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On Geodesic Mappings Preserving the Einstein Tensor*

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Abstract

In this paper there are discussed the geodesic mappings which preserved the Einstein tensor. We proved that the tensor of concircular curvature is invariant under Einstein tensor-preserving geodesic mappings.

Key words: Geodesic mapping, Einstein tensor.

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1 Introduction

From the very beginning, the theory of geodesic mappings attracted attention by a wide scale of possibilities for applications, not only in geometry itself, but also as a useful tool of modeling various processes in mechanics and physics.

If we distinguish some class of mappings between spaces from a fixed class, a natural questions arises, what objects and properties of spaces are preserved, invariant, under all mappings under consideration.

As far as invariant objects under geodesic mappings are concerned, let us mention Thomas' parameters and the Weyl tensor of projective curvature. To mention some invariant properties, note that the class of spaces of constant curvature and the class of Einstein spaces are closed under geodesic mappings.

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In this paper, we examine nontrivial geodesic mappings of pseudo-Riemannian spaces preserving the Einstein tensor. We prove that the tensor of concircular curvature is an invariant of geodesic mappings. Further, we examine some geometric properties of such spaces.

2 Basic concepts

A diffeomorphism (or a bijection of sufficiently high differentiability class) between two pseudo-Riemannian spaces V_n and $\bar{V}_n$, equipped with a metric tensor g and $\bar{g}$, respectively, is called *geodesic* if it is geodesic-preserving, that is, when it maps any geodesic of V_n into an arbitrarily parametrized geodesic of $\bar{V}_n$ again.

A necessary and sufficient condition for existence of a geodesic mapping between V_n and $\bar{V}_n$ is that the conditions [3, 5, 6, 7]

$$\bar{\Gamma}^h_{ij} = \Gamma^h_{ij} + \psi_i \delta^h_j + \psi_j \delta^h_i, \tag{1}$$

are satisfied where Γ^h_{ij} and $(\bar{\Gamma}^h_{ij})$ are components of the Christoffel symbols in V_n and $\bar{V}_n$, respectively (the object in $\bar{V}_n$ corresponding to the given object under a geodesic mapping will be denoted by bar), δ^h_i are Kronecker symbols.

The condition (1) is equivalent of the following Levi-Civita equation [3, 5, 6, 7]

$$\bar{g}_{ij},_k = 2\psi_k \bar{g}_{ij} + \psi_i \bar{g}_{jk} + \psi_j \bar{g}_{ik} \tag{2}$$

where “,” denotes a covariant derivative in V_n, ψ_i is some gradient-like vector, i.e. $\psi_i = \psi_{,i}$.

If $\psi_i \neq 0$ holds, the mapping is called a *non-trivial geodesic mapping.* The following conditions are necessary for a geodesic mapping:

$$\bar{R}^h_{ijk} = R^h_{ijk} + \psi_{ij} \delta^h_k - \psi_{ik} \delta^h_j, \tag{3}$$

$$\bar{R}_{ij} = R_{ij} + (n-1)\psi_{ij}, \tag{4}$$

Here R^h_{ijk} is the Riemannian curvature tensor, R_{ij} is the Ricci tensor, and $\psi_{ij} = \psi_{i,j} - \psi_i \psi_j$.

On the other hand, necessary and sufficient condition for existence of non-trivial geodesic mappings of the given pseudo-Riemannian space onto pseudo-Riemannian spaces is existence of a solution for the system of equations [3, 7]

$$a_{ij,k} = \lambda_i g_{jk} + \lambda_j g_{ik}, \tag{5}$$

$$n\lambda_{i,j} = \mu g_{ij} + a_{\alpha i} R^\alpha_j - a_{\alpha\beta} R^{\alpha}_{\cdot\, ij\, \cdot}{}^{\beta}, \tag{6}$$

$$(n-1)\mu_{,k} = 2(n+1)\lambda_\alpha R^\alpha_k + a_{\alpha\beta}(2R^{\alpha}_{\cdot\, k, \cdot}{}^{\beta} - R^{\alpha\beta}_{\cdot\,\cdot\,,k}) \tag{7}$$

with respect to a regular symmetric tensor a_{ij}, a co-vector $\lambda_i \neq 0$ and a function μ. Here $R^i_j = R_{\alpha j} g^{\alpha i}$; $R^{k}_{\cdot\, ij\, \cdot}{}^{h} = R_{\alpha i j \beta} g^{\alpha k} g^{\beta h}$; $R^{ij}_{\cdot\,\cdot\,,k} = R_{\alpha\beta,k} g^{\alpha i} g^{\beta j}$; $R^{i}_{\cdot j, \cdot}{}^{k} = R_{\alpha j,\beta} g^{\alpha i} g^{\beta k}$; g^{ij} are elements of the matrix inverse to g_{ij}.

