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Multivariate Statistical Models; Solvability of Basic Problems*

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Abstract

Multivariate models frequently used in many branches of science have relatively large number of different structures. Sometimes the regularity condition which enable us to solve statistical problems are not satisfied and it is reasonable to recognize it in advance. In the paper the model without constraints on parameters is analyzed only, since the greatness of the class of such problems in general is out of the size of the paper.

Key words: Multivariate model, estimation, testing hypotheses, sensitivity.

2000 Mathematics Subject Classification: 62J05, 62H12

1 Introduction

The aim of the paper is to attract an attention to solvability of basic statistical problems in multivariate models. They are estimation, construction of confidence regions and testing statistical hypotheses.

Plenty of nice solutions of such problems are given in statistical monographs [1], [6], [7], [24], however comments on situations where some of given problems are unsolvable, are rather rare. The monograph [12] is devoted to such problems.

An attempt in this paper is to give an overview of basic statistical problems which either can be or cannot be solved in a standard way on the basis of [12]. Therefore majority of statements given here are given without proofs (they

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are given in [12]). Exception are the proof of Lemma (Rao) and proof of the statements in a consideration on insensitivity regions, since these approaches are not commonly used.

The multivariate linear models without constraints are considered only, because of the greatness of the class of such problems in models with constraints (see in more detail in [12]).

2 Structures of the models; structures of the covariance matrices

In the following text $\underline{\mathbf{Y}}$ means a random matrix which realization gives us a set of measured data, $\mathbf{X}$ is a known design matrix, $\mathbf{B}$ is a matrix of unknown parameters and $\mathbf{\Sigma}$ is a covariance matrix either of the rows, or the columns of the matrix $\underline{\mathbf{Y}}$. The symbol $\mathbf{I}$ means the identity matrix.

The following models without constraints on the matrix $\mathbf{B}$ will be considered.

$$\operatorname{vec}(\underline{\mathbf{Y}}_{n,m}) \sim_{nm} [(\mathbf{I}_{m,m} \otimes \mathbf{X}_{n,k}) \operatorname{vec}(\mathbf{B}_{k,m}), \mathbf{\Sigma}_{m,m} \otimes \mathbf{I}_{n,n}], \tag{1}$$

$$\operatorname{vec}(\underline{\mathbf{Y}}_{n,m}) \sim_{nm} [(\mathbf{I}_{m,m} \otimes \mathbf{X}_{n,k}) \operatorname{vec}(\mathbf{B}_{k,m}), \mathbf{I}_{m,m} \otimes \mathbf{\Sigma}_{n,n}]. \tag{2}$$

Here $\otimes$ denotes the Kronecker multipication of matrices and $\operatorname{vec}(\mathbf{B})$ means the vector composed of the columns of the matrix $\mathbf{B}$. The notation in (1) means that the mean of the vector $\underline{\mathbf{Y}}$ is $(\mathbf{I} \otimes \mathbf{X}) \operatorname{vec}(\mathbf{B})$ and its covariance matrix is $\mathbf{\Sigma} \otimes \mathbf{I}$.

The model (2) is in fact m univariate not correlated models

$$\mathbf{Y}_i \sim_n (\mathbf{X}\boldsymbol{\beta}_i, \mathbf{\Sigma}), \quad i = 1, \dots, m,$$
$$\underline{\mathbf{Y}} = (\mathbf{Y}_1, \dots, \mathbf{Y}_m), \quad \mathbf{B} = (\boldsymbol{\beta}_1, \dots, \boldsymbol{\beta}_m)$$

The model (2) is considered here because of purposes given in the following text.

$$\operatorname{vec}(\underline{\mathbf{Y}}_{n,r}) \sim_{nr} [(\mathbf{Z}'_{r,m} \otimes \mathbf{X}_{n,k}) \operatorname{vec}(\mathbf{B}_{k,m}), \mathbf{\Sigma}_{r,r} \otimes \mathbf{I}_{n,n}], \tag{3}$$

$$\operatorname{vec}(\underline{\mathbf{Y}}_{n,r}) \sim_{nr} [(\mathbf{Z}'_{r,m} \otimes \mathbf{X}_{n,k}) \operatorname{vec}(\mathbf{B}_{k,m}), \mathbf{I}_{r,r} \otimes \mathbf{\Sigma}_{n,n}]. \tag{4}$$

The models (3) and (4) are the same from the viewpoint of theory however from the viewpoint of application it is reasonable to analyze them separately.

It is to be said some comments to these four models. The first one occurs frequently when a group of patients is investigated. The i-th row of the observation matrix $\underline{\mathbf{Y}}$ means the observation vector of the i-th patient, where m values of some physiological quantities are measured. The j-th column of the matrix $\mathbf{B}$ are parameters of the j-th quantity measured in the experiment and they are common for all patients. It is assumed that the design matrix $\mathbf{X}$ is the same for all quantities. The matrix $\mathbf{\Sigma}$ is the covariance matrix of the rows of the observation matrix $\underline{\mathbf{Y}}$ (it is assumed that patients are stochastically independent). The i-th row of the matrix $\mathbf{X}$ can be given, e.g. by the quantities as age, weight, body mass index, etc., of the i-th patient.

The model (2) is typical for deformation measurement in m epochs. The j-th column of $\underline{\mathbf{Y}}$ is the observation vector of the j-th epoch, the j-th column of the matrix $\mathbf{B}$ is the vector parameters of the investigated object (e.g. the 3D coordinates of a group of points in the j-th epoch of measurement, which positions are measured on the investigated object m-times in order to detect their changes during the time of deformation process), $\mathbf{\Sigma}$ is the covariance matrix of the columns of the matrix $\underline{\mathbf{Y}}$ (the measurements of two different epochs are stochastically independent) and $\mathbf{X}$ characterizes the design of measurement, which is assumed to be the same in all m-epochs.

The third model is the growth curve model. It arises, e.g. when a time course of a physiological parameter in investigated for n patients. The indirectly measured values of them for the i-th patient are $\phi_i(t_1), \phi_i(t_2), \dots, \phi_i(t_r)$. If the polynomial time course is assumed, i.e.

$$\phi_i(t) = \Theta_{i,0} + \Theta_{i,1}t + \ldots + \Theta_{i,m-1}t^{m-1},$$

then

$$E(\underline{\mathbf{Y}}) = \mathbf{X} \begin{pmatrix} \Theta_{1,0}, & \Theta_{1,1}, & \ldots, & \Theta_{1,m-1} \\ \ldots & \ldots & \ldots & \ldots \\ \Theta_{k,0}, & \Theta_{k,1}, & \ldots, & \Theta_{k,m-1} \end{pmatrix} \begin{pmatrix} 1, & 1, & \ldots, & 1 \\ t_1, & t_2, & \ldots, & t_r \\ \ldots & \ldots & \ldots & \ldots \\ t_1^{m-1}, & t_2^{m-1}, & \ldots, & t_r^{m-1} \end{pmatrix}.$$

Thus $\mathbf{X}$ characterizes the design of indirect measurement,

$$\mathbf{B} = \begin{pmatrix} \Theta_{1,0}, & \Theta_{1,1}, & \ldots, & \Theta_{1,m-1} \\ \ldots & \ldots & \ldots & \ldots \\ \Theta_{k,0}, & \Theta_{k,1}, & \ldots, & \Theta_{k,m-1} \end{pmatrix}, \quad \mathbf{Z} = \begin{pmatrix} 1, & 1, & \ldots, & 1 \\ t_1, & t_2, & \ldots, & t_r \\ \ldots & \ldots & \ldots & \ldots \\ t_1^{m-1}, & t_2^{m-1}, & \ldots, & t_r^{m-1} \end{pmatrix}.$$

The matrix $\mathbf{\Sigma}$ is the covariance matrix of the rows of the observation matrix $\underline{\mathbf{Y}}$.

The model (4) is typical for deformation measurement. Let a time course of coordinates of a characteristic point of the investigated object is investigated, i.e.

$$\{\mathbf{B}\}_{i,j} = \Theta_{i,1}\varphi_1(t_j) + \ldots + \Theta_{i,m}\varphi_m(t_j)$$

(the matrix $\mathbf{B}$ is from the model (2)), where $\varphi_1(\cdot), \dots, \varphi_m(\cdot)$ are suitable functions for characterizing the deformation process. Thus the following model

$$E(\underline{\mathbf{Y}}) = \mathbf{X} \begin{pmatrix} \Theta_{1,1}, & \Theta_{1,2}, & \ldots, & \Theta_{1,m} \\ \ldots & \ldots & \ldots & \ldots \\ \Theta_{k,1}, & \Theta_{k,2}, & \ldots, & \Theta_{k,m} \end{pmatrix} \begin{pmatrix} \varphi_1(t_1), & \varphi_1(t_2), & \ldots, & \varphi_1(t_r) \\ \ldots & \ldots & \ldots & \ldots \\ \varphi_m(t_1), & \varphi_m(t_2), & \ldots, & \varphi_m(t_r) \end{pmatrix},$$

is obtained.

