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ORTHOCOMPLEMENTED DIFFERENCE LATTICES WITH FEW GENERATORS

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The algebraic theory of quantum logics overlaps in places with certain areas of cybernetics, notably with the field of artificial intelligence (see, e. g., [19, 20]). Recently an effort has been exercised to advance with logics that possess a symmetric difference ([13, 14]) – with so called orthocomplemented difference lattices (ODLs). This paper further contributes to this effort. In [13] the author constructs an ODL that is not set-representable. This example is quite elaborate. A main result of this paper somewhat economizes on this construction: There is an ODL with 3 generators that is not set-representable (and so the free ODL with 3 generators cannot be set-representable). The result is based on a specific technique of embedding orthomodular lattices into ODLs. The ODLs with 2 generators are always set-representable as we show by characterizing the free ODL with 2 generators – this ODL is $\mathrm{MO}_3 \times 2^4$.

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1. INTRODUCTION. BASIC NOTIONS

The notion of ODL has been introduced in [13] and further studied in [14] and [15]. The axiomatic setup of ODLs came into existence by taking an abstract form of set theoretic *symmetric difference* as a primitive operation (see Def. 1.1). As it turns out, an ODL is automatically orthomodular and therefore it forms an orthomodular lattice (an OML). This situates the variety of ODLs between OMLs and Boolean algebras. In a potential application, the ODLs add to the instances considered previously as quantum logics (see [4, 5, 6, 7, 10, 17, 18, 19] etc.). In this paper we find a minimal number of generators of an ODL that is not set-representable. This number is 3. We shall make use of the Greechie's paste job for OMLs together with certain techniques of embeddings of OMLs into ODLs. An acquitance with the theory of OMLs is assumed in places (see, e. g., [1, 12, 19] for basics on OMLs). For some specific properties of ODLs, let us refer the reader to [13].

Let us first recall the definition of an ODL.

Definition 1.1. Let $L = (X, \wedge, \vee, ^{\perp}, 0, 1, \triangle)$, where $(X, \wedge, \vee, ^{\perp}, 0, 1)$ is an orthocomplemented lattice (abbr., an OCL) and $\triangle : X^2 \to X$ is a binary operation.

Then L is said to be an *orthocomplemented difference lattice* (abbr., an ODL) if the following identities hold in L:

(D$_1$) $x \triangle (y \triangle z) = (x \triangle y) \triangle z$,

(D$_2$) $x \triangle 1 = x^\perp$, $1 \triangle x = x^\perp$,

(D$_3$) $x \triangle y \leq x \vee y$.

Obviously, the class of all ODLs forms a variety. We will denote it by $\mathcal{ODL}$.

Let $L = (X, \wedge, \vee, {}^\perp, 0, 1, \triangle)$ be an ODL. Then the OCL $(X, \wedge, \vee, {}^\perp, 0, 1)$ will be denoted by L_{supp} and called the *support* of L. Occasionally, the ODL L will be identified with the couple $(L_{\mathrm{supp}}, \triangle)$. Let us list basic properties of ODLs as we shall use them in the sequel.

Proposition 1.2. Let L be an ODL and let us suppose that $x, y \in L$. Then the following statements hold true:

(1) $x \triangle 0 = x$, $0 \triangle x = x$,

(2) $x \triangle x = 0$,

(3) $x \triangle y = y \triangle x$,

(4) $x \triangle y^\perp = x^\perp \triangle y = (x \triangle y)^\perp$,

(5) $x^\perp \triangle y^\perp = x \triangle y$,

(6) $x \triangle y = 0 \Leftrightarrow x = y$,

(7) $(x \wedge y^\perp) \vee (y \wedge x^\perp) \leq x \triangle y \leq (x \vee y) \wedge (x \wedge y)^\perp$.

Proof. Let us first observe that the property (D$_2$) yields $1 \triangle 1 = 1^\perp = 0$. Let us verify the properties (1)–(7).

(1) $x \triangle 0 = x \triangle (1 \triangle 1) = (x \triangle 1) \triangle 1 = x^\perp \triangle 1 = (x^\perp)^\perp = x$. Further, $0 \triangle x = (1 \triangle 1) \triangle x = 1 \triangle (1 \triangle x) = 1 \triangle x^\perp = (x^\perp)^\perp = x$.

(2) Let us first show that $x^\perp \triangle x^\perp = x \triangle x$. We consecutively obtain $x^\perp \triangle x^\perp = (x \triangle 1) \triangle (1 \triangle x) = (x \triangle (1 \triangle 1)) \triangle x = (x \triangle 0) \triangle x = x \triangle x$. Moreover, we have $x \triangle x \leq x$ as well as $x \triangle x = x^\perp \triangle x^\perp \leq x^\perp$. This implies that $x \triangle x \leq x \wedge x^\perp = 0$.

(3) $x \triangle y = (x \triangle y) \triangle 0 = (x \triangle y) \triangle [(y \triangle x) \triangle (y \triangle x)] = x \triangle (y \triangle y) \triangle x \triangle (y \triangle x) = x \triangle 0 \triangle x \triangle (y \triangle x) = x \triangle x \triangle (y \triangle x) = 0 \triangle (y \triangle x) = y \triangle x$.

(4) $x \triangle y^\perp = x \triangle (y \triangle 1) = (x \triangle y) \triangle 1 = (x \triangle y)^\perp$. The equality $x^\perp \triangle y = (x \triangle y)^\perp$ follows from $x \triangle y^\perp = (x \triangle y)^\perp$ by applying the equality (3).

(5) Using (4) we obtain $x^\perp \triangle y^\perp = (x^\perp \triangle y)^\perp = (x \triangle y)^{\perp\perp} = x \triangle y$.

(6) If $x = y$, then $x \triangle y = 0$ by the condition (2). Conversely, suppose that $x \triangle y = 0$. Then $x = x \triangle 0 = x \triangle (y \triangle y) = (x \triangle y) \triangle y = 0 \triangle y = y$.

(7) The property (D$_3$) together with the properties (4), (5) imply that $x \triangle y \leq x \vee y$, $x \triangle y \leq x^\perp \vee y^\perp = (x \wedge y)^\perp$, $x \wedge y^\perp \leq x \triangle y$, $x^\perp \wedge y \leq x \triangle y$. $\square$

Theorem 1.3. Let L be an ODL. Then its support L_{supp} is an OML.