According to the known solutions of the above system of differential equations the metrics of the resulting image spaces under geodesic mappings can be determined from the equations [3, 7]:

$$a_{ij} = e^{2\psi}\bar{g}^{\alpha\beta}g_{\alpha i}g_{\beta j}; \tag{8}$$

$$\lambda_i = -e^{2\psi}\psi_\alpha\bar{g}^{\alpha\beta}g_{\beta i}. \tag{9}$$

The important invariants under geodesic mappings are the Thomas' parameters

$$\bar{T}^h_{ij} = T^h_{ij}; \quad T^h_{ij} = \Gamma^h_{ij} - \frac{1}{n-1}(\delta^h_i\Gamma^\alpha_{j\alpha} + \delta^h_j\Gamma^\alpha_{i\alpha}) \tag{10}$$

and the Weyl tensor of projective curvature

$$\bar{W}^h_{ijk} = W^h_{ijk}; \quad W^h_{ijk} = R^h_{ijk} - \frac{1}{n-1}(\delta^h_k R_{ij} - \delta^h_j R_{ik}). \tag{11}$$

3 Basic equations for Einstein tensor-preserving geodesic mappings

We call a geodesic mapping *Einstein tensor-preserving* if it satisfies:

$$\bar{E}_{ij} = E_{ij}, \tag{12}$$

where

$$E_{ij} = R_{ij} - \frac{R}{n}g_{ij} \tag{13}$$

is the Einstein tensor and $R = R_{\alpha\beta}g^{\alpha\beta}$ is the scalar curvature.

If this is the case, the deformation tensor for the Ricci tensor takes the form:

$$T_{ij} = \bar{R}_{ij} - R_{ij} = \frac{\bar{R}}{n}\bar{g}_{ij} - \frac{R}{n}g_{ij}. \tag{14}$$

On the other hand, accounting (4) we obtain

$$T_{ij} = \bar{R}_{ij} - R_{ij} = (n-1)\psi_{ij}. \tag{15}$$

Comparing we get:

$$\psi_{ij} = \frac{\bar{R}}{n(n-1)}\bar{g}_{ij} - \frac{R}{n(n-1)}g_{ij}. \tag{16}$$

Substituting the last expression into (3) and using the notation

$$Y^h_{ijk} = R^h_{ijk} - \frac{R}{n(n-1)}(\delta^h_k g_{ij} - \delta^h_j g_{ik}) \tag{17}$$

(and similarly with bar) we find

$$\bar{Y}^h_{ijk} = Y^h_{ijk}. \tag{18}$$

Here Y^h_{ijk} are components of the tensor of *concircular curvature* [3, 5, 6, 7, 8].

Hence we have proved:

Theorem 1 *The tensor of concircular curvature is invariant under Einstein tensor-preserving geodesic mappings.*

Let us apply covariant differentiation to the formula (9):

$$\lambda_{i,j} = -e^{2\psi}\psi_{\alpha,j}\,\bar{g}^{\alpha\beta}g_{\beta i} + e^{2\psi}\psi_\alpha\psi_\beta\bar{g}^{\alpha\beta}g_{ji} + e^{2\psi}\psi_j\psi_\alpha\bar{g}^{\alpha\beta}g_{\beta i}. \tag{19}$$

By (8) and (16), we get

$$\lambda_{i,j} = \mu g_{ij} + \frac{R}{n(n-1)}a_{ij}, \tag{20}$$

where

$$\mu = e^{2\psi}\left(\psi_\alpha\psi_\beta\bar{g}^{\alpha\beta} - \frac{\bar{R}}{n(n-1)}\right). \tag{21}$$

Obviously using (8), (9), from (19) and (20) we get (16), and consequently also (12), hence we have proved:

Theorem 2 *A pseudo-Riemannian space admits an Einstein tensor-preserving geodesic mapping if and only if the conditions (5), (20) and (21) are satisfied.*

We say that a pseudo-Riemannian space V_n *belongs to the class* $V_n(B)$ if it admits a geodesic mapping and the corresponding vector satisfies [3, 4, 5, 6]

$$\lambda_{i,j} = \mu g_{ij} + B a_{ij} \tag{22}$$

for some function B.

So we have actually proved that a pseudo-Riemannian space V_n admitting Einstein tensor-preserving geodesic mappings belongs to the class $V_n(B)$ where $B = \frac{R}{n(n-1)}$.

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