Thus in the model (4) we have

$$\mathbf{B} = \begin{pmatrix} \Theta_{1,1}, & \Theta_{1,2}, & \ldots, & \Theta_{1,m} \\ \ldots & \ldots & \ldots & \ldots \\ \Theta_{k,1}, & \Theta_{k,2}, & \ldots, & \Theta_{k,m} \end{pmatrix}, \quad \mathbf{Z} = \begin{pmatrix} \varphi_1(t_1), & \varphi_1(t_2), & \ldots, & \varphi_1(t_r) \\ \ldots & \ldots & \ldots & \ldots \\ \varphi_m(t_1), & \varphi_m(t_2), & \ldots, & \varphi_m(t_r) \end{pmatrix}.$$

Regularity conditions:

$$r(\mathbf{X}_{n,k}) = k < n, \quad r(\mathbf{Z}_{m,r}) = m < r, \quad \mathbf{\Sigma} \text{ positive definite.}$$

In the following text no regularity is assumed.

Structures of the covariance matrix is assumed to be of the forms:

$\mathbf{\Sigma}$ ... completely known,

$\mathbf{\Sigma} = \sigma^2\mathbf{V}, \quad \sigma^2 \in (0,\infty)$ unknown, $\mathbf{V}$ known positive definite,

$\mathbf{\Sigma} = \sum_{i=1}^{p} \vartheta_i \mathbf{V}_i, \quad \boldsymbol{\vartheta} = (\vartheta_1, \dots, \vartheta_p)'$ unknown, $\vartheta_i > 0,\ i = 1, \dots, p,$

$\mathbf{V}_1, \dots, \mathbf{V}_p$ known, symmetric and positive semidefinite,

$\mathbf{\Sigma}$ ... completely unknown.

3 Estimability of the first order parameters

The statements of this section are proved in [12].

Model (1): Let $\mathcal{M}(\mathbf{H}'_{u,k}) \subset \mathcal{M}(\mathbf{X}')$. (The notation $\mathcal{M}(\mathbf{X}')$ means the space $\{\mathbf{X}'\mathbf{v} \colon \mathbf{v} \in R^n\}$ and R^n is the n-dimensional linear space.) Then $\mathbf{HB}$ is unbiasedly estimable and there exists the BLUE of it independently on the knowledge of the matrix $\mathbf{\Sigma}$.

Model (2):

$\mathbf{\Sigma}$ known: if $\mathcal{M}(\mathbf{H}') \subset \mathcal{M}(\mathbf{X}')$ the BLUE of $\mathbf{HB}$ exists,

$\mathbf{\Sigma} = \sigma^2\mathbf{V}$: dtto,

$\mathbf{\Sigma} = \sum_{i=1}^{p} \vartheta_i \mathbf{V}_i$: if $\mathcal{M}(\mathbf{H}') \subset \mathcal{M}(\mathbf{X}')$

the $\boldsymbol{\vartheta}_0$-LBLUE exists only; if "plug-in" estimator is used, then

the sensitivity analysis is necessary, see section 6,

$\mathbf{\Sigma}$ unknown: the BLUE does not exist.

Model (3):

$\mathbf{\Sigma}$ known: if $\mathcal{M}(\mathbf{H}'_1) \subset \mathcal{M}(\mathbf{X}')$ & $\mathcal{M}(\mathbf{H}_2) \subset \mathcal{M}(\mathbf{Z})$

the BLUE of $\mathbf{H}_1\mathbf{B}\mathbf{H}_2$ exists,

$\mathbf{\Sigma} = \sigma^2\mathbf{V}$: dtto,

$\mathbf{\Sigma} = \sum_{i=1}^{p} \vartheta_i \mathbf{V}_i$: if $\mathcal{M}(\mathbf{H}'_1) \subset \mathcal{M}(\mathbf{X}')$ & $\mathcal{M}(\mathbf{H}_2) \subset \mathcal{M}(\mathbf{Z})$

the $\boldsymbol{\vartheta}_0$-LBLUE exists only; if "plug-in" estimator is used, then

the sensitivity analysis is necessary, see section 6,

$\mathbf{\Sigma}$ unknown: the BLUE does not exist.

Model (4): The same is valid as in the model (3). (See also the paper [2], where insensitivity regions in multivariate statistical model (4) is constructed.)

4 Estimability of the second order parameters

The statements in this section are proved in [12].

(i) The case $\mathbf{\Sigma} = \sigma^2\mathbf{V}$; the parameter σ^2 is estimable if

Model (1): $r(\mathbf{V}) > 0 \;\&\; n - r(\mathbf{X}) > 0$,

Model (2): $r(\mathbf{V}, \mathbf{X}) - r(\mathbf{X}) > 0$,

Model (3): $\big((r(\mathbf{V}) > 0 \;\&\; n - r(\mathbf{X}) > 0)\big) \vee \big((r(\mathbf{X}) > 0 \;\&\; r(\mathbf{V}, \mathbf{Z}') - r(\mathbf{Z}') > 0)\big)$,

Model (4): $\big((r(\mathbf{V}) > 0 \;\&\; r - r(\mathbf{Z}) > 0)\big) \vee \big((r(\mathbf{Z}) > 0 \;\&\; r(\mathbf{V}, \mathbf{X}) - r(\mathbf{X}) > 0)\big)$.

(ii) The case $\mathbf{\Sigma} = \sum_{i=1}^{p} \vartheta_i \mathbf{V}_i$; the parameters $\vartheta_1, \dots, \vartheta_p$, are unbiasedly eastimable if

Model (1): $\mathbf{S}_{\Sigma_0^+}$ regular, $\left\{\mathbf{S}_{\Sigma_0^+}\right\}_{i,j} = \mathrm{Tr}(\mathbf{V}_i \mathbf{\Sigma}_0^+ \mathbf{V}_j \mathbf{\Sigma}_0^+), \quad i, j = 1, \dots, p,$

Model (2): $\mathbf{S}_{(M_X \Sigma_0 M_X)^+}$ regular,

$\left\{\mathbf{S}_{(M_X \Sigma_0 M_X)^+}\right\}_{i,j} = \mathrm{Tr}[\mathbf{V}_i (\mathbf{M}_X \mathbf{\Sigma}_0 \mathbf{M}_X)^+ \mathbf{V}_j (\mathbf{M}_X \mathbf{\Sigma}_0 \mathbf{M}_X)^+], \quad j = 1, \dots, p,$

Model (3): $[n - r(\mathbf{X})]\mathbf{S}_{\Sigma_0^+} + r(\mathbf{X})\mathbf{S}_{(M_X \Sigma_0 M_X)^+}$ regular,

Model (4): $[r - r(\mathbf{Z})]\mathbf{S}_{\Sigma_0^+} + r(\mathbf{Z})\mathbf{S}_{(M_X \Sigma_0 M_X)^+}$ regular.

Here the following notations are used. $\mathbf{\Sigma}_0 = \sum_{i=1}^{p} \vartheta_{0,i} \mathbf{V}_i$, $\boldsymbol{\vartheta}_0 = (\vartheta_{0,1} \dots, \vartheta_{0,p})'$ is an approximate value of the parameter $\boldsymbol{\vartheta}$, $\mathbf{M}_X = \mathbf{I} - \mathbf{P}_X$, $\mathbf{P}_X = \mathbf{X}\mathbf{X}^+$ and $^+$ means the Moore–Penrose generalized inverse of a matrix, i.e.

$$\mathbf{A}\mathbf{A}^+\mathbf{A} = \mathbf{A},\ \mathbf{A}^+\mathbf{A}\mathbf{A}^+ = \mathbf{A}^+,\ (\mathbf{A}\mathbf{A}^+)' = \mathbf{A}\mathbf{A}^+,\ (\mathbf{A}^+\mathbf{A})' = \mathbf{A}^+\mathbf{A}.$$

(iii) The case of the completely unknown $\mathbf{\Sigma}$; the covariance matrix $\mathbf{\Sigma}$ is estimable by the matrix $\widehat{\mathbf{\Sigma}}$

Model (1): $\widehat{\mathbf{\Sigma}}_{m,m} = \underline{\mathbf{Y}}'\mathbf{M}_X\underline{\mathbf{Y}}/[n - r(\mathbf{X})]$ & if $m < n - r(\mathbf{X})$,

Model (2): estimator does not exist,

Model (3): $\widehat{\mathbf{\Sigma}}_{r,r} = \underline{\mathbf{Y}}'\mathbf{M}_X\underline{\mathbf{Y}}/[n - r(\mathbf{X})]$ & if $r < n - r(\mathbf{X})$,

Model (4): $\widehat{\mathbf{\Sigma}}_{n,n} = \underline{\mathbf{Y}}\mathbf{M}_{Z'}\underline{\mathbf{Y}}'/[r - r(\mathbf{Z})]$ & if $n < r - r(\mathbf{Z})$.