Proof. Suppose that $x, y \in L$, $x \leq y$, $y \wedge x^{\perp} = 0$. Let us prove that $x = y$. Since $x \leq y$, we conclude that $(x \wedge y^{\perp}) \vee (y \wedge x^{\perp}) = y \wedge x^{\perp} = 0$ and $(x \vee y) \wedge (x \wedge y)^{\perp} = y \wedge x^{\perp} = 0$. By Prop. 1.2, (6), (7) we see that $x \triangle y = 0$ and therefore $x = y$. □

In view of the above proposition, all notions of OMLs can be referred to in ODLs, too. In particular, we shall say that two elements x, y in an ODL L *commute* (in symbols, $x \, C \, y$) if they commute in L_{supp}. Similarly, we shall denote by $C(L)$ the set of all elements of L that commute with all elements of L. Let us call $C(L)$ the *centre* of L. It can be easily shown that $C(L)$ is a subalgebra of L ([13]).

Let us suppose that B is a Boolean algebra. Let us denote by Δ_B the standard symmetric difference on B. Thus, if $x, y \in B$ then $x \ \Delta_B \ y = (x \wedge y^{\perp}) \vee (y \wedge x^{\perp}) = (x \vee y) \wedge (x \wedge y)^{\perp}$.

Proposition 1.4. Let L be an ODL. Let $x, y \in L$ with $x \, C \, y$. Then $x \triangle y = (x \wedge y^{\perp}) \vee (y \wedge x^{\perp}) = (x \vee y) \wedge (x \wedge y)^{\perp}$.
A consequence: If B is a Boolean sub-algebra of L and $x_1, \ldots, x_n \in B$, then $x_1 \triangle \cdots \triangle x_n = x_1 \Delta_B \cdots \Delta_B x_n$.

Proof. According to Prop. 1.2, (7), we have the inequalities $(x \wedge y^{\perp}) \vee (y \wedge x^{\perp}) \leq x \triangle y \leq (x \vee y) \wedge (x \wedge y)^{\perp}$. Since the elements x, y commute, the left-hand side of the previous inequality coincides with the right-hand side and therefore $x \triangle y = (x \wedge y^{\perp}) \vee (y \wedge x^{\perp}) = (x \vee y) \wedge (x \wedge y)^{\perp}$. □

Let us exhibit some simple examples of ODLs. Firstly, each Boolean algebra can be understood as an ODL which the following proposition shows.

Proposition 1.5. Let B be a BA. Then there exists exactly one mapping $\triangle : B \times B \to B$ which fulfils the conditions (D_1), (D_2) and (D_3) of Def. 1.1.

Proof. To prove the existence, take for the operation $\triangle$ the standard symmetric difference Δ_B in B. The properties (D_1), (D_2) and (D_3) of Def. 1.1 are then obviously fulfilled.

Let us prove the uniqueness of $\triangle$. Let $\triangle_1 : B \times B \to B$ be a mapping that fulfils the conditions (D_1), (D_2) and (D_3). So the couple $(B, \triangle_1)$ is an ODL. If $x, y \in B$, then $x \, C \, y$, and therefore $x \triangle_1 y = x \ \Delta_B \ y = x \triangle y$ (Prop. 1.4). □

Example 1.6. Let MO_3 be the OML obtained as the horizontal sum of three 4-element BA's (see, e. g., [12]). Write $MO_3 = \{0, 1, x, x^{\perp}, y, y^{\perp}, z, z^{\perp}\}$. Then one can easily show that there is exactly one mapping $\triangle : MO_3 \times MO_3 \to MO_3$ such that $x \triangle y = z$ and $(MO_3, \triangle)$ is an ODL. The ODL obtained in this way will again be denoted by MO_3. Obviously, the ODL MO_3 is generated by the elements x, y. (It might be noted that MO_k can be viewed as an ODL exactly when $k = 2^n - 1$, [13]. We shall only use MO_3 in this paper.)

Proposition 1.7. Let L be an ODL and let $x, y \in L$. Then

(a) $x \vee (x \triangle y) = x \vee y$,

(b) $x \wedge (x \triangle y) = x \wedge y^{\perp}$.

Proof. Before verifying the equalities, recall the convention of the preference of $\triangle$ over the operations $\wedge$ and $\vee$ (thus, for instance, $x \vee y \triangle z$ means $x \vee (y \triangle z)$ etc.).
(a) The inequality $x \vee x \triangle y \leq x \vee y$ is obvious. We have to show that $x \vee y \leq x \vee x \triangle y$. But $x \leq x \vee x \triangle y$ and therefore we need to check $y \leq x \vee x \triangle y$. According to ($D_3$), we have $x \vee x \triangle y \geq x \triangle (x \triangle y) = y$. (It is worthwhile observing that this equality can be viewed as a strengthening of the condition (D_3) from the definition of ODL's.)
(b) The equality follows from (a) via the following calculation: $x \wedge x \triangle y = (x \wedge x \triangle y)^{\perp\perp} = (x^{\perp} \vee x^{\perp} \triangle y)^{\perp} = (x^{\perp} \vee y)^{\perp} = x \wedge y^{\perp}$. □

Proposition 1.8. Let L be an ODL and let $x, y \in L$. Then $x \perp y \Leftrightarrow x \triangle y = x \vee y$ (where $x \perp y$ stands for $x \leq y^{\perp}$).

Proof. Let us suppose that $x \perp y$. Then both sides in Prop. 1.2, (7) equal to $x \vee y$, and therefore $x \triangle y = x \vee y$. Conversely, if $x \triangle y = x \vee y$, then Prop. 1.7, (b) implies that $x \wedge (x \vee y) = x \wedge y^{\perp}$. Thus, $x = x \wedge y^{\perp}$, and this gives us $x \leq y^{\perp}$. □

We shall need the following simple fact on OMLs.

Lemma 1.9. Let L be an OML. Let $x, y, x_1, x_2 \in L$ and let $y = x_1 \vee x_2$, $x_1 \leq x$, $x_2 \leq x^{\perp}$. Then $x\ C\ y$ and $x_1 = y \wedge x$, $x_2 = y \wedge x^{\perp}$.

