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PLANAR FLOWS OF INCOMPRESSIBLE HEAT-CONDUCTING SHEAR-THINNING FLUIDS—EXISTENCE ANALYSIS*

Miroslav Bulíček, Oldřich Ulrych, Praha

Dedicated to Professor K. R. Rajagopal on the occasion of his 60th birthday

Abstract. We study the flow of an incompressible homogeneous fluid whose material coefficients depend on the temperature and the shear-rate. For large class of models we establish the existence of a suitable weak solution for two-dimensional flows of fluid in a bounded domain. The proof relies on the reconstruction of the globally integrable pressure, available due to considered Navier's slip boundary conditions, and on the so-called L^∞-truncation method, used to obtain the strong convergence of the velocity gradient. The important point of the approach consists in the choice of an appropriate form of the balance of energy.

Keywords: heat-conducting fluid, non-Newtonian fluid, shear-thinning fluid, existence, weak solution, suitable weak solution, L^∞-truncation method, balance of energy

MSC 2010: 35Q30, 35Q80, 76D03, 76A05

1. Introduction

This paper focuses on the existence analysis for an unsteady flow of an incompressible homogeneous heat-conducting non-Newtonian fluid in a bounded two-dimensional Lipschitz domain $\Omega \subset \mathbb{R}^2$. Such a flow is governed by the following system of partial differential equations (PDE's):

$$\operatorname{div} \boldsymbol{v} = 0, \tag{1.1}$$

$$\boldsymbol{v}_{,t} + \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v}) - \operatorname{div} \mathbf{S} = -\nabla p + \boldsymbol{f}, \tag{1.2}$$

$$E_{,t} + \operatorname{div}(\boldsymbol{v}(E + p)) - \operatorname{div}(\mathbf{S}\boldsymbol{v} - \boldsymbol{q}) = \boldsymbol{f} \cdot \boldsymbol{v}, \tag{1.3}$$

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which is supposed to be satisfied in $Q := (0,T)\times\Omega$, where $T > 0$ denotes the length of time interval. In (1.1)–(1.3), $\boldsymbol{v}\colon Q \to \mathbb{R}^2$ denotes the velocity of the fluid; $p\colon Q \to \mathbb{R}$ is the pressure; $\mathbf{S}\colon Q \to \mathbb{R}^{2\times 2}_{\mathrm{sym}}$ stands for the constitutively determined (deviatoric) part of the Cauchy stress; $E\colon Q \to \mathbb{R}$ denotes the density of the total energy; $\boldsymbol{q}\colon Q \to \mathbb{R}^2$ is the heat flux and finally $\boldsymbol{f}\colon Q \to \mathbb{R}^2$ is the density of the external body forces. For simplicity (but without any essential changes in the proof) we assume that $\boldsymbol{f} \equiv \mathbf{0}$ in what follows. Note that (1.1) represents the incompressibility constraint that is, for the homogeneous fluid considered, equivalent to the balance of mass, (1.2) is the balance of linear momentum and (1.3) expresses the balance of energy.

To formulate an appropriate initial-boundary value problem relevant to (1.1)–(1.3) we have to set up the initial and boundary conditions. For the initial data we assume that

$$\boldsymbol{v}(0,x) = \boldsymbol{v}_0(x) \quad \text{and} \quad E(0,x) = E_0(x) \quad \text{for } x \in \Omega. \tag{1.4}$$

For simplicity, we set $\Gamma := (0,T) \times \partial\Omega$ and assume that the boundary data for the velocity field are given by the so-called Navier's slip boundary conditions, i.e., we assume that for some $\gamma \in [0,1)$

$$\boldsymbol{v}\cdot\boldsymbol{n} = 0 \quad \text{and} \quad \gamma\boldsymbol{v}_\tau + (1-\gamma)(\mathbf{S}\boldsymbol{n})_\tau = \mathbf{0} \quad \text{on } \Gamma. \tag{1.5}$$

Finally, we assume that there is no input or output of the energy through the boundary. Thus

$$(\mathbf{S}\boldsymbol{v} - \boldsymbol{q})\cdot\boldsymbol{n} = 0 \quad \text{on } \Gamma. \tag{1.6}$$

In (1.5)–(1.6), $\boldsymbol{n}$ denotes the unit outward normal vector to $\partial\Omega$, and $\boldsymbol{w}_\tau$ is the projection of any vector $\boldsymbol{w}$ to the tangent line at the considered point of the boundary. The boundary conditions (1.5) describe an internal flow (no outflow/inflow is allowed) that slips along the boundary according to $(1.5)_2$. In fact, $(1.5)_2$ states that the velocity of the fluid is proportional to the tangent component of the traction on the boundary $\partial\Omega$. Note that setting $\gamma = 0$ in $(1.5)_2$ we obtain the no-stick (or slip) boundary conditions and setting $\gamma = 1$ we obtain the no-slip boundary conditions for the velocity. It should be mentioned at the very beginning that while the case $\gamma \in [0,1)$ is included in our analysis, the case $\gamma = 1$ is not covered since we are not able to construct the globally integrable pressure p, which seems to be natural requirement in order to make an appropriate weak formuation of (1.3) meaningful. To simplify further notation, we also define a slip parameter $\alpha \geqslant 0$ as

$$\alpha := \frac{\gamma}{1-\gamma}.$$

Regarding the boundary condition relevant to the balance of energy we consider (1.6), since it guarantees that the total energy is a conserved quantity. Moreover, such a condition is also the most natural one if one deals with steady problems, as explained in [7]. On the other hand, from the point of view of the existence analysis of unsteady flows, the condition (1.6) can be relaxed and we are also able to treat the thermally isolated part of the boundary ($\boldsymbol{q} \cdot \boldsymbol{n} = 0$) or Dirichlet boundary conditions for the temperature, see [6] for details.