5 Confidence regions of the first order parameters (normality is assumed)

The statements of this section are proved in [12].

(i) The case of the known $\mathbf{\Sigma}$. If $\mathcal{M}(\mathbf{H}') \subset \mathcal{M}(\mathbf{X}')$, then there exists confidence regions for $\mathbf{HB}$ in the models (1) and (2).

If $\mathcal{M}(\mathbf{H}_1') \subset \mathcal{M}(\mathbf{X}')$ & $\mathcal{M}(\mathbf{H}_2) \subset \mathcal{M}(\mathbf{Z})$, then there exists confidence regions for $\mathbf{H}_1\mathbf{B}\mathbf{H}_2$ in the models (3) and (4).

(ii) The case $\mathbf{\Sigma} = \sigma^2\mathbf{V}$. The same is valid as in the case "$\mathbf{\Sigma}$ is known", however σ^2 must be estimable.

(iii) The case $\mathbf{\Sigma} = \sum_{i=1}^{p} \vartheta_i \mathbf{V}_i$. "Plug-in" determination of the confidence regions for estimable linear functions of $\mathbf{B}$ can be used, however an additional sensitivity analysis should be used.

(iv) The case $\mathbf{\Sigma}$ is completely unknown.

Model (1): If $\mathcal{M}(\mathbf{H}') \subset \mathcal{M}(\mathbf{X}')$ and $\mathbf{l} \in R^m$ is arbitrary, then there exists a confidence region for the vector $\mathbf{HBl}$.

If $\mathbf{h} \in \mathcal{M}(\mathbf{X}')$, then there exists a confidence region for the vector $\mathbf{h}'\mathbf{B}$.

Model (2): Confidence regions based on $\underline{\mathbf{Y}}$ do not exist for any functions of $\mathbf{B}$. If the Wishart matrix $\mathbf{W} \sim W_n(f, \mathbf{\Sigma})$ independent of $\mathbf{Y}$ is at our disposal, then there exist confidence regions for the vector $\mathbf{HBl}$, where

$$\mathcal{M}(\mathbf{H}') \subset \mathcal{M}(\mathbf{X}')$$

and $\mathbf{l} \in R^m$, however the condition

$$f > r[(\mathbf{M}_X \mathbf{\Sigma} \mathbf{M}_X)^+] + r(\mathbf{V}_H) - 1,$$

where $\mathbf{V}_H = \mathbf{H}\{[\mathbf{X}'(\mathbf{\Sigma} + \mathbf{XX}')^+\mathbf{X}]^+ - \mathbf{I}\}\mathbf{H}'$, must be satisfied. More on this approach cf. [23].

The confidence region exists also for the vector $\mathrm{vec}(\mathbf{k}'\mathbf{B})$, where $\mathbf{k} \in \mathcal{M}(\mathbf{X}')$. However the estimator of the vector $\mathrm{vec}(\mathbf{k}'\mathbf{B})$, is based on

$$[\mathbf{k}'(\mathbf{X}'\mathbf{X})^+\mathbf{X}'\underline{\mathbf{Y}}]$$

only, i.e. the estimator is not the BLUE. The degrees of freedom f of the Wishart matrix must satisfy the condition $f > m$.

Model (3): If the Wishart matrix $\mathbf{W} \sim W_r(f, \mathbf{\Sigma})$ independent of $\underline{\mathbf{Y}}$ is at our disposal, then the confidence region exists for the vector $\mathrm{vec}(\mathbf{h}_1'\mathbf{B}\mathbf{H}_2)$, where $\mathbf{h}_1 \in \mathcal{M}(\mathbf{X}')$ and $\mathcal{M}(\mathbf{H}_2) \subset \mathcal{M}(\mathbf{Z})$. In this case the condition

$$f > r(\mathbf{\Sigma}, \mathbf{Z}') - r(\mathbf{Z}') + r(\mathbf{V}_{H_2}) - 1$$

must be satisfied. Here

$$\mathbf{V}_{H_2} = \mathbf{H}_2'\{[\mathbf{Z}(\mathbf{\Sigma} + \mathbf{ZZ}')^+\mathbf{Z}']^+ - \mathbf{I}\}\mathbf{H}_2.$$

Model (4): The confidence region for the function $\mathbf{H}_1\mathbf{B}\mathbf{h}_2$,

$$\mathcal{M}(\mathbf{H}_1') \subset \mathcal{M}(\mathbf{X}'), \quad \mathbf{h}_2 \in \mathcal{M}(\mathbf{Z}),$$

exists if the Wishart matrix $\mathbf{W} = f\mathbf{S} \sim W_n(f, \mathbf{\Sigma})$ independent of $\underline{\mathbf{Y}}$ is at our disposal. The inequality

$$f > r(\mathbf{\Sigma}, \mathbf{X}) - r(\mathbf{X}) + r[\mathrm{Var}(\widehat{\mathbf{H}_1\mathbf{B}\mathbf{h}_2})] - 1$$

must be satisfied. Here

$$\mathbf{V}_{H_1} = \mathbf{H}_1[(\mathbf{X}'\mathbf{T}^+\mathbf{X})^+ - \mathbf{I}]\mathbf{H}_1', \quad \mathbf{T} = \mathbf{\Sigma} + \mathbf{XX}', \quad \tilde{\mathbf{T}} = \mathbf{S} + \mathbf{XX}'.$$

In the following text an example is given how to proceed in the model (4) when an additional Wishart matrix is at our disposal.

The following lemma, which is a nonessential modification (see in [12]) of the Rao's statement given in [23], is necessary.

Lemma 1 (Rao, C. R.) *Let an observation vector* $\mathbf{Y} \sim N_n(\mathbf{X}_{n,k}\boldsymbol{\beta}, \boldsymbol{\Sigma})$, $\boldsymbol{\beta} \in R^k$, *and the Wishart matrix* $f\mathbf{S} = \mathbf{W} = \sum_{\alpha=1}^{f} \mathbf{U}_\alpha \mathbf{U}'_\alpha \sim W_n(f, \boldsymbol{\Sigma})$, $f > r(\boldsymbol{\Sigma})$ *be stochastically independent. Let* $\mathbf{H}_{h,k}$ *be a given matrix with the property* $\mathcal{M}(\mathbf{H}') \subset \mathcal{M}(\mathbf{X}')$ *(i.e. the vector* $\mathbf{H}\boldsymbol{\beta}$ *is unbiasedly estimable). The BLUE of the function* $\mathbf{H}\boldsymbol{\beta}$, $\boldsymbol{\beta} \in R^k$, *is*

$$\widehat{\mathbf{H}\boldsymbol{\beta}} = \mathbf{H}[(\mathbf{X}')^-_{m(\Sigma)}]'\mathbf{Y}.$$

Here $(\mathbf{X}')^-_{m(\Sigma)}$ *means the matrix with the following property.*

$$\forall\{\mathbf{y} \in \mathcal{M}(\mathbf{X}')\}\{\mathbf{X}'(\mathbf{X}')^-_{m(\Sigma)}\mathbf{y} = \mathbf{y}\} \ \&\ \forall\{\mathbf{x}\colon \mathbf{X}'\mathbf{x} = \mathbf{y}\}\{\mathbf{x}'\boldsymbol{\Sigma}\mathbf{x} \\ \geq \big[(\mathbf{X}')^-_{m(\Sigma)}\mathbf{y}\big]'\boldsymbol{\Sigma}(\mathbf{X}')^-_{m(\Sigma)}\mathbf{y}\}.$$

Let

$$\widehat{\boldsymbol{\tau}} = \mathbf{H}[(\mathbf{X}')^-_{m(S)}]'\mathbf{Y}$$

and $\mathbf{t}_2 = \mathbf{M}_X\mathbf{Y}$ *(the vector* $\mathbf{t}_2$ *represents all linear unbiased estimators, based on the vector* $\mathbf{Y}$, *of zero function). The upper index* $^{(p)}$ *means that a random vector is conditioned by the random vector* $\mathbf{t}_2$ *and simultenously by the random matrix* $\widehat{\boldsymbol{\Lambda}}_{2,2}$, *where* $f\widehat{\boldsymbol{\Lambda}}_{2,2} = \mathbf{M}_X \sum_{\alpha=1}^{f} \mathbf{U}_\alpha \mathbf{U}'_\alpha \mathbf{M}_X$.