Proof. Since $x_1 \leq x$ and $x_2 \leq x^{\perp}$, we see that $x_1 \leq x \leq x_2^{\perp}$. Thus, the elements x_1, x_2, x are mutually commutative. As known, $x\ C\ (x_1 \vee x_2)$ and therefore $x\ C\ y$. Moreover,

$y \wedge x = (x_1 \vee x_2) \wedge x = (x_1 \wedge x) \vee (x_2 \wedge x) = x_1 \vee 0 = x_1$, and
$y \wedge x^{\perp} = (x_1 \vee x_2) \wedge x^{\perp} = (x_1 \wedge x^{\perp}) \vee (x_2 \wedge x^{\perp}) = 0 \vee x_2 = x_2$. □

Proposition 1.10. Let L be an ODL. Let $x, y, z \in L$ with $x\ C\ y$ and $x\ C\ z$. Then $x\ C\ (y \triangle z)$ and $x \wedge (y \triangle z) = (x \wedge y) \triangle (x \wedge z)$.

Proof. The commutativity of the pair $x\ C\ y$ and $x\ C\ z$ yields the equations $y = (y \wedge x) \vee (y \wedge x^{\perp})$, $z = (z \wedge x) \vee (z \wedge x^{\perp})$. Since $(y \wedge x) \perp (y \wedge x^{\perp})$ and $(z \wedge x) \perp (z \wedge x^{\perp})$, we see by Prop. 1.8 that $y = (y \wedge x) \triangle (y \wedge x^{\perp})$ and $z = (z \wedge x) \triangle (z \wedge x^{\perp})$. But we also have $y \triangle z = [(y \wedge x) \triangle (y \wedge x^{\perp})] \triangle [(z \wedge x) \triangle (z \wedge x^{\perp})] = [(y \wedge x) \triangle (z \wedge x)] \triangle [(y \wedge x^{\perp}) \triangle (z \wedge x^{\perp})]$. Let us write $x_1 = (y \wedge x) \triangle (z \wedge x)$, $x_2 = (y \wedge x^{\perp}) \triangle (z \wedge x^{\perp})$. Then $x_1 \leq (y \wedge x) \vee (z \wedge x) \leq x$. Analogously, $x_2 \leq x^{\perp}$. This implies that $x_1 \perp x_2$. By Prop. 1.8, $y \triangle z = x_1 \vee x_2$. The proof is completed by using Lemma 1.9. □

Let us take up the intervals in ODLs. We will need them for the decomposition property with respect to a central element. Consider first the situation in OMLs.

Let K be an OML and let $a \in K$. Let us write $[0,a]_K = \{x \in K;\ x \le a\}$. As known, the interval $[0,a]$ constitutes an OML. We will denote it by K^a. Let us shortly recall the construction of K^a (see, for example, [12], p. 20): If $x, y \in [0,a]$, then $x \wedge y \in [0,a]$ and $x \vee y \in [0,a]$. The element 0, resp. a, is a least, resp. a greatest, element of K^a. The orthocomplement of x in K^a, $x^{\perp_a}$, is defined by setting $x^{\perp_a} = x^{\perp_K} \wedge a$. It can be easily seen that $K^a = ([0,a], \wedge, \vee, ^{\perp_a}, 0, a)$ is an OML.

Let L be an ODL and let $a \in L$. If $x, y \in [0,a]$ then $x \triangle y \in [0,a]$. Let us consider the algebra $L^a = ([0,a], \wedge, \vee, ^{\perp_a}, 0, a, \triangle) = ((L_{\text{supp}})^a, \triangle)$.

Proposition 1.11. Let L be an ODL and let $a \in L$. Then the algebra L^a is again an ODL. Moreover, if $a \in C(L)$, then the mapping $\pi_a : L \to [0,a]$ defined by putting $\pi_a(x) = x \wedge a$ is a surjective homomorphism of L onto L^a.

P r o o f. In order for L^a to be an ODL, it is sufficient to check that the conditions (D_1), (D_2) and (D_3) of Def. 1.1 hold in L^a. The conditions (D_1) and (D_3) can be easily verified . It remains to check the condition (D_2). For that, suppose $x \in [0,a]$. Then $x \triangle_{L^a} 1_{L^a} = x \triangle a$. Since $x \le a$, we have xCa and Prop. 1.4 implies that $x \triangle a = a \wedge x^{\perp} = x^{\perp_a}$. The equality $1_{L^a} \triangle_{L^a} x = x^{\perp_a}$ follows from the commutativity of $\triangle$.

Suppose further that $a \in C(L)$. Then the mapping π_a is an OML-homomorphism $L_{\text{supp}} \to (L^a)_{\text{supp}}$ (see [12], p. 20). It remains to show that the mapping π_a preserves the operation $\triangle$. Suppose that $x, y \in L$. Then by Prop. 1.10 we consecutively obtain $\pi_a(x \triangle_L y) = (x \triangle_L y) \wedge a = (x \wedge a) \triangle_L (y \wedge a) = \pi_a(x) \triangle_{L^a} \pi_a(y)$. This completes the proof. $\square$

In the final auxiliary result, let us show that an ODL can be decomposed with the help of a central element in the way analogous to the situation known in OMLs.

Proposition 1.12. Suppose that L is an ODL and $a \in C(L)$. Then the mapping $i : L \to [0,a] \times [0,a^{\perp}]$ defined by putting $i(x) = (\pi_a(x), \pi_{a^{\perp}}(x))$ is an isomorphism of L onto $L^a \times L^{a^{\perp}}$.

P r o o f. The mapping i is an isomorphism between the OMLs L_{supp} and $(L^a)_{\text{supp}} \times (L^{a^{\perp}})_{\text{supp}}$ (see again [12], p. 20). Since both the mappings $\pi_a, \pi_{a^{\perp}}$ preserve the operation $\triangle$, so does the mapping i and the proof is done. $\square$

In the conclusion of preliminaries, let us recall an important class of ODLs – the ODLs that are *set-representable.* They form a variety ([13]) and represent some 'nearly Boolean' ODLs. Though the name itself suggests their definition, let us recall it in more formal terms. Let X be a set and let $\mathcal{D}$ be a family of subsets of X such that

(1) $X \in \mathcal{D}$,

(2) the family $\mathcal{D}$ forms a lattice with respect to the inclusion relation, and

(3) $\mathcal{D}$ is closed under the formation of the set symmetric difference.

Obviously, $\mathcal{D}$ constitutes an ODL. Let us call it *concrete*. If L is an ODL that is isomorphic with a concrete one, then L is said to be set-representable.