Next, using (1.5)–(1.6) we can deduce the boundary condition for $\boldsymbol{q} \cdot \boldsymbol{n}$ on Γ. Indeed, using the symmetry of $\mathbf{S}$ and the fact that $\boldsymbol{v} \cdot \boldsymbol{n} = 0$ on Γ, we observe that $(\mathbf{S}\boldsymbol{v}) \cdot \boldsymbol{n} = (\mathbf{S}\boldsymbol{n})_\tau \cdot \boldsymbol{v}$ on Γ. Consequently, using (1.5)–(1.6) we find that

$$\boldsymbol{q} \cdot \boldsymbol{n} = -\alpha |\boldsymbol{v}|^2 \quad \text{on } \Gamma. \tag{1.7}$$

Further, we describe the constitutive relations for $\mathbf{S}$, E, $\boldsymbol{q}$ we are interested in. First of all, the density of the total energy E is given as the sum of the kinetic and the internal energy

$$E := \frac{1}{2}|\boldsymbol{v}|^2 + e, \tag{1.8}$$

where $e\colon Q \to \mathbb{R}_+$ is the density of the internal energy. For the heat flux $\boldsymbol{q}$ we consider the generalized Fourier law

$$\boldsymbol{q} = \hat{\boldsymbol{q}}(\theta, \mathbf{D}(\boldsymbol{v}), \nabla\theta) = -\hat{\kappa}(\theta, |\mathbf{D}(\boldsymbol{v})|)\nabla\theta; \tag{1.9}$$

here $\hat{\kappa}\colon \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ denotes the heat conductivity of the fluid and $\theta\colon Q \to \mathbb{R}_+$ is the temperature. We also employ the notation for the symmetric part of the velocity gradient $\mathbf{D}(\boldsymbol{v}) := \frac{1}{2}(\nabla\boldsymbol{v} + (\nabla\boldsymbol{v})^T)$. Note that the flux $\boldsymbol{q}$ is allowed to be dependent on the shear rate $|\mathbf{D}(\boldsymbol{v})|$, we refer to [6] for more details concerning the physical importance of such a setting.

In order to cover a general class of fluids without making any other assumptions on the heat capacity of the fluid, we prefer to reformulate (1.9) in such a way that $\boldsymbol{q}$ is a function of the internal energy and its gradient rather than of the temperature and the temperature gradient. To do so, we assume that the internal energy e is a smooth function of θ with a smooth inverse,[1] i.e.

$$e = \tilde{e}(\theta) \quad \text{and} \quad \theta = \tilde{\theta}(e),$$

[1] The assumption that e is an invertible function of θ is valid for most fluids, since the energy e is usually a strictly increasing function of the temperature.

where $\tilde{\theta} = (\tilde{e})^{-1}$. Using this assumption, we can view e as a primitive quantity. Then (1.9) can be rewritten as

$$\boldsymbol{q} = \boldsymbol{q}^*(e, \mathbf{D}(\boldsymbol{v}), \nabla e) = -\kappa^*(e, |\mathbf{D}(\boldsymbol{v})|)\nabla e, \tag{1.10}$$

where we set

$$\kappa^*(e, |\mathbf{D}(\boldsymbol{v})|) := \hat{\kappa}(\tilde{\theta}(e), |\mathbf{D}(\boldsymbol{v})|)\tilde{\theta}'(e). \tag{1.11}$$

We will also assume that the constitutively determined part of the Cauchy stress $\mathbf{S}$ depends on the internal energy (to be precise, on the temperature that is however a function of the energy) and on the symmetric part of the velocity gradient:

$$\mathbf{S} = \mathbf{S}^*(e, \mathbf{D}(\boldsymbol{v})). \tag{1.12}$$

Our main interest is to develop an existence theory for models where the relations (1.10) and (1.12) are of the form

$$\begin{aligned} \mathbf{S}^*(e, \mathbf{D}) &= \nu_0(e)(1 + \nu_1(e) + |\mathbf{D}|^2)^{(r-2)/2}\mathbf{D}, \\ \boldsymbol{q}^*(e, \mathbf{D}, \nabla e) &= -\kappa_0(e, \mathbf{D})e^{\beta}\nabla e, \end{aligned} \tag{1.13}$$

whereas the model parameters r and β fulfil

$$r \in (1, 2) \quad \text{and} \quad \beta \leqslant 0, \tag{1.14}$$

and ν_0, ν_1, and κ_0 are continuous bounded functions that are bounded from below by a positive constant, say ε. There are three main reasons why we focus on the range of model parameters fulfilling (1.14). First, if r and β fulfil (1.14) then $\boldsymbol{v}$ is not an admissible test function in the weak formulation of (1.2) and consequently the chosen form (1.3) of the balance of enery is essential for our analysis. We discuss this point below in detail. Second, (1.13)–(1.14) covers models interesting from the point of view of their applications in various areas of science such as geophysics, bioengineering, material science, food industry, coloid mechanics, etc., see [6], [17], [18] for details. Third, the model characterized by the constitutive equations (1.13) satisfies the assumptions of the paper given in Section 2 and we are able to develop large data existence theory for all $r > \frac{3}{2}$ and $\beta > -\frac{1}{2}$, see assumptions of Theorem 2.1, where the range for β can be even relaxed depending on r.

This paper extends the paper [6] where the authors establish similar analysis to the same problem in three space dimensions. The results established here give a theoretical background to numerical analysis and computer experiments in two-dimensional setting. We are interested in verifying the effect of the chosen form of the energy balance on the stability of computer simulations. While such experiments, for models described above, are very demanding in three-dimensional setting, the two-dimensional case can be handled more easily. We aim at analyzing the relevant numerical approaches for planar flows in a forthcoming study.