Then

$$\widehat{\boldsymbol{\tau}}^{(p)} \sim N_h\left\{\mathbf{H}\boldsymbol{\beta}, \mathbf{H}[(\mathbf{X}')^-_{m(\Sigma)}]'\boldsymbol{\Sigma}(\mathbf{X}')^-_{m(\Sigma)}]\mathbf{H}'\left(1 + \frac{1}{f}\mathbf{t}'_2\widehat{\boldsymbol{\Lambda}}^-_{2,2}\mathbf{t}_2\right)\right\},$$

$$f\Big\{\mathbf{H}[(\mathbf{X}')^-_{m(S)}]'\mathbf{S}(\mathbf{X}')^-_{m(S)}\mathbf{H}'\Big\}^{(p)} \\ \sim W_h\Big\{f - r[\mathrm{Var}(\mathbf{t}_2)], \mathbf{H}[(\mathbf{X}')^-_{m(\Sigma)}]'\boldsymbol{\Sigma}(\mathbf{X}')^-_{m(\Sigma)}\mathbf{H}'\Big\}.$$

Proof The statistic $\mathbf{t}_1 = \mathbf{H}\mathbf{X}^-\mathbf{Y}$ is an unbiased linear estimator of the vector $\mathbf{H}\boldsymbol{\beta}$, $\boldsymbol{\beta} \in R^k$. The matrix $\mathbf{X}^-$ is an arbitrary g-inverse of the matrix $\mathbf{X}$. Then the BLUE of $\mathbf{H}\boldsymbol{\beta}$ is

$$\widehat{\mathbf{H}\boldsymbol{\beta}} = \mathbf{t}_1 - \mathrm{cov}(\mathbf{t}_1, \mathbf{t}_2)\,\mathrm{Var}(\mathbf{t}_2)^-\mathbf{t}_2,$$

since

$$\mathrm{cov}\Big\{[\mathbf{t}_1 - \mathrm{cov}(\mathbf{t}_1, \mathbf{t}_2)[\mathrm{Var}(\mathbf{t}_2)]^-\mathbf{t}_2, \mathbf{t}_2\Big\} = \mathbf{0}.$$

Let

$$\mathbf{V}_\alpha = \begin{pmatrix} \mathbf{V}_{\alpha,1} \\ \mathbf{V}_{\alpha,2} \end{pmatrix} = \begin{pmatrix} \mathbf{H}\mathbf{X}^- \\ \mathbf{M}_X \end{pmatrix}\mathbf{U}_\alpha,$$

i.e.

$$\mathbf{V}_\alpha \sim N_{h+n}\left[\begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}, \begin{pmatrix} \boldsymbol{\Lambda}_{1,1}, & \boldsymbol{\Lambda}_{1,2} \\ \boldsymbol{\Lambda}_{2,1}, & \boldsymbol{\Lambda}_{2,2} \end{pmatrix}\right],$$

$$\begin{pmatrix} \boldsymbol{\Lambda}_{1,1}, & \boldsymbol{\Lambda}_{1,2} \\ \boldsymbol{\Lambda}_{2,1}, & \boldsymbol{\Lambda}_{2,2} \end{pmatrix} = \begin{pmatrix} \mathbf{H}\mathbf{X}^-\boldsymbol{\Sigma}(\mathbf{X}^-)'\mathbf{H}', & \mathbf{H}\mathbf{X}^-\boldsymbol{\Sigma}\mathbf{M}_X \\ \mathbf{M}_X\boldsymbol{\Sigma}(\mathbf{X}^-)'\mathbf{H}', & \mathbf{M}_X\boldsymbol{\Sigma}\mathbf{M}_X \end{pmatrix}.$$

It is valid that

$$\sum_{\alpha=1}^{f}\begin{pmatrix}\mathbf{HX}^-\\ \mathbf{M}_X\end{pmatrix}\mathbf{U}_\alpha\mathbf{U}'_\alpha[(\mathbf{X}^-)'\mathbf{H}',\mathbf{M}_X]\sim W_{h+n}\left[f,\begin{pmatrix}\mathbf{\Lambda}_{1,1}, & \mathbf{\Lambda}_{1,2}\\ \mathbf{\Lambda}_{2,1}, & \mathbf{\Lambda}_{2,2}\end{pmatrix}\right],$$

$$f\widehat{\mathbf{\Lambda}}_{i,j}=\sum_{\alpha=1}^{f}\mathbf{V}_{\alpha,i}\mathbf{V}'_{\alpha,j},\ i,j=1,2,$$

and

$$\mathbf{V}_{\alpha,1}^{(p)}\sim N_h\Big(\mathbf{\Lambda}_{1,2}\mathbf{\Lambda}_{2,2}^-\mathbf{V}_{\alpha,2},\mathbf{\Lambda}_{1,1}-\mathbf{\Lambda}_{1,2}\mathbf{\Lambda}_{2,2}^-\mathbf{\Lambda}_{2,1}\Big).$$

Since

$$\begin{aligned}\widehat{\mathbf{H}\boldsymbol{\beta}}&=\mathbf{H}\big[(\mathbf{X}')^-_{m(\Sigma)}\big]'\mathbf{Y}\\ &=\mathbf{HX}^-\mathbf{Y}-\operatorname{cov}(\mathbf{HX}^-\mathbf{Y},\mathbf{M}_X\mathbf{Y})\big[\operatorname{Var}(\mathbf{M}_X\mathbf{Y})\big]^-\mathbf{M}_X\mathbf{Y}\\ &=\mathbf{HX}^-\mathbf{Y}-\mathbf{HX}^-\mathbf{\Sigma}\mathbf{M}_X(\mathbf{M}_X\mathbf{\Sigma}\mathbf{M}_X)^+\mathbf{M}_X\mathbf{Y},\end{aligned}$$

$$\begin{aligned}\widehat{\boldsymbol{\tau}}&=\mathbf{H}\big[(\mathbf{X}')^-_{m(S)}\big]'\mathbf{Y}=\mathbf{HX}^-\mathbf{Y}-\mathbf{HX}^-\mathbf{S}\mathbf{M}_X(\mathbf{M}_X\mathbf{S}\mathbf{M}_X)^+\mathbf{M}_X\mathbf{Y}\\ &=\mathbf{HX}^-\mathbf{Y}-\sum_{\alpha=1}^{f}\mathbf{V}_{\alpha,1}\mathbf{V}'_{\alpha,2}\left(\sum_{\alpha=1}^{p}\mathbf{V}_{\alpha,2}\mathbf{V}'_{\alpha,2}\right)^-\mathbf{t}_2,\end{aligned}$$

it is valid that

$$\widehat{\boldsymbol{\tau}}^{(p)}=\mathbf{HX}^-\mathbf{Y}^{(p)}-\sum_{\alpha=1}^{f}\mathbf{V}_{\alpha,1}^{(p)}\mathbf{V}'_{\alpha,2}\left(\sum_{\alpha=1}^{p}\mathbf{V}_{\alpha,2}\mathbf{V}'_{\alpha,2}\right)^-\mathbf{t}_2,$$