2. EACH ODL WITH TWO GENERATORS IS SET-REPRESENTABLE (A CHARACTERIZATION OF THE FREE ODL WITH TWO GENERATORS)

In analysing structural properties of varieties, free objects play a fundamental role ([3]). We show in this section that the free ODL on 2 generators coincides with $2^4 \times \mathrm{MO}_3$ (where, as usual, 2^4 stands for the Boolean algebra with 4 atoms). Since the ODL $2^4 \times \mathrm{MO}_3$ is set-representable, and since a homomorphic image of a set-representable ODL is again set-representable ([13]), we see that any ODL with two generators is set-representable.

In order to characterize the free ODL with 2 generators, we shall need two auxiliary results. For the sake of a transparent formulation of these results, let us assume that the generators of the Boolean algebra 2^4 are elements x_1, y_1 and the generators of the ODL MO_3 are elements x_2, y_2 (compare with Example 1.6 – we have renamed x, y of Example 1.6 with x_2, y_2).

Proposition 2.1. Let L be an ODL and let $a, b \in L$. Let us suppose that $a \wedge b = a \wedge b^\perp = a^\perp \wedge b = a^\perp \wedge b^\perp = 0$. Then there exists a homomorphism $h : \mathrm{MO}_3 \to L$ with $h(x_2) = a$, $h(y_2) = b$.

Proof. Let us denote $z_2 = x_2 \triangle y_2$ in MO_3. Let us set $h(0_{\mathrm{MO}_3}) = 0_L$, $h(1_{\mathrm{MO}_3}) = 1_L$, $h(x_2) = a$, $h(x_2^\perp) = a^\perp$, $h(y_2) = b$, $h(y_2^\perp) = b^\perp$, $h(z_2) = a \triangle b$ and $h(z_2^\perp) = a \triangle b^\perp$. The definition of h implies that h preserves the least and greatest element. Also, the operations $^\perp$ and $\triangle$ are obviously preserved. Let us check that h preserves the operation $\wedge$, too. Suppose therefore that $x, y \in \mathrm{MO}_3$ and let us ask whether or not we have $h(x \wedge y) = h(x) \wedge h(y)$. If x, y commute in MO_3, this equality is obvious. Suppose that x, y do not commute. Without any loss of generality, it is sufficient to consider the images of the elements $x_2 \wedge y_2$ and $x_2 \wedge z_2$. We firstly see that $h(x_2 \wedge y_2) = h(x_2 \wedge z_2) = h(0_{\mathrm{MO}_3}) = 0_L$, and further we have $h(x_2) \wedge h(y_2) = a \wedge b = 0_L$ as well as, by Prop. 1.7, $h(x_2) \wedge h(z_2) = a \wedge (a \triangle b) = a \wedge b^\perp = 0_L$. The preservation of the operation $\vee$ is a simple consequence of de Morgan's law. The proof is complete. □

Proposition 2.2. Let L be an ODL with two generators s, t. Let us set $a = (s \wedge t) \vee (s \wedge t^\perp) \vee (s^\perp \wedge t) \vee (s^\perp \wedge t^\perp)$. Then $a \in C(L)$ and there exist homomorphisms $g : 2^4 \to L^a$, $h : \mathrm{MO}_3 \to L^{a^\perp}$ such that

$$\begin{aligned} g(x_1) &= \pi_a(s), g(y_1) = \pi_a(t), \\ h(x_2) &= \pi_{a^\perp}(s), h(y_2) = \pi_{a^\perp}(t). \end{aligned}$$

Proof. It is obvious that the element $s \wedge t$ commutes with both s and t. Since s, t generate the ODL L, we see that $s \wedge t \in C(L)$. Analogously, all the elements $s \wedge t^\perp$, $s^\perp \wedge t$ and $s^\perp \wedge t^\perp$ belong to $C(L)$. As a consequence, $(s \wedge t) \vee (s \wedge t^\perp) \vee (s^\perp \wedge t) \vee (s^\perp \wedge t^\perp) = a \in C(L)$.

Let us go on with the proof. Since the elements s, t generate L and since π_a is a surjective homomorphism onto L^a, it follows that the elements $\pi_a(s)$, $\pi_a(t)$ generate the ODL L^a. Making use of the Foulis–Holland theorem ([12]) we infer that

$$\begin{aligned}\pi_a(s) &= s \wedge a = (s \wedge t) \vee (s \wedge t^\perp),\\ \pi_a(t) &= t \wedge a = (s \wedge t) \vee (s^\perp \wedge t).\end{aligned}$$

As a consequence of the above identities we see that the elements $\pi_a(s), \pi_a(t)$ commute and therefore L^a is a Boolean algebra. Since 2^4 is a free Boolean algebra on the set $\{x_1, y_1\}$, the existence of the homorphism g is evident.

Let us take up the construction of the morphism h. It is sufficient to check (Prop. 2.1) that
$\pi_{a^\perp}(s) \wedge \pi_{a^\perp}(t) = \pi_{a^\perp}(s) \wedge (\pi_{a^\perp}(t))^\perp = (\pi_{a^\perp}(s))^\perp \wedge \pi_{a^\perp}(t) = (\pi_{a^\perp}(s))^\perp \wedge (\pi_{a^\perp}(t))^\perp = 0$.
Let us prove that $\pi_{a^\perp}(s) \wedge (\pi_{a^\perp}(t))^\perp = 0$, the other equalities can be derived analogously. Since $\pi_{a^\perp} : L \to L^{a^\perp}$ preserves the operation $^\perp$, we see that $(\pi_{a^\perp}(t))^\perp = (\pi_{a^\perp}(t))^{\perp_{L^{a^\perp}}} = \pi_{a^\perp}(t^{\perp_L})$. As a consequence we obtain
$\pi_{a^\perp}(s) \wedge (\pi_{a^\perp}(t))^\perp = \pi_{a^\perp}(s) \wedge \pi_{a^\perp}(t^{\perp_L}) = \pi_{a^\perp}(s \wedge t^{\perp_L}) = (s \wedge t^\perp) \wedge a^\perp = (s \wedge t^\perp) \wedge (s^\perp \vee t^\perp) \wedge (s^\perp \vee t) \wedge (s \vee t^\perp) \wedge (s \vee t) = (s \wedge t^\perp) \wedge (s^\perp \vee t) = (s \wedge t^\perp) \wedge (s \wedge t^\perp)^\perp = 0$. □

Theorem 2.3. Suppose that the elements x_1, y_1 are generators of the free Boolean algebra 2^4 and suppose that the elements x_2, y_2 are generators of the ODL MO_3. Then the product $2^4 \times \mathrm{MO}_3$ is a free ODL on the set $\{x, y\}$, where $x = (x_1, x_2)$ and $y = (y_1, y_2)$.