At this point, we discuss the chosen form of the balance of energy, the formulation of the entropy inequality and its equivalence to the energy inequality, and we also introduce the notion of the suitable weak solution to the generalized Navier-Stokes-Fourier system (see [6], [5] for details). First, we take the scalar product of (1.2) with $\boldsymbol{v}$ and subtract the result from (1.3) to obtain the balance of energy in the form

$$e_t + \operatorname{div}(e\boldsymbol{v}) + \operatorname{div}\boldsymbol{q} = \mathbf{S}\cdot\mathbf{D}(\boldsymbol{v}). \tag{1.15}$$

However, as we only deal with weak solutions in general, such a procedure is rigorous if $\boldsymbol{v}$ is an admissible test function in a weak formulation of (1.2). If this is not the case, then the validity of (1.15) is open in general and one is forced to replace the equality (1.15) by the inequality

$$e_t + \operatorname{div}(e\boldsymbol{v}) + \operatorname{div}\boldsymbol{q} \geqslant \mathbf{S}\cdot\mathbf{D}(\boldsymbol{v}). \tag{1.16}$$

To justify the possible inequality sign in (1.16), we show and it is interesting to show that any (weak) solution to (1.1)–(1.3) and (1.16) can be considered the physical one. If the entropy S, the internal energy e and the temperature θ are related through the condition

$$\frac{1}{\theta} := \frac{\partial S}{\partial e},$$

it is of interest to observe that (1.16) is in fact equivalent to the entropy inequality provided the temperature is strictly positive. Indeed, multiplying (1.16) by a positive quantity $1/\theta$ and using (1.1) leads to

$$S_{,t} + \operatorname{div}(S\boldsymbol{v}) + \operatorname{div}\Big(\frac{\boldsymbol{q}}{\theta}\Big) \geqslant \frac{1}{\theta}\Big(\mathbf{S}\cdot\mathbf{D}(\boldsymbol{v}) - \frac{\boldsymbol{q}\cdot\nabla\theta}{\theta}\Big) \quad (\geqslant 0), \tag{1.17}$$

which is nothing else than the entropy inequality. Although this step was again formal, it can be deduced rigorously since $1/\theta$ will be always a possible test function in (1.16) for the class of fluids we are interested in.

On the other hand, subtracting (1.16) from (1.3), we obtain the inequality

$$(|\boldsymbol{v}|^2)_{,t} + \operatorname{div}(\boldsymbol{v}(|\boldsymbol{v}|^2 + 2p)) - 2\operatorname{div}(\mathbf{S}\boldsymbol{v}) + 2\mathbf{S}\cdot\mathbf{D}(\boldsymbol{v}) \leqslant 2\boldsymbol{f}\cdot\boldsymbol{v}. \tag{1.18}$$

Note that weak solutions to (1.1)–(1.2) satisfying also (1.18) are called suitable weak solutions, as introduced in [9], where the inequality (1.18) plays the important role in proving partial regularity results for suitable weak solutions to three-dimensional Navier-Stokes equations. Thus, having the equivalence of (1.18) and (1.16), it is natural to call a weak solution the *suitable* one if it solves (1.1)–(1.3) in a weak sense and if it in addition satisfies in a weak sense the energy inequality (1.16) (that is equivalent to the entropy inequality (1.17)). Therefore, our goal in this paper is to find $(\boldsymbol{v}, p, e, \mathbf{S}, \boldsymbol{q}, E)$ that solve in a weak sense (1.1)–(1.6) and (1.16), and solve (1.8), (1.10), and (1.12) pointwise in Q. Such $(\boldsymbol{v}, p, e, \mathbf{S}, \boldsymbol{q}, E)$ we call the suitable weak solution.

The structure of the paper is the following. In Section 2 we introduce the structural assumptions on $\mathbf{S}^*$ and κ^* that are considered in the paper. Then we explain the notation and formulate the main existence theorem of the paper. Next, in Section 3 we recall several important tools and auxiliary results used in the proof of the main theorem. Then in Section 4 we formally derive *a priori* estimates on $(\boldsymbol{v}, p, e, \boldsymbol{q}, \mathbf{S})$ and show that the weak formulation introduced in the main theorem is really meaningful. Finally, in Section 5 we prove the main theorem. We mainly follow the paper [6], where the three-dimensional case is treated but with more restrictive assumptions on $\mathbf{S}^*$ and $\boldsymbol{q}^*$.

2. Assumptions on $\mathbf{S}^*$ and κ^*

In the paper we assume that $\mathbf{S}^*\colon \mathbb{R}_+ \times \mathbb{R}^{2\times 2}_{\mathrm{sym}} \to \mathbb{R}^{2\times 2}_{\mathrm{sym}}$ and $\kappa^*\colon \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ are continuous mappings. Moreover, we assume that there are $r \in (1, \infty)$ and $\beta \in \mathbb{R}$ such that for all $e \in \mathbb{R}_+$ and all $\mathbf{B}, \mathbf{D} \in \mathbb{R}^{2\times 2}_{\mathrm{sym}}$, $\mathbf{B} \neq \mathbf{D}$,

$$C_1(|\mathbf{D}|^r - 1) \leqslant \mathbf{S}^*(e, \mathbf{D}) \cdot \mathbf{D}, \qquad |\mathbf{S}^*(e, \mathbf{D})| \leqslant C_2(|\mathbf{D}|^{r-1} + 1), \tag{2.1}$$

$$[\mathbf{S}^*(e, \mathbf{D}) - \mathbf{S}^*(e, \mathbf{B})] \cdot (\mathbf{D} - \mathbf{B}) > 0, \tag{2.2}$$

$$C_3 e^{\beta} \leqslant \kappa^*(e, |\mathbf{D}|) \leqslant C_4 e^{\beta}, \tag{2.3}$$

i.e., we assume that $\mathbf{S}^*$ is r-coercive, has $(r-1)$-growth and is strictly monotone w.r.t. $\mathbf{D}$. Since we will always have $e \geqslant e_{\min} > 0$, where $e_{\min}$ is the minimum (infimum) of the initial internal energy e_0, we can relax the assumptions (2.1)–(2.3) and consider them to be valid only for $e \geqslant e_{\min} > 0$. Note that the examples (1.13) satisfy (2.1)–(2.3). We also refer to [6], [5], [12], [18], [17] for other models of $\mathbf{S}^*$ and $\boldsymbol{q}^*$ satisfying (2.1)–(2.3).