$$\begin{aligned}E(\widehat{\boldsymbol{\tau}}^{(p)})&=E[(\mathbf{HX}^-\mathbf{Y})^{(p)}]-\sum_{\alpha=1}^{f}E[\mathbf{V}_{\alpha,1}^{(p)}]\mathbf{V}'_{\alpha,2}\left(\sum_{\alpha=1}^{f}\mathbf{V}_{\alpha,2}\mathbf{V}'_{\alpha,2}\right)^-\mathbf{t}_2\\ &=\mathbf{H}\boldsymbol{\beta}+\mathbf{\Lambda}_{1,2}\mathbf{\Lambda}_{2,2}^-\mathbf{t}_2-\sum_{\alpha=1}^{f}\mathbf{\Lambda}_{1,2}\mathbf{\Lambda}_{2,2}^-\mathbf{V}_{\alpha,2}\mathbf{V}'_{\alpha,2}\sum_{\alpha=1}^{f}(\mathbf{V}_{\alpha,2}\mathbf{V}'_{\alpha,2})^-\mathbf{t}_2\\ &=\mathbf{H}\boldsymbol{\beta},\\ \operatorname{Var}(\widehat{\boldsymbol{\tau}}^{(p)})&=\operatorname{Var}[(\mathbf{HX}^-\mathbf{Y})^{(p)}]+\sum_{\alpha=1}^{f}\operatorname{Var}\left(\mathbf{V}_{\alpha,1}^{(p)}\right)\mathbf{t}'_2\left(\sum_{\alpha=1}^{f}\mathbf{V}_{\alpha,2}\mathbf{V}'_{\alpha,2}\right)^-\mathbf{t}_2\\ &=(\mathbf{\Lambda}_{1,1}-\mathbf{\Lambda}_{1,2}\mathbf{\Lambda}_{2,2}^-\mathbf{\Lambda}_{2,1})\left(1+\frac{1}{f}\mathbf{t}'_2\widehat{\mathbf{\Lambda}}_{2,2}^-\mathbf{t}_2\right).\end{aligned}$$

As far as the matrix $\widehat{\mathbf{\Lambda}}_{11.2}^{(p)}=\widehat{\mathbf{\Lambda}}_{1,1}^{(p)}-\widehat{\mathbf{\Lambda}}_{1,2}^{(p)}(\widehat{\mathbf{\Lambda}}_{2,2})^-\widehat{\mathbf{\Lambda}}_{2,1}^{(p)}$ is concerned, it is valid that

$$f\widehat{\mathbf{\Lambda}}_{11.2}^{(p)}=\sum_{\alpha=1}^{f}\mathbf{V}_{\alpha,1}^{(p)}(\mathbf{V}_{\alpha,1}^{(p)})'-\sum_{\alpha=1}^{f}\mathbf{V}_{\alpha,1}^{(p)}\mathbf{V}'_{\alpha,2}\left(\sum_{\alpha=1}^{f}\mathbf{V}_{\alpha,2}(\mathbf{V}_{\alpha,2})'\right)^-\sum_{\alpha=1}^{f}\mathbf{V}_{\alpha,2}(\mathbf{V}_{\alpha,1}^{(p)})'$$

what means that $f\widehat{\mathbf{\Lambda}}_{11.2}^{(p)} \sim W_h[f - r(\mathbf{\Lambda}_{2,2}), \mathbf{\Lambda}_{11.2}]$. Here

$$\mathbf{\Lambda}_{11.2} = \mathbf{\Lambda}_{1,1} - \mathbf{\Lambda}_{1,2}\mathbf{\Lambda}_{2,2}^{-}\mathbf{\Lambda}_{2,1} = \mathbf{H}\big[(\mathbf{X}')_{m(\Sigma)}^{-}\big]'\mathbf{\Sigma}(\mathbf{X}')_{m(\Sigma)}^{-}\mathbf{H}'.$$

□

Corollary 1 *With respect to the preceding Lemma and the relations*

$$\begin{aligned} r(\mathbf{S}) &= r(\mathbf{\Sigma}), \\ r(\mathbf{\Lambda}_{11.2}) &= r\{\mathbf{H}[(\mathbf{X}'\tilde{\mathbf{T}}^{+}\mathbf{X})^{+} - \mathbf{I}]\mathbf{H}'\}, \\ r(\mathbf{\Lambda}_{2,2}) &= r(\mathbf{M}_X\mathbf{S}) = r(\mathbf{S},\mathbf{X}) - r(\mathbf{X}), \\ \tilde{\mathbf{T}} &= \mathbf{S} + \mathbf{X}\mathbf{X}', \end{aligned}$$

the $(1-\alpha)$*-confidence region for the vector* $\mathbf{H}\boldsymbol{\beta}$ *is*

$$\begin{aligned} \mathcal{C}_{H\beta} &= \Bigg\{\mathbf{u}\colon \mathbf{u} \in R^h, \frac{(\mathbf{u}-\widehat{\boldsymbol{\tau}})'\{\mathbf{H}[(\mathbf{X}'\tilde{\mathbf{T}}^{+}\mathbf{X})^{+} - \mathbf{I}]\mathbf{H}'\}^{+}(\mathbf{u}-\widehat{\boldsymbol{\tau}})}{1 + \frac{1}{f}\tilde{\mathbf{v}}'\mathbf{S}^{+}\tilde{\mathbf{v}}} \\ &\qquad \times \frac{f - r(\mathbf{S},\mathbf{X}) + r(\mathbf{X}) - r(\tilde{\mathbf{V}}_H) + 1}{f r(\mathbf{V}_H)} \le F_{f_1,f_2}(1-\alpha)\Bigg\}, \\ f_1 &= r(\mathbf{V}_H),\ f_2 = f - r(S,X) + r(X) - r(V_H) + 1 \end{aligned}$$

and $F_{f_1,f_2}(1-\alpha)$ *is the* $(1-\alpha)$*-quantile of the central Fisher–Snedecor distribution with* f_1 *and* f_2 *degrees of freedom.*

Here

$$\tilde{\mathbf{v}} = \mathbf{Y} - \mathbf{X}[(\mathbf{X}')_{m(S)}^{-}]'\mathbf{Y}, \quad \tilde{\mathbf{V}}_H = \mathbf{H}[(\mathbf{X}'\tilde{\mathbf{T}}^{+}\mathbf{X})^{+} - \mathbf{I}]\mathbf{H}'.$$

The proof that $\mathbf{t}_2'\widehat{\mathbf{\Lambda}}_{2,2}^{-}\mathbf{t}_2 = \tilde{\mathbf{v}}'\mathbf{S}^{+}\tilde{\mathbf{v}}$, is implied by the following consideration.

$$\begin{gathered} \mathbf{t}_2'\widehat{\mathbf{\Lambda}}_{2,2}^{-}\mathbf{t}_2 = \mathbf{Y}\mathbf{M}_X(\mathbf{M}_X\mathbf{S}\mathbf{M}_X)^{+}\mathbf{M}_X\mathbf{Y} \\ = \mathbf{Y}'[\tilde{\mathbf{T}}^{+} - \tilde{\mathbf{T}}^{+}\mathbf{X}(\mathbf{X}'\tilde{\mathbf{T}}^{+}\mathbf{X})^{+}\mathbf{X}'\tilde{\mathbf{T}}^{+}]\mathbf{Y} = \mathbf{Y}'\tilde{\mathbf{T}}^{+}[\mathbf{I} - \mathbf{X}(\mathbf{X}'\tilde{\mathbf{T}}^{+}\mathbf{X})^{+}\mathbf{X}'\tilde{\mathbf{T}}^{+}]\mathbf{Y} \\ = \mathbf{Y}'\tilde{\mathbf{T}}^{+}\tilde{\mathbf{v}} = \tilde{\mathbf{v}}'\tilde{\mathbf{T}}^{+}\tilde{\mathbf{v}} = \tilde{\mathbf{v}}'\mathbf{S}^{+}\tilde{\mathbf{v}}. \end{gathered}$$

The last equality follows from the fact that $\tilde{\mathbf{T}}^{+}$ and also $\mathbf{S}^{+}$ are g-inverses of the matrix $\mathbf{S} - \mathbf{S}(\mathbf{X}')_{m(S)}^{-}\mathbf{X}'$ and $\widehat{\mathbf{v}} \in \mathcal{M}[\mathbf{S} - \mathbf{S}(\mathbf{X}')_{m(S)}^{-}\mathbf{X}']$ with probability one.