Proof. Write $F = 2^4 \times \mathrm{MO}_3$. Let us first show that the set $\{x, y\}$ generates F. Let us denote by S the subalgebra of F generated by $\{x, y\}$. Suppose that $a \in F$. Then $a = (a_1, a_2) = (a_1, 0) \vee (0, a_2)$, where $a_1 \in 2^4$ and $a_2 \in \mathrm{MO}_3$. We therefore have to show that all elements of the form $(a_1, 0), (0, a_2)$ lie in S. Since $x, x^\perp, y, y^\perp$ are elements of S, so are the elements $x \wedge y$, $x \wedge y^\perp$, $x^\perp \wedge y$, $x^\perp \wedge y^\perp$. Taking into account that $x_2 \wedge y_2 = x_2 \wedge y_2^\perp = x_2^\perp \wedge y_2 = x_2^\perp \wedge y_2^\perp = 0$, we infer that all elements $(x_1 \wedge y_1, 0), (x_1 \wedge y_1^\perp, 0), (x_1^\perp \wedge y_1, 0), (x_1^\perp \wedge y_1^\perp, 0)$ belong to S. But $x_1 \wedge y_1$, $x_1 \wedge y_1^\perp$, $x_1^\perp \wedge y_1$, $x_1^\perp \wedge y_1^\perp$ are precisely all atoms of the Boolean algebra 2^4. This implies that $(a_1, 0) \in S$. As a consequence, $(a_1, 1) \in S$. Further, observing $(0, x_2) = (x_1^\perp, 1) \wedge (x_1, x_2)$, we see that $(0, x_2) \in S$. Analogously, $(0, y_2) \in S$ and, also, $(0, z_2) = (0, x_2) \triangle (0, y_2) \in S$. We have shown that $S = F$.

In order to show that F is free, let K be an ODL and let $f_0 : \{x, y\} \to K$ be a mapping. We have to show that f_0 can be extended as a homomorphism $f : F \to K$. Write $s = f_0(x)$, $t = f_0(y)$ and suppose that L is the subalgebra of K generated by the set $\{s, t\}$. Set $a = (s \wedge t) \vee (s \wedge t^\perp) \vee (s^\perp \wedge t) \vee (s^\perp \wedge t^\perp)$. By Prop. 2.2 we have that $a \in C(L)$ and, moreover, there exist homomorphisms $g : 2^4 \to L^a$, $h : \mathrm{MO}_3 \to L^{a^\perp}$ such that

$g(x_1) = \pi_a(s), g(y_1) = \pi_a(t)$,
$h(x_2) = \pi_{a^\perp}(s), h(y_2) = \pi_{a^\perp}(t)$.

Let $i : L \to L^a \times L^{a^\perp}$ be the isomorphism of Prop. 1.12. Let us consider the mapping

$g \times h$ defined by setting $(g \times h)(p, q) = (g(p), h(q))$, where $(p, q) \in 2^4 \times \mathrm{MO}_3$. Obviously, $g \times h : 2^4 \times \mathrm{MO}_3 \to L^a \times L^{a^\perp}$ is a homomorphism. Let us set $f = (g \times h) \circ i^{-1}$, i. e. for any $(p, q) \in 2^4 \times \mathrm{MO}_3$ let us set $f(p, q) = i^{-1}(g(p), h(q))$. Then $f : 2^4 \times \mathrm{MO}_3 \to L$ is a homomorphism and since L is a subalgebra of K, we see that $f : 2^4 \times \mathrm{MO}_3 \to K$ is a homomorphism, too. Moreover, $f(x) = i^{-1}(g(x_1), h(x_2)) = i^{-1}(\pi_a(s), \pi_{a^\perp}(s)) = i^{-1}(i(s)) = s = f_0(x)$. Analogously, $f(y) = f_0(y)$. We have verified that f extends f_0 and the proof is complete. □

It should be noted in the conclusion of this paragraph that the result of Thm. 2.3 has already been obtained in [11] (a student thesis under the supervision of the authors of this paper). However, the methods used here differ considerably from those of [11] and allow us to prove the result in a simpler way.

3. THERE IS AN ODL WITH THREE GENERATORS THAT IS NOT SET-REPRESENTABLE (SO THE FREE ODL WITH THREE GENERATORS IS NOT SET-REPRESENTABLE)

In this section we develop an embedding technique of OMLs into ODLs. This will allow us to prove the assertion stated in the heading of this paragraph. Let us start with a few conventions.

Let N stand for the set of all natural numbers, $N = \{0, 1, 2, \ldots\}$. Let $\mathcal{B}$ be the Boolean algebra of all finite and cofinite subsets of N. Let us denote by Δ the standard set-theoretic difference on $\mathcal{B}$. In considering countable ODLs we can visualise, with the help of $\mathcal{B}$, the operation $\triangle$ set-theoretically. The following proposition formalizes it.

Proposition 3.1. If $L = (X, \wedge, \vee, ^\perp, 0, 1, \triangle)$ is at most countable ODL, then the algebra $(X, \triangle, 0, 1)$ can be embedded into the algebra $(\mathcal{B}, \Delta, 0_{\mathcal{B}}, 1_{\mathcal{B}})$, where $0_{\mathcal{B}} = \emptyset$, $1_{\mathcal{B}} = N$.

P r o o f. Let us choose a Boolean algebra B' such that B' is a sub-algebra of $\mathcal{B}$ and $\mathrm{card}(B') = \mathrm{card}(X)$. Obviously, the algebra $(B', \Delta, 0_{\mathcal{B}}, 1_{\mathcal{B}})$ is a sub-algebra of the algebra $(\mathcal{B}, \Delta, 0_{\mathcal{B}}, 1_{\mathcal{B}})$. Moreover, the study of ultrafilters in $(X, \triangle, 0, 1)$ made in [13], Prop. 7.7 implies that the algebras $(B', \Delta, 0_{\mathcal{B}}, 1_{\mathcal{B}})$ and $(X, \triangle, 0, 1)$ are isomorphic. □

The above result will be frequently used in the sequel. First, let us agree on a convention. In order to avoid rather inconvenient referring to finite and cofinite subsets of N, let us make use of the standard coding of finite subsets of N by natural numbers. If A is finite, $A \subset N$, let us assign to A the number $k(A)$ as follows: $k(\emptyset) = 0$, $k(\{a_1, \ldots, a_n\}) = 2^{a_1} + \ldots + 2^{a_n}$. Thus, so defined k is nothing but the famous *Gödel's coding*. As known, the assignment k is injective. Let us choose some new elements $\bar{0}, \bar{1}, \bar{2}, \ldots$ (intuitively, these elements play a role of certain copies of $0, 1, 2, \ldots$). For any cofinite B, $B = N \setminus A$ with A finite, let us set $k(B) = \overline{k(A)}$. Denoting by D_∞ the set $\{0, \bar{0}, 1, \bar{1}, 2, \bar{2}, \ldots\}$, we see that k is a bijection of $\mathcal{B}$ onto D_∞. Further, let us introduce an operation, $\oplus$, on the set D_∞ by setting $x \oplus y =$

$k(k^{-1}(x)\ \Delta\ k^{-1}(y))$. The following two assertions bring out the properties of the operation $\oplus$. The proofs are not difficult and we omit them.