Next, we introduce a notation of function spaces that is suitable for describing the problem (1.1)–(1.3) and (1.16). We use the standard notation for Sobolev, Lebesgue, and Bochner spaces. For the vector-valued functions having zero normal part on the boundary we employ the notation[2]

$$
\begin{aligned}
W^{1,q}_{\boldsymbol{n}} &:= \{\boldsymbol{v} \in W^{1,q}(\Omega)^2;\ \boldsymbol{v}\cdot\boldsymbol{n} = 0 \text{ on } \partial\Omega\},\ W^{-1,q'}_{\boldsymbol{n}} := (W^{1,q}_{\boldsymbol{n}})^*,\\
W^{1,q}_{\boldsymbol{n},\mathrm{div}} &:= \{\boldsymbol{v} \in W^{1,q}_{\boldsymbol{n}};\ \operatorname{div}\boldsymbol{v} = 0\},\ W^{-1,q'}_{\boldsymbol{n},\mathrm{div}} := (W^{1,q}_{\boldsymbol{n},\mathrm{div}})^*,\ L^q_{\boldsymbol{n},\mathrm{div}} := \overline{\{\boldsymbol{v} \in W^{1,q}_{\boldsymbol{n},\mathrm{div}}\}}^{\|\cdot\|_q}.
\end{aligned}
$$

Note that the above spaces are separable for $q \in [1,\infty)$ and reflexive for $q \in (1,\infty)$. In order to simplify the presentation we also denote for any $r, q \in [1,+\infty]$

$$
\begin{aligned}
X^{r,q} &:= \{\boldsymbol{u} \in L^r(0,T;W^{1,r}_{\boldsymbol{n}}) \cap L^q(0,T;L^q(\Omega)^2);\ \boldsymbol{u} \in L^2(0,T;L^2(\partial\Omega)^2)\},\\
X^{r,q}_{\mathrm{div}} &:= \{\boldsymbol{u} \in X^{r,q};\ \operatorname{div}\boldsymbol{u} = 0\},
\end{aligned}
$$

and also recall these spaces are separable for $r, q \in [1,\infty)$ and reflexive for $r, q \in (1,\infty)$. The Lebesgue spaces of functions having zero mean value are denoted by L^q_0, i.e., we define $L^q_0 := \{p \in L^q(\Omega);\ \int_\Omega p \,\mathrm{d}x = 0\}$. Finally, for f and g being scalar-, vector- or tensor-valued functions we define $(f,g)_O := \int_O fg$ whenever $fg \in L^1(O)$. Since the most frequent setting in the paper is $O = \Omega$, we shorten the notation for Ω as $(f,g) := (f,g)_\Omega$. Moreover, for any $g \in X$ and $f \in X^*$ we set $\langle f,g\rangle := \langle f,g\rangle_{X^*,X}$ whenever it is clear from the context which duality pairing is taken into account.

The constant C in the whole paper depends only on the data, i.e., on $\boldsymbol{v}_0$, e_0, Ω, T, r, β and α. If there is any dependence on other quantities it is clearly marked in the text.

Having the above definitions of function spaces, we can formulate the main theorem of the paper.

Theorem 2.1. *Let $\Omega \in \mathcal{C}^{1,1}$ be an open bounded domain in $\mathbb{R}^2$, let $r > \frac{3}{2}$ and $\beta > \max(-1, \frac{1}{2}r/(r-1) - 2)$ be arbitrary. Assume that the initial data satisfy*

$$
\boldsymbol{v}_0 \in L^2_{\boldsymbol{n},\mathrm{div}} \quad \text{and} \quad e_0 \in L^1(\Omega); \quad e_0(x) \geqslant e_{\min} > 0 \ \text{ in } \Omega, \tag{2.4}
$$

$\mathbf{S}^*$ *and* $\boldsymbol{q}^*$ *satisfy* (2.1)–(2.3) *with r and β, and let $m_E > 1$ be defined as*

$$
m_E := \begin{cases}
\dfrac{2(\beta+2)}{3} & \text{for } \beta \geqslant r-2 \text{ and } \beta \in \left(-\dfrac{1}{2}, 1\right),\\[2mm]
\dfrac{2r(\beta+1)}{2r+\beta-1} & \text{for } \beta \geqslant r-2 \text{ and } \beta \geqslant 1,\\[2mm]
\dfrac{2(\beta+2)(r-1)}{2r-1+\beta} & \text{for } \beta < r-2 \text{ and } r > 2,\\[2mm]
\dfrac{2r(\beta+2)}{2\beta+r+4} & \text{for } \beta < r-2 \text{ and } \dfrac{3}{2} < r \leqslant 2.
\end{cases} \tag{2.5}
$$

[2] For simplicity we write $\boldsymbol{v}$ instead of $\operatorname{tr}\boldsymbol{v}$ whenever it is clear from the context that we are restricted to $\partial\Omega$.