Theorem 1 *Let in the model (4) the Wishart matrix*

$$f\mathbf{S} = \mathbf{W} \sim W_r(f,\mathbf{\Sigma}), \quad f > r(\mathbf{\Sigma},\mathbf{X}) - r(\mathbf{X}) + r[\mathrm{Var}(\widehat{\mathbf{H}_1\mathbf{B}\mathbf{h}_2})] - 1,$$

independent on the observation matrix $\underline{\mathbf{Y}}$*, be at our disposal. Then the* $(1-\alpha)$*-confidence region for the vector* $\mathbf{H}_1\mathbf{B}\mathbf{h}_2$*, where* $\mathcal{M}(\mathbf{H}_1') \subset \mathcal{M}(\mathbf{X}')$ *and* $\mathbf{h}_2 \in \mathcal{M}(\mathbf{Z})$*, is*

$$\mathcal{C}_{H_1Bh_2} = \left\{\mathbf{u}\colon \mathbf{u} \in R^n, \frac{Q_1(\mathbf{u})}{Q_2(\underline{\mathbf{Y}})}\frac{d_2}{d_1} \le F_{d_1,d_2}(1-\alpha)\right\},$$

where

$$
\begin{aligned}
Q_1(\mathbf{u}) &= (\mathbf{u} - \widehat{\mathbf{H}_1\tilde{\mathbf{B}}\mathbf{h}_2})'\{\mathbf{H}_1[(\mathbf{X}'\tilde{\mathbf{T}}^+\mathbf{X})^+ - \mathbf{I}]\mathbf{H}_1'\}^+(\mathbf{u} - \widehat{\mathbf{H}_1\tilde{\mathbf{B}}\mathbf{h}_2}),\\
Q_2(\underline{\mathbf{Y}}) &= \mathbf{h}_2'(\mathbf{Z}'\mathbf{Z})^+\mathbf{h}_2\left\{1 + \frac{1}{f}\frac{1}{\mathbf{h}_2'(\mathbf{Z}\mathbf{Z}')^+\mathbf{h}_2}\mathbf{h}_2'(\mathbf{Z}\mathbf{Z}')^+\mathbf{Z}\underline{\mathbf{Y}}'(\mathbf{M}_X\mathbf{S}\mathbf{M}_X)^+\right.\\
&\qquad \left.\times\, \underline{\mathbf{Y}}\mathbf{Z}'(\mathbf{Z}\mathbf{Z}')^+\mathbf{h}_2\right\},\\
d_1 &= r[\mathrm{Var}(\widehat{\mathbf{H}_1\mathbf{B}\mathbf{h}_2})] = r\{\mathbf{H}_1[(\mathbf{X}'\tilde{\mathbf{T}}\mathbf{X})^+ - \mathbf{I}]\mathbf{H}_1'\},\\
d_2 &= f - r(\boldsymbol{\Sigma}, \mathbf{X}) + r(\mathbf{X}) - r(\mathrm{Var}(\widehat{\mathbf{H}_1\mathbf{B}\mathbf{h}_2}) + 1,\\
\tilde{\mathbf{T}} &= \mathbf{S} + \mathbf{X}\mathbf{X}',\\
\widehat{\mathbf{H}_1\tilde{\mathbf{B}}\mathbf{h}_2} &= \mathbf{H}_1(\mathbf{X}'\tilde{\mathbf{T}}\mathbf{X})^+\mathbf{X}'\tilde{\mathbf{T}}\underline{\mathbf{Y}}\mathbf{Z}'(\mathbf{Z}\mathbf{Z}')^+\mathbf{h}_2.
\end{aligned}
$$

Proof The BLUE of the vector $\mathbf{H}_1\mathbf{B}\mathbf{h}_2$ in the model (4) is the same as in the model

$$\underline{\mathbf{Y}}\mathbf{Z}'(\mathbf{Z}\mathbf{Z}')^+\mathbf{h}_2 \sim N_n[(\mathbf{h}_2' \otimes \mathbf{X})\,\mathrm{vec}(\mathbf{B}), \mathbf{h}_2'(\mathbf{Z}\mathbf{Z}')^+\mathbf{h}_2\boldsymbol{\Sigma}],$$

i.e.

$$\widehat{\mathbf{H}_1\mathbf{B}\mathbf{h}_2} = \mathbf{H}_1(\mathbf{X}'\mathbf{T}^+\mathbf{X})^+\mathbf{X}'\mathbf{T}^+\underline{\mathbf{Y}}\mathbf{Z}'(\mathbf{Z}\mathbf{Z}')^+\mathbf{h}_2.$$

Now regarding preceding Lemma let

$$
\begin{aligned}
\mathbf{t}_2 &= \mathbf{M}_{h_2'\otimes X}\underline{\mathbf{Y}}\mathbf{Z}'(\mathbf{Z}\mathbf{Z}')^+\mathbf{h}_2\\
&= \mathbf{M}_X\underline{\mathbf{Y}}\mathbf{Z}'(\mathbf{Z}\mathbf{Z}')^+\mathbf{h}_2 \sim N_n(\mathbf{0}, \mathbf{h}_2'(\mathbf{Z}\mathbf{Z}')^+\mathbf{h}_2\mathbf{M}_X\boldsymbol{\Sigma}\mathbf{M}_X).
\end{aligned}
$$

Thus

$$
\begin{aligned}
\mathbf{t}_2'\widehat{\boldsymbol{\Lambda}}_{2,2}^-\mathbf{t}_2 &= [\mathrm{vec}(\underline{\mathbf{Y}})]'\left\{[\mathbf{Z}'(\mathbf{Z}\mathbf{Z}')^+\mathbf{h}_2] \otimes \mathbf{M}_X\right\}\frac{1 \otimes (\mathbf{M}_X\mathbf{S}\mathbf{M}_X)^+}{\mathbf{h}_2'(\mathbf{Z}\mathbf{Z}')^+\mathbf{h}_2}\\
&\quad \times \left\{[\mathbf{h}_2'(\mathbf{Z}\mathbf{Z}')^+\mathbf{Z}] \otimes \mathbf{M}_X\right\}\mathrm{vec}(\underline{\mathbf{Y}})\\
&= \frac{1}{\mathbf{h}_2'(\mathbf{Z}\mathbf{Z}')^+\mathbf{h}_2}\mathbf{h}_2'(\mathbf{Z}\mathbf{Z}')^+\mathbf{Z}\underline{\mathbf{Y}}'(\mathbf{M}_X\mathbf{S}\mathbf{M}_X)^+\underline{\mathbf{Y}}\mathbf{Z}'(\mathbf{Z}\mathbf{Z}')^+\mathbf{h}_2.
\end{aligned}
$$

Regarding Lemma and Corollary we have

$$
\begin{aligned}
&\widehat{\mathbf{H}_1\tilde{\mathbf{B}}\mathbf{h}_2}^{(p)} \sim\\
&\sim N_n\left(\mathbf{H}_1\mathbf{B}\mathbf{h}_2, \mathbf{h}_2'(\mathbf{Z}\mathbf{Z}')^+\mathbf{h}_2\left\{\mathbf{H}_1[(\mathbf{X}'\mathbf{T}^+\mathbf{X})^+ - \mathbf{I}]\mathbf{H}_1'\right\}\left\{1 + \frac{1}{f}\mathbf{t}_2'\widehat{\boldsymbol{\Lambda}}_{2,2}^-\mathbf{t}_2\right\}\right),
\end{aligned}
$$

$$\left\{\mathbf{H}_1[(\mathbf{X}'\tilde{\mathbf{T}}^+\mathbf{X})^+ - \mathbf{I}]\mathbf{H}_1'\right\}^{(p)} \sim W\left\{f - r(\boldsymbol{\Lambda}_{2,2}), \mathbf{H}_1[(\mathbf{X}'\mathbf{T}^+\mathbf{X})^+ - \mathbf{I}]\mathbf{H}_1'\right\}$$

and the proof can be easily finished. □

6 Linear hypotheses on the first order parameters (normality is assumed)

The statements of this section are proved in [12].

(i) The case of the known $\mathbf{\Sigma}$.

Model (1). The hypothesis $H_0\colon \mathbf{HB} + \mathbf{H}_0 = \mathbf{0}$ versus $H_a\colon \mathbf{HB} + \mathbf{H}_0 \neq \mathbf{0}$ can be tested if $\mathcal{M}(\mathbf{H}') \subset \mathcal{M}(\mathbf{X}')$.

Model (2). dtto

Model (3). The hypothesis $H_0\colon \mathbf{H}_1\mathbf{B}\mathbf{H}_2 + \mathbf{H}_0 = \mathbf{0}$ versus $H_a\colon \mathbf{H}_1\mathbf{B}\mathbf{H}_2 + \mathbf{H}_0 \neq \mathbf{0}$ can be tested if $\mathcal{M}(\mathbf{H}_1') \subset \mathcal{M}(\mathbf{X}')$ & $\mathcal{M}(\mathbf{H}_2) \subset \mathcal{M}(\mathbf{Z})$.

Model (4). dtto

(ii) The case $\mathbf{\Sigma} = \sigma^2\mathbf{V}$. If σ^2 is estimable, the situation is analogous as in "$\mathbf{\Sigma}$ is known".