Lemma 3.2. The mapping k is an isomorphism of the algebra $(\mathcal{B}, \Delta, 0_{\mathcal{B}}, 1_{\mathcal{B}})$ onto the algebra $(D_\infty, \oplus, 0, \bar{0})$.

Lemma 3.3. If $n \in N$, then the set $\{0, \bar{0}, 1, \bar{1}, \ldots, 2^n - 1, \overline{2^n - 1}\}$ is a subalgebra of the algebra $(D_\infty, \oplus, 0, \bar{0})$.

We shall utilize the following mapping $^\star : D_\infty \to D_\infty$. If $n \in N$, we set $n^\star = \bar{n}$ and $(\bar{n})^\star = n$.

Lemma 3.4. If $x \in D_\infty$, then $x^\star = x \oplus \bar{0} = \bar{0} \oplus x$.

Proof. We have $x \oplus \bar{0} = k(\overline{k^{-1}(x)\ \Delta\ k^{-1}}(\bar{0})) = k(k^{-1}(x)\ \Delta\ N) = k(N \setminus k^{-1}(x))$. If $x = n$, then $k(N \setminus k^{-1}(x)) = \overline{k(k^{-1}(x))} = \bar{x} = x^\star$. Alternatively, suppose that $x = \bar{n}$. Take a set A such that $k(A) = n$. We then see that $x = \bar{n} = \overline{k(A)} = k(N \setminus A)$, i.e. $k^{-1}(x) = N \setminus A$. Consequently, $k(N \setminus k^{-1}(x)) = k(N \setminus (N \setminus A)) = k(A) = n = x^\star$.

Since $\oplus$ is commutative, we infer that $\bar{0} \oplus x = x^\star$ and this completes the proof. □

The objective of the following consideration is to show that mappings into D_∞ allow us to embed certain OMLs into ODLs. Let us first introduce a few new notions.

Definition 3.5. Let K be an OML and let L be an ODL. Let us agree to write $K \ll L$ if K is a sub-OML of L_{supp} and $a \triangle b \in K$ for any $a, b \in K$ with $a \vee b < 1$.

Lemma 3.6.

(1) Let K be an OML and let L be an ODL. Let us suppose that there is an OML M such that L_{supp} is a horizontal sum of OMLs K and M. Then $K \ll L$.

(2) Let L be an ODL and let K be a sub-ODL of L. Then $K_{\mathrm{supp}} \ll L$.

Proof. It is routine and we omit it. □

Let K be an OML and let $\mathrm{Bl}(K)$ be the set of all blocks (= the set of all maximal Boolean subalgebras) of K. Let $\mathrm{At}(K)$ stand for the set of all atoms of K. Let us denote by $\mathcal{OML}_8$ the class of all OMLs K such that $\mathrm{card}(B) = 8$ for any $B \in \mathrm{Bl}(K)$. So, for instance, each horizontal sum of 8-element Boolean algebras belongs to $\mathcal{OML}_8$ and so does the projection lattice $L(R^3)$. More involved examples will be encountered in the sequel.

Let $K \in \mathcal{OML}_8$ and let $p, q \in \mathrm{At}(K)$. Let us write $p \approx q$ provided $p \neq q$ and $p\ C\ q$. Further, let us write $p \sim q$ if there exists an $r \in \mathrm{At}(K)$ such that $p\ C\ r$ and $r\ C\ q$ and, moreover, p does not commute with q.

Lemma 3.7. Let $K \in \mathcal{OML}_8$ and let $p, q \in \mathrm{At}(K)$ with $p \sim q$. Let $r \in \mathrm{At}(K)$ such that $r\ C\ p$ and $r\ C\ q$. Then $r = p^\perp \wedge q^\perp$.

P r o o f. Since p does not commute with q, we have $p \neq q$. Further, the element $r \in \mathrm{At}(K)$ with $r\ C\ p$ and $r\ C\ q$ must be different from both p and q. Since $p\ C\ r$ and $p \neq r$ we see that $r \leq p^\perp$. For an analogous reason, $r \leq q^\perp$. As a result, $r \leq p^\perp \wedge q^\perp < p^\perp$. We infer that $r = p^\perp \wedge q^\perp$ and this completes the proof. □

Definition 3.8. Let $K \in \mathcal{OML}_8$ be finite and let $l : \mathrm{At}(K) \to D_\infty$ be a mapping. We say that l is a *labelling* of the atoms in K if

(1) for any pair $a, b \in \mathrm{At}(K)$ with $a \neq b$ we have $l(a) \neq l(b)$, $l(a) \neq l(b)^\star$,

(2) for any $B \in \mathrm{Bl}(K)$ such that $\mathrm{At}(B) = \{a, b, c\}$ we have $l(a) \oplus l(b) \oplus l(c) = \bar{0}$,

(2) for any pair $a, b \in \mathrm{At}(K)$ with $a \sim b$ there is an $s \in \mathrm{At}(K)$ such that $s \approx a^\perp \wedge b^\perp$ and $l(s) = l(a) \oplus l(b)$.

Before justifying this definition in the next theorem, let us explicitely formulate the following simple fact.

Lemma 3.9. Suppose that l is a labelling of the atoms of K and suppose that $a \in \mathrm{At}(K)$. Then $l(a) \notin \{0, \bar{0}\}$.