Then there exists $(\boldsymbol{v}, p, e, \mathbf{S}, \boldsymbol{q}, E)$ *satisfying* (1.8), (1.10), *and* (1.12) *such that*

$$\boldsymbol{v} \in \mathcal{C}_{\mathrm{weak}}(0,T;L^2_{\boldsymbol{n},\mathrm{div}}) \cap X^{r,2r}_{\mathrm{div}}, \tag{2.6}$$

$$\boldsymbol{v}_{,t} \in \big(X^{r,2r/(2r-3)}_{\mathrm{div}}\big)^* \cap L^{\min(r,r')}(0,T;W^{-1,\min(r,r')}_{\boldsymbol{n}}), \tag{2.7}$$

$$\mathbf{S} \in L^{r'}(0,T;L^{r'}(\Omega)^{2\times 2}), \tag{2.8}$$

$$p \in L^{\min(r,r')}(0,T;L^{\min(r,r')}_0), \tag{2.9}$$

$$e \in L^\infty(0,T;L^1(\Omega)), \quad e \geqslant e_{\min} \text{ in } Q, \tag{2.10}$$

$$e^{\frac{1}{2}(\beta+\lambda+1)} \in L^2(0,T;W^{1,2}(\Omega)) \quad \text{for all } \lambda < 0, \tag{2.11}$$

$$E \in L^q(0,T;L^q(\Omega)) \cap W^{1,s'}(0,T;(W^{1,s})^*) \tag{2.12}$$

for all $q < \min(2+\beta, r)$ *and all* $s' < \min(\frac{2}{3}r, 2r/(2r-1), m_E)$,

$$\boldsymbol{q} \in L^m(0,T;L^m(\Omega)^2) \quad \text{for all } 1 \leqslant m < \frac{4+2\beta}{3+2\beta}, \tag{2.13}$$

such that they satisfy (1.8), (1.10), *and* (1.12) *a.e. in* Q, *and such that they satisfy* (1.2)–(1.3) *and* (1.16) *in the following sense:*

$$\begin{aligned} &\langle \boldsymbol{v}_{,t}, \boldsymbol{\varphi}\rangle - (\boldsymbol{v}\otimes\boldsymbol{v}, \nabla\boldsymbol{\varphi})_Q + \alpha(\boldsymbol{v},\boldsymbol{\varphi})_\Gamma + (\mathbf{S}, \mathbf{D}(\boldsymbol{\varphi}))_Q = (p, \operatorname{div}\boldsymbol{\varphi})_Q \\ &\qquad\qquad \text{for all } \boldsymbol{\varphi} \in L^{\max(r,r')}(0,T;W^{1,\max(r,r')}_{\boldsymbol{n}}), \end{aligned} \tag{2.14}$$

$$\begin{aligned} &\langle E_{,t}, \varphi\rangle - ((E+p)\boldsymbol{v}, \nabla\varphi)_Q + (\mathbf{S}\boldsymbol{v} - \boldsymbol{q}, \nabla\varphi)_Q = 0 \\ &\qquad\qquad \text{for all } \varphi \in L^\infty(0,T;W^{1,\infty}(\Omega)), \end{aligned} \tag{2.15}$$

$$\begin{aligned} &-(e,\psi_{,t})_Q - (\boldsymbol{v}e, \nabla\psi)_Q - (\boldsymbol{q}, \nabla\psi)_Q \geqslant (\mathbf{S}, \mathbf{D}(\boldsymbol{v})\psi)_Q + \alpha(|\boldsymbol{v}|^2, \psi)_\Gamma \\ &\qquad\qquad \text{for all nonnegative } \psi \in \mathcal{D}(0,T;W^{1,\infty}(\Omega)), \end{aligned} \tag{2.16}$$

and the initial condition (1.4) *is attained in the following sense:*

$$\lim_{t\to 0+} \|\boldsymbol{v}(t) - \boldsymbol{v}_0\|_2^2 + \|e(t) - e_0\|_1 = 0. \tag{2.17}$$

In addition, we see that for $r \geqslant 2$ we can set $\boldsymbol{\varphi} := \boldsymbol{v}$ in (2.14) and therefore, following the argumentation in Section 1 and the procedure described in [6] we can strengthen the statement of Theorem 2.1 in the following way.

Corollary 2.1. *Let all assumptions of Theorem* 2.1 *be satisfied. Moreover, assume that* $r \geqslant 2$. *Then there exists* $(\boldsymbol{v}, p, e, \mathbf{S}, \boldsymbol{q}, E)$ *satisfying* (1.8), (1.10), (1.12), *and* (2.6)–(2.17) *and, in addition, also*

$$e \in \mathcal{C}(0,T;L^1(\Omega)) \cap W^{1,1}(0,T;(W^{1,q})^*) \quad \text{for sufficiently large } q \gg 1 \tag{2.18}$$

and

$$\langle e_{,t}, \psi \rangle - (\boldsymbol{v}e, \nabla\psi)_Q - (\boldsymbol{q}, \nabla\psi)_Q = (\mathbf{S}, \mathbf{D}(\boldsymbol{v})\psi)_Q + \alpha(|\boldsymbol{v}|^2, \psi)_\Gamma \quad \textit{for all } \psi \in L^\infty(0,T; W^{1,\infty}(\Omega)). \tag{2.19}$$

The Navier-Stokes or Navier-Stokes-Fourier-like systems are ones of the most studied systems of PDE's coming from mathematical physics. These systems of equations can be split into two cases. The first, called subcritical, takes place for $r \geqslant 2$ in two dimensions and $r \geqslant \frac{11}{5}$ in dimension three, and in this case the velocity can be used as a test function. In the second, called supercritical, the velocity cannot be used as a test function. While the subcritical cases were studied by many authors, we refer to [11], [10], and the existence theory can be established with help of the Minty method, the supercritical cases have been solved very recently. First such result for Newtonian fluid in spatially periodic setting was established in [12] and then extended to Navier's boundary conditions in [5]. The existence analysis for the fully nonlinear model satisfying (2.1)–(2.3) was developed for $r \in (\frac{9}{5}, 2]$ and for three dimensions in [6], where in addition the dependence of $\mathbf{S}^*$ on the pressure p is allowed. On the other hand, some uniform monotonicity is required there. This paper uses the method developed in [6] for extending the existence theory to sub- and supercritical cases in dimension two provided that (2.1)–(2.3) hold. The main novelty consists in introducing the optimal function spaces that are natural for the problem considered and follows from *a priori* estimates, and in establishing the existence of a weak solution whenever *a priori* estimates guarantee that (2.14)–(2.16) are meaningful.