(iii) The case $\mathbf{\Sigma} = \sum_{i=1}^{p} \vartheta_i \mathbf{V}_i$. "Plug-in" test statistics for the cases "$\mathbf{\Sigma}$ is known" exist however the sensitivity analysis is to be made. The vector $\boldsymbol{\vartheta} = (\vartheta_1, \dots, \vartheta_p)'$ must be estimable.

(iv) The case $\mathbf{\Sigma}$ is completely unknown.

Model (1). The hypothesis $H_0\colon \mathbf{h}'\mathbf{B} + \mathbf{h}_0 = \mathbf{0}$ versus $H_a\colon \mathbf{h}'\mathbf{B} + \mathbf{h}_0 \neq \mathbf{0}$ can be tested if $\mathbf{h} \in \mathcal{M}(\mathbf{X}')$. The hypothesis $H_0\colon \mathbf{H}_1\mathbf{B}\mathbf{l} + \mathbf{h}_0 = \mathbf{0}$ versus $H_a\colon \mathbf{H}_1\mathbf{B}\mathbf{l} + \mathbf{h}_0 \neq \mathbf{0}$ can be tested if $\mathcal{M}(\mathbf{H}_1') \subset \mathcal{M}(\mathbf{X}'), \mathbf{l} \notin \mathcal{K}er(\mathbf{\Sigma})$. The hypothesis $H_0\colon \mathbf{h}_1'\mathbf{B}\mathbf{H}_2 + \mathbf{h}_0' = \mathbf{0}$ versus $H_a\colon \mathbf{h}_1'\mathbf{B}\mathbf{H}_2 + \mathbf{h}_0' \neq \mathbf{0}$ can be tested if $\mathbf{h}_1 \in \mathcal{M}(\mathbf{X}')$ and $n - r(\mathbf{X}) > r(\mathbf{H}_2'\mathbf{\Sigma}\mathbf{H}_2) + 1$, where $\mathbf{H}_2$ is arbitrary.

Model (2). The matrix $\mathbf{\Sigma}$ cannot be estimated and thus test statistics cannot be constructed.

Models (3) and (4). The BLUEs of linear functions of $\mathbf{B}$ do not exist and thus test statistics cannot be constructed.

Thus in the models (2), (3) and (4) it must be assumed that a Wishart matrix with proper dimension and degrees of freedom independent of the observation matrix $\underline{\mathbf{Y}}$ is at our disposal.

7 The sensitivity approach for the case $\mathbf{\Sigma} = \sum_{i=1}^{p} \vartheta_i \mathbf{V}_i$

The statements of this section are proved in [12].

In the case $\mathbf{\Sigma} = \sum_{i=1}^{p} \vartheta_i \mathbf{V}_i$ with the unknown parameters ϑ_i, $i = 1, \dots, p$, the BLUEs cannot be determined and neither linear hypothesis can be tested, nor the $(1-\alpha)$-confidence region can be determined.

The plug-in approach can be used, i.e. statistical inference is realized with the matrix

$$\widehat{\mathbf{\Sigma}} = \sum_{i=1}^{p} \widehat{\vartheta}_i \mathbf{V}_i$$

instead the matrix $\mathbf{\Sigma}$, however some analysis is necessary. In some cases the sensitivity approach can be used, i.e. if we know that the actual value $\boldsymbol{\vartheta}^*$ of the parameter $\boldsymbol{\vartheta}$ is with sufficiently high probability in so called insensitivity region, then, e.g. the plug-in estimator is almost BLUE, or the power function of the test is destroyed less than in advance prescribed sufficiently small $\varepsilon > 0$, etc.

The only problem is to find the insensitivity region. It is to be remarked that for different statistical inference, the insensitivity regions are different. More on the sensitivity approach cf. [2], [4], [5], [15], [22], [16], [8], [9], [21], [10], [20], [17], [18], [11], [19], [13], [14].

In the following text the insensitivity regions are demonstrated for the BLUE of estimable linear functions of the parameter matrix $\mathbf{B}$.

It is to be remarked that in the model (1) the insensitivity region for the BLUE is not necessary, since estimators do not depend on the covariance matrix $\mathbf{\Sigma}$.

Let us consider the model (2). Let the matrix $\mathbf{H}$, $\mathcal{M}(\mathbf{H}') \subset \mathcal{M}(\mathbf{X}')$, be given. In the following text the symbol $\mathbf{T} = \mathbf{\Sigma} + \mathbf{X}\mathbf{X}'$ will be used.

Theorem 2 *Let in the model (2) the function* $\mathrm{Tr}(\mathbf{HB}), \mathbf{B} \in \mathcal{M}_{k\times m}$, *must be estimated* ($\mathcal{M}(\mathbf{H}') \subset \mathcal{M}(\mathbf{X}')$)*. Let* $\mathbf{W}_H$ *be* $p \times p$ *matrix with the* (i,j)*-th entry equal to*

$$\begin{aligned}\{\mathbf{W}_H\}_{i,j} &= \mathrm{Tr}\left[\mathbf{H}(\mathbf{X}'\mathbf{T}_0^-\mathbf{X})^-\mathbf{X}'\mathbf{T}_0^-\mathbf{V}_i(\mathbf{M}_X\mathbf{\Sigma}_0\mathbf{M}_X)^+\mathbf{V}_j\mathbf{T}_0^-\mathbf{X}(\mathbf{X}'\mathbf{T}_0^-\mathbf{X})^-\mathbf{H}'\right],\\ &\quad i,j = 1,\dots,p,\end{aligned}$$

where $\mathbf{\Sigma}_0 = \sum_{i=1}^p \vartheta_{0,i}\mathbf{V}_i$, $\boldsymbol{\vartheta}_0 = (\vartheta_{0,1},\dots,\vartheta_{0,p})'$ *is an approximate value of the true value* $\boldsymbol{\vartheta}^*$ *of the parameter* $\boldsymbol{\vartheta}$ *and* $\mathbf{T}_0 = \mathbf{\Sigma}_0 + \mathbf{X}\mathbf{X}'$*. Let* $\delta\boldsymbol{\vartheta} = \boldsymbol{\vartheta}^* - \boldsymbol{\vartheta}_0$*. Then*

$$\delta\boldsymbol{\vartheta} \in \mathcal{N}_H \Rightarrow \sqrt{\mathrm{Var}_{\vartheta^*}\left\{\mathrm{Tr}[\widehat{\mathbf{HB}}(\boldsymbol{\vartheta}_0)]\right\}} \le (1+\varepsilon)\sqrt{\mathrm{Var}_{\vartheta^*}\left\{\mathrm{Tr}[\widehat{\mathbf{HB}}(\boldsymbol{\vartheta}^*)]\right\}},$$

where

$$\begin{aligned}\mathrm{Tr}\left[\widehat{\mathbf{HB}}(\boldsymbol{\vartheta}_0)\right] &= \mathrm{Tr}\left[\mathbf{H}(\mathbf{X}'\mathbf{T}_0^-\mathbf{X})^-\mathbf{X}'\mathbf{T}_0^-\underline{\mathbf{Y}}\right],\\ \mathrm{Var}_{\vartheta^*}\left\{\mathrm{Tr}[\widehat{\mathbf{HB}}(\boldsymbol{\vartheta}_0)]\right\} &= \mathrm{Tr}\left[\mathbf{H}(\mathbf{X}'\mathbf{T}_0^-\mathbf{X})^-\mathbf{X}'\mathbf{T}_0^-\mathbf{T}^*\mathbf{T}_0^-\mathbf{X}(\mathbf{X}'\mathbf{T}_0^-\mathbf{X})^-\mathbf{H}'\right]\\ &\quad - \mathrm{Tr}(\mathbf{H}\mathbf{H}'),\\ \mathrm{Tr}\left[\widehat{\mathbf{HB}}(\boldsymbol{\vartheta}^*)\right] &= \mathrm{Tr}\left\{\mathbf{H}\left[\mathbf{X}'(\mathbf{T}^*)^-\mathbf{X}\right]^-\mathbf{X}'(\mathbf{T}^*)^-\underline{\mathbf{Y}}\right\},\\ \mathrm{Var}_{\vartheta^*}\left\{\mathrm{Tr}[\widehat{\mathbf{HB}}(\boldsymbol{\vartheta}^*)]\right\} &= \mathrm{Tr}\left\{\mathbf{H}\left[\mathbf{X}'(\mathbf{T}^*)^-\mathbf{X}\right]^-\mathbf{H}'\right] - \mathrm{Tr}(\mathbf{H}\mathbf{H}'),\end{aligned}$$

$\varepsilon > 0$ *is sufficiently small real number and*

$$\mathcal{N}_H = \left\{\delta\boldsymbol{\vartheta} \colon \delta\boldsymbol{\vartheta}'\mathbf{W}_H\delta\boldsymbol{\vartheta} \le (2\varepsilon+\varepsilon^2)\,\mathrm{Var}_{\vartheta^*}\left\{\mathrm{Tr}[\widehat{\mathbf{HB}}(\boldsymbol{\vartheta}^*)]\right\}\right\}.$$

In the model (3) we obtain analogously the following theorem.