P r o o f. Let B be a block in K with $a \in B$. Let $\mathrm{At}(B) = \{a, b, c\}$. Then $l(a) \oplus l(b) \oplus l(c) = \bar{0}$. If $l(a) = 0$, then $l(b) \oplus l(c) = \bar{0}$. This means that $l(b) = l(c)^\star$ which is absurd. If $l(a) = \bar{0}$, then $l(b) \oplus l(c) = 0$. This means that $l(b) = l(c)$ which is again absurd. □

Theorem 3.10. Let $K \in \mathcal{OML}_8$ be finite. Then the following two statements are equivalent:

(1) There is a finite ODL, L, such that $K \ll L$,

(2) there is a labelling of the atoms of K.

P r o o f. Suppose first that there is a finite ODL, L, such that $K \ll L$. Then there is an embedding, f, of the algebra $(L, \triangle, 0, 1)$ into the algebra $(D_\infty, \oplus, 0, \bar{0})$ (Prop. 3.1 and Lemma 3.2). Let l be the restriction of f to the set $\mathrm{At}(K)$. In order to show that l is a labelling, we are to verify three conditions.

(1) Suppose that $a, b \in \mathrm{At}(K)$ with $a \neq b$. Then $a \neq b^\perp$ and the rest follows from the injectivity of f.

(2) Let $B \in \mathrm{Bl}(K)$. Write $\mathrm{At}(B) = \{a, b, c\}$. Then $a\ \Delta_B\ b\ \Delta_B\ c = 1$. Since B is a Boolean sub-algebra of L we have $a \triangle b \triangle c = a\ \Delta_B\ b\ \Delta_B\ c$ (Prop. 1.4). It means that $a \triangle b \triangle c = 1$, and therefore $f(a) \oplus f(b) \oplus f(c) = \bar{0}$.

(3) Suppose that $a, b \in \mathrm{At}(K)$ with $a \sim b$. Obviously, $a \vee b < 1$. Set $s = a \triangle b \in L$. Then $s \in K$ (compare the Def. 3.5). If $s = a \vee b$, then $a \perp b$ in view of $a \vee b = a \triangle b$ (Prop. 1.8). This is a contradiction. If $s = 0$, then $a = b$ and this is again a contradiction. Summarizing the previous considerations, we conclude that $0 < s < a \vee b$. And this implies that $s \in \mathrm{At}(K)$. Since $s \leq a \vee b$, we have $s\ C\ (a \vee b)$ and therefore $s\ C\ (a^\perp \wedge b^\perp)$. If $s = a^\perp \wedge b^\perp$, then $(a \vee b)^\perp \leq a \vee b$ which cannot be the case since this would imply $a \vee b = 1$. We conclude that $s \approx a^\perp \wedge b^\perp$ and therefore

$l(s) = f(s) = f(a \triangle b) = f(a) \oplus f(b) = l(a) \oplus l(b)$. So the implication (1) $\Rightarrow$ (2) has been verified.

Conversely, assume that there is a labelling $l : \mathrm{At}(K) \to D_\infty$. We can suppose that $K \cap D_\infty = \emptyset$. Choose an n, $n \in N$, such that $l[\mathrm{At}(K)] \subseteq \{1, \bar{1}, 2, \bar{2}, \dots, 2^n - 1, \overline{2^n - 1}\}$. Rewrite the set $\{1, \bar{1}, 2, \bar{2}, \dots, 2^n - 1, \overline{2^n - 1}\} \setminus \{l(a), l(a)^\star;\ a \in \mathrm{At}(K)\}$ as $\{i_1, \overline{i_1}, i_2, \overline{i_2}, \dots, i_m, \overline{i_m}\}$. Let M be the copy of MO_m, where $M = \{0, \overline{0}, i_1, \overline{i_1}, i_2, \overline{i_2}, \dots, i_m, \overline{i_m}\}$ under the understanding of $\overline{i_k} = i_k^\perp$, $1 \le k \le m$. Consider the horizontal sum K with M and denote it by L'. It remains to show that there is an ODL L such that $L_{\mathrm{supp}} = L'$.

Let $e : L' \to \{0, \bar{0}, 1, \bar{1}, \dots, 2^n - 1, \overline{2^n - 1}\}$ be the mapping that is defined as follows:

$e(0_{L'}) = 0$, $e(1_{L'}) = \bar{0}$,

$e(a) = l(a)$, $e(a^\perp) = l(a)^\star$ for $a \in \mathrm{At}(K)$,

and e acts as identity on $\{i_1, \overline{i_1}, i_2, \overline{i_2}, \dots, i_m, \overline{i_m}\}$.

Obviously, e is a bijection of L' onto $\{0, \bar{0}, 1, \bar{1}, \dots, 2^n - 1, \overline{2^n - 1}\}$. Let us set, for $x, y \in L'$, $x \triangle y = e^{-1}(e(x) \oplus e(y))$ and verify that $L = (L', \triangle)$ is an ODL.

The associativity as well as the commutativity follow immediately from the algebra isomorphism of $(L', \triangle)$ on $(\{0, \bar{0}, 1, \bar{1}, \dots, 2^n - 1, \overline{2^n - 1}\}, \oplus)$. Further, $x \triangle 1_{L'} = e^{-1}(e(x) \oplus e(1_{L'})) = e^{-1}(e(x) \oplus \bar{0}) = e^{-1}(e(x)^\star)$. Consider now the possibilities for x in order. If $x \in K$, then $e^{-1}(e(x)^\star) = e^{-1}(l(x)^\star) = e^{-1}(l(x^\perp)) = e^{-1}(e(x^\perp)) = x^\perp$. If $x = i_k$, then $x \triangle 1_{L'} = e^{-1}(e(i_k)^\star) = e^{-1}(i_k^\star) = e^{-1}(\overline{i_k})) = \overline{i_k} = i_k^\perp$. Finally, if $x = \overline{i_k}$, then $x \triangle 1_{L'} = e^{-1}(e(\overline{i_k})^\star) = e^{-1}(\overline{i_k}^\star) = e^{-1}(i_k) = i_k = \overline{i_k}^\perp$. Thus, $x \triangle 1_{L'} = x^\perp$ for any $x \in L'$.

It remains to check the last axiom of ODL's, $x \triangle y \le x \vee y$. Let $x, y \in L'$. If $x \vee y = 1$, there is nothing to check. If $x = y$, then $x \triangle y = 0_{L'}$ and the inequality in question is clear. Let us finally suppose that $x \vee y < 1$ with $x \ne y$. Then $x, y \in K$. Let us discuss the possibilities for $x, y \in K$ which may occur.