We would also like to emphasize that our result is nontrivial. It is well known that for non-Newtonian fluids (without coupling with the energy) the regularity of the solution can be established for any $r > 1$ at least in the spatially periodic setting (see [16]) and therefore one could think that the same method works also for the full Navier-Stokes-Fourier system. However, once the equations are coupled, i.e., once $\mathbf{S}^*$ depends on e, the resulting system falls into the so-called class of PDE's with critical growth on the right-hand side, for which the regularity theory does not hold in general and therefore the method developed in [16] cannot be used and one is forced to use a different procedure.

3. Auxiliary tools

In this section we recall several auxiliary results and important tools that will be used in the proof of Theorem 2.1. First, we recall the theory for the Laplace equation.

Lemma 3.1. *Let $\Omega \in \mathcal{C}^{1,1}$ be an open bounded domain and $1 < q < \infty$. There exists a continuous linear operator $\mathcal{N}^{-1}\colon L^q_0 \to W^{2,q}(\Omega)$ such that*

$$\Delta \mathcal{N}^{-1}(z) = z \text{ in } \Omega, \quad \nabla \mathcal{N}^{-1}(z) \cdot \boldsymbol{n} = 0 \text{ on } \partial\Omega, \quad \int_\Omega \mathcal{N}^{-1}(z)\,\mathrm{d}x = 0. \tag{3.1}$$

Moreover, if we define $\boldsymbol{v}_{\mathrm{div}} := \boldsymbol{v} - \nabla \mathcal{N}^{-1}(\operatorname{div} \boldsymbol{v})$ for any $\boldsymbol{v} \in W^{1,q}_{\boldsymbol{n}} \cap L^s(\Omega)^2$ then the following inequalities hold:

$$\|\mathcal{N}^{-1}(\operatorname{div} \boldsymbol{v})\|_{2,q} \leqslant C\|\operatorname{div} \boldsymbol{v}\|_q, \qquad \|\boldsymbol{v}_{\mathrm{div}}\|_{1,q} \leqslant C\|\boldsymbol{v}\|_{1,q}, \tag{3.2}$$

$$\|\mathcal{N}^{-1}(\operatorname{div} \boldsymbol{v})\|_{1,s} \leqslant C\|\boldsymbol{v}\|_s, \qquad \|\boldsymbol{v}_{\mathrm{div}}\|_{s} \leqslant C\|\boldsymbol{v}\|_{s}. \tag{3.3}$$

Note that due to the definition of $\boldsymbol{v}_{\mathrm{div}}$ we have $\operatorname{div} \boldsymbol{v}_{\mathrm{div}} = \operatorname{div} \boldsymbol{v} - \Delta \mathcal{N}^{-1}(\operatorname{div} \boldsymbol{v}) = 0$ and $\boldsymbol{v}_{\mathrm{div}} \in W^{1,q}_{\boldsymbol{n},\mathrm{div}}$.

Proof. The proof can be found in [15, Proposition 2.5.2.3, page 131]. □

Next, since $\mathbf{S}^*$ depends only on the symmetric part of the velocity gradient we need to recall the Korn inequality.

Lemma 3.2 (Korn inequality)**.** *Let $\Omega \in \mathcal{C}^{0,1}$ be an open bounded domain and $q \in (1, \infty)$. Then for all $\boldsymbol{v} \in W^{1,q}(\Omega)^2 \cap L^2(\Omega)^2$ the following inequality holds:*

$$\|\boldsymbol{v}\|_{1,q} \leqslant C(\|\mathbf{D}(\boldsymbol{v})\|_q + \|\boldsymbol{v}\|_2). \tag{3.4}$$

Proof. We refer to [16] for a detailed proof. □

Since we deal with Navier's boundary conditions, we need to control compactness of the trace operator in a proper space. Therefore, we recall the following lemma.

Lemma 3.3. *Let $\Omega \in \mathcal{C}^{0,1}$ be an open bounded domain. For $1 < q_1, q_2 < \infty$ and $r > \frac{3}{2}$ let us define the set*

$$\mathcal{S} := \{\boldsymbol{v};\ \boldsymbol{v} \in L^\infty(0,T;L^2(\Omega)^2) \cap L^r(0,T;W^{1,r}_{\boldsymbol{n},\mathrm{div}}),\ \boldsymbol{v}_{,t} \in L^{q_1}(0,T;W^{-1,q_2}_{\boldsymbol{n},\mathrm{div}})\}.$$

Assume that $\{\boldsymbol{v}^i\}_{i=1}^\infty$ is bounded in $\mathcal{S}$. Then the sequence $\{\boldsymbol{v}_i|_{\partial\Omega}\}_{i=1}^\infty$ is precompact in $L^2(0,T;L^2(\partial\Omega)^2)$.

P r o o f. We refer to [6, Lemma 1.4] for a detailed proof. $\square$

Finally, we introduce two interpolation inequalities that are frequently used in the proof of the main theorem.

Lemma 3.4. *Let $\Omega \in \mathcal{C}^{0,1}$ be an open bounded domain in $\mathbb{R}^2$. Assume that $1 < r < \infty$ and $\gamma > 0$ are fixed. Then for any $s \in [2, 2r/(2-r)]$ if $r < 2$, for any $s \in [2, \infty)$ if $r \geqslant 2$, and for any q such that*

$$\frac{1}{q} + \frac{1}{s(r-1)} = \frac{1}{2(r-1)}$$

the following interpolation holds:

$$\|f\|_{L^q(0,T;L^s(\Omega))} \leqslant C\|f\|^{\beta}_{L^\infty(0,T;L^2(\Omega))}\|f\|^{1-\beta}_{L^r(0,T;W^{1,r}(\Omega))} \tag{3.5}$$

with

$$\beta := \frac{r}{s(r-1)} + \frac{r-2}{2(r-1)}.$$

Moreover, for any $1 < s, q < \infty$ such that

$$\frac{2}{q} + \frac{1}{s\gamma} = \frac{1}{\gamma}$$

the following inequality holds:

$$\|f\|_{L^q(0,T;L^s(\Omega))} \leqslant C\|f\|^{1/s}_{L^\infty(0,T;L^1(\Omega))}\|f^\gamma\|^{(s-1)/s\gamma}_{L^2(0,T;W^{1,2}(\Omega))}, \tag{3.6}$$

provided that $f \geqslant 0$.