Theorem 3 *Let in the model (3) the function* $\mathrm{Tr}(\mathbf{H}_1\mathbf{B}\mathbf{H}_2), \mathbf{B} \in \mathcal{M}_{k\times m}$ *must be estimated. The matrices* $\mathbf{H}_1, \mathbf{H}_2$ *satisfy the conditions* $\mathcal{M}(\mathbf{H}_1') \subset \mathcal{M}(\mathbf{X}')$

and $\mathcal{M}(\mathbf{H}_2) \subset \mathcal{M}(\mathbf{Z})$*. Let the matrix* $\mathbf{W}_{H_1BH_2}$ *be defined as follows*

$$\begin{aligned}
&\{\mathbf{W}_{H_1BH_2}\}_{i,j} = \\
&= \mathrm{Tr}\left(\mathbf{H}_2'\left[\mathbf{Z}(\boldsymbol{\Sigma}_0 + \mathbf{Z}'\mathbf{Z})^-\mathbf{Z}'\right]\mathbf{Z}(\boldsymbol{\Sigma}_0 + \mathbf{Z}'\mathbf{Z})^-\mathbf{V}_j(\mathbf{M}_{Z'}\boldsymbol{\Sigma}_0\mathbf{M}_{Z'})^+\mathbf{V}_i \right.\\
&\quad \left. \times (\boldsymbol{\Sigma}_0 + \mathbf{Z}'\mathbf{Z})^-\mathbf{Z}'\left[\mathbf{Z}(\boldsymbol{\Sigma}_0 + \mathbf{Z}'\mathbf{Z})^-\mathbf{Z}'\right]\mathbf{H}_2\mathbf{H}_1(\mathbf{X}'\mathbf{X})^-\mathbf{H}_1'\right),\\
&\quad i,j = 1,\dots,p,
\end{aligned}$$

where $\boldsymbol{\Sigma}_0 = \sum_{i=1}^p \vartheta_{0,i}\mathbf{V}_i$, $\boldsymbol{\vartheta}_0 = (\vartheta_{0,1},\dots,\vartheta_{0,p})'$ *is an approximate value of the true value* $\boldsymbol{\vartheta}^*$ *of the parameter* $\boldsymbol{\vartheta}$*. Then*

$$\begin{aligned}
&\delta\boldsymbol{\vartheta} \in \mathcal{N}_{H_1BH_2}\\
&\quad \Rightarrow \sqrt{\mathrm{Var}_{\vartheta^*}\left\{\mathrm{Tr}[\widehat{\mathbf{H}_1\mathbf{B}\mathbf{H}_2}(\boldsymbol{\vartheta}_0)]\right\}} \le (1+\varepsilon)\sqrt{\mathrm{Var}_{\vartheta^*}\left\{\mathrm{Tr}[\widehat{\mathbf{H}_1\mathbf{B}\mathbf{H}_2}(\boldsymbol{\vartheta}^*)]\right\}},
\end{aligned}$$

where $\varepsilon > 0$ *is sufficiently small real number and*

$$\mathcal{N}_{H_1BH_2} = \left\{\delta\boldsymbol{\vartheta}\colon \delta\boldsymbol{\vartheta}'\mathbf{W}_{H_1BH_2}\delta\boldsymbol{\vartheta} \le (2\varepsilon + \varepsilon^2)\,\mathrm{Var}_{\vartheta^*}\left\{\mathrm{Tr}[\widehat{\mathbf{H}_1\mathbf{B}\mathbf{H}_2}(\boldsymbol{\vartheta}^*)]\right\}\right\}.$$

In the following example it is shown the insensitivity region for the estimator of a linear function of the matrix $\mathbf{B}$ in the model (4), which is used in the paper [2].

Theorem 4 *Let in the model (4)*

$$\mathrm{vec}(\underline{\mathbf{Y}}) \sim_{nr} \left[(\mathbf{Z}' \otimes \mathbf{X})\mathrm{vec}(\mathbf{B}), \mathbf{I} \otimes \sum_{i=1}^p \vartheta_i\mathbf{V}_i\right],$$

the function $\mathrm{Tr}(\mathbf{H}_1\mathbf{B}\mathbf{H}_2), \mathbf{B} \in \mathcal{M}_{k\times m}$ *must be estimated. The matrices* $\mathbf{H}_1, \mathbf{H}_2$ *satisfy the conditions* $\mathcal{M}(\mathbf{H}_1') \subset \mathcal{M}(\mathbf{X}')$ *and* $\mathbf{H}_2, \mathcal{M}(\mathbf{H}_2) \subset \mathcal{M}(\mathbf{Z})$ *(i.e. the matrix function of the parameter matrix* $\mathbf{B}$ *is unbiasedly estimable).*

Let the matrix $\mathbf{W}_{H_1BH_2}$ *be defined as follows*

$$\begin{aligned}
\left\{\mathbf{W}_{H_1BH_2}\right\}_{i,j} &= \mathrm{Tr}\left[\mathbf{H}_1(\mathbf{X}'\mathbf{T}_0\mathbf{X})^-\mathbf{X}'\mathbf{T}_0^-\mathbf{V}_i(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\mathbf{V}_j\mathbf{T}_0^-\mathbf{X}\right.\\
&\quad \left.\times (\mathbf{X}'\mathbf{T}_0\mathbf{X})^-\mathbf{H}_1'\mathbf{H}_2'(\mathbf{Z}\mathbf{Z}')^-\mathbf{H}_2\right], \quad i,j = 1,\dots,p,
\end{aligned}$$

where $\boldsymbol{\Sigma}_0 = \sum_{i=1}^p \vartheta_{0,i}\mathbf{V}_i$, $\boldsymbol{\vartheta}_0 = (\vartheta_{0,1},\dots,\vartheta_{0,p})'$ *is an approximate value of the true value* $\boldsymbol{\vartheta}^*$ *of the parameter* $\boldsymbol{\vartheta}$ *and* $\mathbf{T}_0 = \boldsymbol{\Sigma}_0 + \mathbf{X}\mathbf{X}'$*. Let* $\delta\boldsymbol{\vartheta} = \boldsymbol{\vartheta}^* - \boldsymbol{\vartheta}_0$*. Then*

$$\begin{aligned}
&\delta\boldsymbol{\vartheta} \in \mathcal{N}_{H_1BH_2}\\
&\quad \Rightarrow \sqrt{\mathrm{Var}_{\vartheta^*}\left\{\mathrm{Tr}[\widehat{\mathbf{H}_1\mathbf{B}\mathbf{H}_2}(\boldsymbol{\vartheta}_0)]\right\}} \le (1+\varepsilon)\sqrt{\mathrm{Var}_{\vartheta^*}\left\{\mathrm{Tr}[\widehat{\mathbf{H}_1\mathbf{B}\mathbf{H}_2}(\boldsymbol{\vartheta}^*)]\right\}},
\end{aligned}$$

where $\varepsilon > 0$ *is sufficiently small real number and the insensitivity region is*

$$\mathcal{N}_{H_1BH_2} = \left\{\delta\boldsymbol{\vartheta}\colon \delta\boldsymbol{\vartheta}'\mathbf{W}_{H_1BH_2}\delta\boldsymbol{\vartheta} \le (2\varepsilon + \varepsilon^2)\,\mathrm{Var}_{\vartheta^*}\left\{\mathrm{Tr}[\widehat{\mathbf{H}_1\mathbf{B}\mathbf{H}_2}(\boldsymbol{\vartheta}^*)]\right\}\right\}.$$

Conclusion

Basic statistical problems, i.e. estimation, determination of confidence regions and testing statistical hypotheses cannot be sometimes solved by standard procedures. It is important to recognize it before realizing experiments which must be evaluated in a statistical way, since an effort and financial investment necessary for an experiment can be lost. If the situation is recognized in advance, then it is possible either to realize some auxiliary experiment, or to gain some additional information, or to find some other way how to make statistical problems solvable.

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