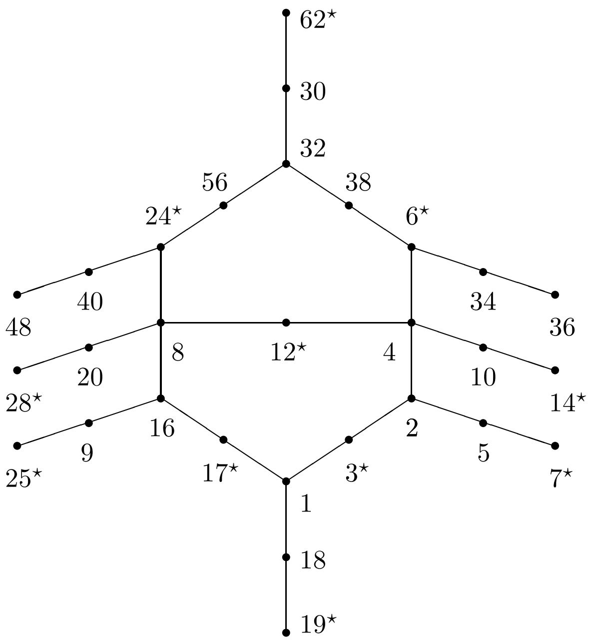
First, suppose that $x\ C\ y$. Let us choose a block B, $B \in \mathrm{Bl}(K)$ such that $x, y \in B$. Then either both x, y are atoms or not. In the former case, when $\mathrm{At}(B) = \{x, y, z\}$, we have $l(x) \oplus l(y) \oplus l(z) = \bar{0}$. It means that $x \triangle y = e^{-1}(e(x) \oplus e(y)) = e^{-1}(l(x) \oplus l(y)) = e^{-1}(l(z)^\star) = e^{-1}(e(z^\perp)) = z^\perp = x \vee y$. In the latter case, when at least one of x and y is not an atom, we have $x \ne y$ and $x \vee y < 1$. Thus, exactly one of x and y is a coatom. Suppose, for instance, that x is an atom and y a coatom. Then $x \le y$. Suppose that $\mathrm{At}(B) = \{x, y^\perp, z\}$. Then $l(x) \oplus l(y^\perp) \oplus l(z) = \bar{0}$ and therefore $l(x) \oplus l(y^\perp)^\star = l(z)$. Further, we obtain $x \triangle y = e^{-1}(e(x) \oplus e(y)) = e^{-1}(e(x) \oplus e(y^{\perp\perp})) = e^{-1}(l(x) \oplus l(y^\perp)^\star) = e^{-1}(l(z)) = e^{-1}(e(z)) = z \le y = x \vee y$. Again, $x \triangle y \le x \vee y$.

Secondly, suppose that x does not commute with y (abbr., $x \neg\, C\ y$). Then neither of x and y coincides with 0 or 1. We are going to show that both x and y are atoms. Looking for a contradiction, suppose that x is a coatom. Then $x \le x \vee y < 1$ and therefore $x = x \vee y$ and this means that $y \le x$ – a contradiction with $x \neg\, C\ y$. We see that both x and y are atoms. So $x < x \vee y < 1$ and therefore $x \vee y$ is a coatom. If we set $z = (x \vee y)^\perp$, we obtain that $x \sim y$. According to the condition (3) in the definition of labelling, an element $s \in \mathrm{At}(K)$ is guaranteed such that $s \approx z$ and $l(s) = l(x) \oplus l(y)$. Consequently, one derives the equalities $x \triangle y = e^{-1}(e(x) \oplus e(y)) = e^{-1}(l(x) \oplus l(y)) = e^{-1}(l(s)) = e^{-1}(e(s)) = s \leq z^{\perp} = x \vee y$. This completes the proof. $\square$

The previous result will be applied in our final construction to provide a proof of a main result of this paper.

Theorem 3.11. There is an ODL L with 3 generators that is not set-representable. A consequence: The free ODL on 3 generators is not set-representable.

Proof. Consider the OML K portrayed by the following figure. Let us make use in the figure the conventions of the Greechie paste job ([9, 12]) and the labelling notation agreed on in Thm. 3.10.



As shown in [8], in each set-representable OML (and, in turn, in each set-representable ODL) the following inequality holds true: $x \wedge (y \vee z) \leq \varphi_x(y) \vee \varphi_{y^\perp}(z)$, where $\varphi_a(b) = (b \vee a^\perp) \wedge a$ is the well-known Sasaki projection ([1, 12]).

Let us see that the above inequality fails in the OML K depicted by the figure. Indeed, let us take $x = 16^\star$, $y = 4$ and $z = 6^\star$. Then $x \wedge (y \vee z) = 16^\star \wedge (4 \vee 6^\star) = 16^\star \wedge 2^\star = 1$ whereas $\varphi_x(y) = (4 \vee 16) \wedge 16^\star = 8^\star \wedge 16^\star = 24^\star$ and $\varphi_{y^\perp}(z) = (6^\star \vee 4) \wedge 4^\star = 2^\star \wedge 4^\star = 6^\star$ which gives us $\varphi_x(y) \vee \varphi_{y^\perp}(z) = 24^\star \vee 6^\star = 32^\star$. But $1 \not\leq 32^\star$.

By Thm. 3.10, there is an ODL, M, such that $K \ll M$. Let L be the sub-ODL of M generated by x, y and z. Then the inequality $x \wedge (y \vee z) \leq \varphi_x(y) \vee \varphi_{y^\perp}(z)$ does not hold true in L and therefore L is not set-representable. Obviously, L has 3 generators and we have completed the proof. $\square$

Remark 3.12. In our opinion, the argument used in the above proof of Thm. 3.11 is reasonably clear and convincing. However, if one wants to have more tangible L one could find the ODL M with the help of the construction in the proof of Thm. 3.10. This procedure would result in M that is isomorphic to the horizontal sum of K and MO_{34}. Moreover, M is then generated by the elements x, y and z and we have $L = M$.

In the series of papers [13] – [16] together with this note we have iniciated a systematic study of axiomatic symmetric difference. The algebras which came into existence, the ODLs, lie between orthomodular lattices and Boolean algebras and might therefore find application in quantum logic theory or elsewhere in algebra. In the former area of application it would be desirable to investigate 'states' on ODLs. In the latter area, a natural step in the effort to understand the intrinsic structure of ODLs is the investigation of free objects in the variety $\mathcal{ODL}$ (the complexity of this problem indicates the analogous study in OMLs, see [2]). A problem linked with the last question is whether this variety is locally finite. Though we conjecture it is not, the problem is still open to us.

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