P r o o f. The proof easily follows from the interpolation inequalities (see [1])

$$\|f\|_s \leqslant C\|f\|_2^{\beta}\|f\|_{1,r}^{1-\beta},$$
$$\|f\|_s \leqslant C\|f\|_1^{1/s}\|f^\gamma\|_{1,2}^{(s-1)/s\gamma},$$

and from the Hölder inequality. $\square$

As a direct consequence of this lemma we obtain

Corollary 3.1. *Let all assumptions of Lemma 3.4 be satisfied. Then*

$$\int_0^T \|\boldsymbol{v}\|_{2r}^{2r}\,\mathrm{d}t \leqslant C\|\boldsymbol{v}\|^r_{L^\infty(0,T;L^2(\Omega)^2)}\int_0^T \|\boldsymbol{v}\|^r_{1,r}\,\mathrm{d}t, \tag{3.7}$$

$$\int_0^T \|f\|_{2\gamma+1}^{2\gamma+1}\,\mathrm{d}t \leqslant C\|f\|_{L^\infty(0,T;L^1(\Omega))}\int_0^T \|f^\gamma\|^2_{1,2}\,\mathrm{d}t. \tag{3.8}$$

4. A priori estimates

This section is devoted to obtaining formal *a priori* estimates for solutions to (2.14)–(2.16). Thus, assuming that the velocity field is smooth enough, we can set $\boldsymbol{\varphi} := \boldsymbol{v}$ in (2.14) to obtain (note that the convective term as well as the term with the pressure vanish, we refer to [6], [16] for details)

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{v}\|_2^2 + \alpha\|\boldsymbol{v}\|_{L^2(\partial\Omega)}^2 + (\mathbf{S}, \mathbf{D}(\boldsymbol{v})) = 0. \tag{4.1}$$

Consequently, using (1.12), (2.1) and (3.4), we find that

$$\sup_{t\in(0,T)} \|\boldsymbol{v}\|_2^2 + \int_0^T \|\boldsymbol{v}\|_{1,r}^r + \|\boldsymbol{v}\|_{L^2(\partial\Omega)}^2 \,\mathrm{d}t \leqslant C\|\boldsymbol{v}_0\|_2^2, \tag{4.2}$$

which gives the space introduced for the velocity field in Theorem 2.1, see (2.6). Note that (2.8) is then a direct consequence of (2.6) and the assumption (2.1). Next, we derive the corresponding estimates also for the internal energy e. Thus, setting $\varphi \equiv 1$ in (2.15), we get (using the definition of E (1.8) and assuming that $e \geqslant e_{\min}$) that

$$\sup_{t\in(0,T)} \|e\|_1 \leqslant C(\|\boldsymbol{v}_0\|_2^2 + \|e_0\|_1), \tag{4.3}$$

which formally gives (2.10). Finally, setting $\psi := e^\lambda$ in (2.16) with arbitrary $-1 < \lambda < 0$, after using the fact that $\operatorname{div}\boldsymbol{v} = 0$ and the nonnegativity of the right-hand side of (2.16), we get that

$$\int_0^T \lambda(e^{\lambda-1}\boldsymbol{q}, \nabla e)\,\mathrm{d}t \leqslant C(e_{\min}, \|e\|_{L^\infty(L^1)}, \|\boldsymbol{v}_0\|_2^2). \tag{4.4}$$

Therefore, using (1.10) and (2.3), we find that

$$\begin{aligned} \int_0^T \|\nabla e^{\frac{1}{2}(\lambda+1+\beta)}\|_2^2 \,\mathrm{d}t &\leqslant \frac{(\lambda+1+\beta)^2}{4}\int_0^T (e^{\lambda-1+\beta}\nabla e, \nabla e)\,\mathrm{d}t \\ &\leqslant C(\lambda^{-1})\lambda\int_0^T (e^{\lambda-1}\boldsymbol{q}, \nabla e)\,\mathrm{d}t \leqslant C(\lambda^{-1}). \end{aligned} \tag{4.5}$$

Thus, we have obtained the corresponding spaces for e as those introduced in Theorem 2.1, see (2.10)–(2.11).

To obtain the desired estimate for the pressure, we first notice that using (2.6) and (4.2), we get

$$\int_0^T \|\boldsymbol{v}\|_{2r}^{2r}\,\mathrm{d}t \leqslant C. \tag{4.6}$$

Thus, setting $\boldsymbol{\varphi} := \nabla\mathcal{N}^{-1}(|p|^{\min(r,r')-2}p - \fint_\Omega |p|^{\min(r,r')-2}p)$ in (2.14) (note that $\boldsymbol{\varphi}\cdot\boldsymbol{n} = 0$ on $\partial\Omega$), we get

$$\int_0^T \|p\|_{\min(r,r')}^{\min(r,r')}\,\mathrm{d}t = \int_0^T (\boldsymbol{v}_{,t},\boldsymbol{\varphi}) + \alpha(\boldsymbol{v},\boldsymbol{\varphi})_{\partial\Omega} + (\mathbf{S},\nabla\boldsymbol{\varphi}) - (\boldsymbol{v}\otimes\boldsymbol{v},\nabla\boldsymbol{\varphi})\,\mathrm{d}t. \tag{4.7}$$

Since $\operatorname{div}\boldsymbol{v} = 0$, the first term vanishes. Next, using Lemma 3.1 we have that $\|\boldsymbol{\varphi}\|_{1,\max(r,r')}^{\max(r,r')} \leqslant C\|p\|_{\min(r,r')}^{\min(r,r')}$ and using the Young inequality and the fact that the trace operator is continuous from $W^{1,2}(\Omega)$ to $L^2(\partial\Omega)$, we get that

$$\int_0^T \|p\|_{\min(r,r')}^{\min(r,r')}\,\mathrm{d}t \leqslant C\left(1 + \int_0^T \|\boldsymbol{v}\|_{L^2(\partial\Omega)}^2 + \|\mathbf{S}\|_{r'}^{r'} + \|\boldsymbol{v}\|_{2r}^{2r}\,\mathrm{d}t\right) \leqslant C, \tag{4.8}$$

where the last inequality follows from (2.6) and (2.8).

