1) \leqslant 0.$$

Then the system (16) *has at most one limit cycle in the semiplane* $\{\varrho > 0\}$.

In particular, the result holds if $\tilde{\gamma}(\theta) \equiv 0$ and $\tilde{\beta}(\theta)$, $\tilde{\delta}(\theta)$ have opposite definite signs.

Similarly, the results contained in Section 3 can be applied to rigid systems when the polynomial $P(x, y)$ is a sum of homogeneous polynomials up to fourth degree, or to a suitable system in the cylinder. We omit further details.

As a last remark, let us comment that the study of non-contractible limit cycles of a general system on the cylinder

$$\begin{cases} \dfrac{\mathrm{d}\varrho}{\mathrm{d}t} = P(\theta, \varrho), \\ \dfrac{\mathrm{d}\theta}{\mathrm{d}t} = Q(\theta, \varrho) \end{cases}$$

where components of the field (P, Q) are periodic in θ and polynomial in ϱ, leads to the study of the existence and multiplicity of periodic solutions of a first order equation with a rational (quotient of two polynomials) nonlinearity. This is a difficult problem which deserves further investigation.

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Authors' address: N. Alkoumi, P. J. Torres, Departamento de Matemática Aplicada, Universidad de Granada, 18071 Granada, Spain, e-mail: ptorres@ugr.es.