In the remaining part of this section we show that the relations (2.14)–(2.16) are well defined for $(\boldsymbol{v}, e, p)$ belonging to the spaces introduced in (2.6), (2.10), and (2.11). The fact that (2.14) is meaningful follows from (2.6), (2.7), (4.6), and (4.8). The corresponding space for the time derivative introduced in (2.7) can be obtained by standard interpolations (see [8], [5], [6] for details). Next, using (3.8), (4.3), and (4.5), it is easy to observe for all $\lambda < 0$ that

$$\int_0^T \|e\|_{\lambda+2+\beta}^{\lambda+2+\beta}\,\mathrm{d}t \leqslant C(\lambda^{-1}). \tag{4.9}$$

Next, using (1.10) and (2.3), we get after using the Hölder inequality, (4.5), and (4.9) that

$$\begin{aligned}\int_0^T \|\boldsymbol{q}\|_{1+\frac{2\lambda+1}{3+2\beta}}^{1+\frac{2\lambda+1}{3+2\beta}}\,\mathrm{d}t &\leqslant C\int_Q |\nabla e^{\frac{1}{2}(\beta+1+\lambda)}|^{1+\frac{2\lambda+1}{3+2\beta}}|e|^{\frac{1}{2}(1-\lambda+\beta)(1+\frac{2\lambda+1}{3+2\beta})}\,\mathrm{d}x\,\mathrm{d}t\\ &\leqslant C(\lambda^{-1})\end{aligned} \tag{4.10}$$

and (2.13) follows. To obtain the first part of (2.12), it is enough to combine (4.6) and (4.9). To obtain also the second part of (2.12), we first deduce by using the Hölder inequality, (4.6), (4.8), and (2.8) that

$$\int_Q (|\boldsymbol{v}|^3 + |p||\boldsymbol{v}| + |\mathbf{S}||\boldsymbol{v}|))^{\min(2r/3,2r/(2r-1))}\,\mathrm{d}x\,\mathrm{d}t \leqslant C. \tag{4.11}$$

Finally, we need to find $m_E > 1$ such that $\int_Q |\boldsymbol{v}e|^q\,\mathrm{d}x\,\mathrm{d}t \leqslant C$ for all $q < m_E$. Once having such m_E we can use (4.11) to obtain the second part of (2.12).

The first natural choice how to find such m_E is to use homogeneous (homogeneous means the same integrability w.r.t. x and t) estimates (4.6) and (4.9). Then m_E is given by the relation

$$\frac{1}{m_E} = \frac{1}{2r} + \frac{1}{\beta+2}.$$

Consequently, to guarantee that $m_E > 1$ we need to restrict ourselves to the case

$$\beta > -2\frac{r-1}{2r-1}.$$

However, in the rest of this section, we show that the use of homogeneous estimates is not optimal and one can get a better relation for m_E than that which is introduced in (2.5) and, consequently, not so restrictive condition for β.

Thus, to find such optimal m_E, we use (2.6), (2.10)–(2.11), and Lemma 3.4. For simplicity, since λ in (2.11) can be arbitrarily close to zero, we do all estimates with $\lambda = 0$ and finally replace the possible equality by the strict inequality as is done in (2.12). Hence, using the Hölder inequality, we get that

$$\|e\boldsymbol{v}\|_{L^{m_E}(Q)} \leqslant \|\boldsymbol{v}\|_{L^{a_1}(0,T;L^{b_1}(\Omega)^2)} \|e\|_{L^{a_2}(0,T;L^{b_2}(\Omega))} \leqslant C,$$

provided that

$$\frac{1}{m_E} = \frac{1}{a_1} + \frac{1}{a_2} = \frac{1}{b_1} + \frac{1}{b_2}, \tag{4.12}$$

$$\frac{1}{a_1} + \frac{1}{b_1(r-1)} = \frac{1}{2(r-1)}, \qquad \frac{1}{a_2} + \frac{1}{b_2(\beta+1)} = \frac{1}{\beta+1} \tag{4.13}$$

and that $1 < a_1, a_2, b_1, b_2 < \infty$. These identities are consequences of (3.5) and (3.6), and the properties (2.6), (2.10)–(2.11). For the last constraint we need to guarantee that

$$a_2 \in (\max(1, \beta+1), \infty), \quad a_1 \in (r^{\#}, \infty), \quad b_1 \in [2, r^*), \quad b_2 \in (1, \infty), \tag{4.14}$$

where $r^* = 2r/(r-2)$ and $r^{\#} = r$ if $r < 2$ and $r^* = \infty$ and $r^{\#} = 2(r-1)$ for $r \geqslant 2$. Thus, solving the system of algebraic equations (4.12)–(4.13), we find that

$$\begin{aligned} \frac{1}{a_1} &= \frac{3}{2r} - \frac{\beta+2}{a_2 r}, \\ \frac{1}{b_1} &= \frac{1}{2} - \frac{3(r-1)}{2r} + \frac{(r-1)(\beta+2)}{a_2 r}, \\ \frac{1}{b_2} &= 1 - \frac{\beta+1}{a_2} \end{aligned} \tag{4.15}$$

and therefore for m_E we get

$$\frac{1}{m_E} = \frac{3}{2r} + \frac{r-\beta-2}{a_2 r}. \tag{4.16}$$

Before we choose a_2 such that m_E is maximal, let us recall the possible ranges for a_1, b_1, a_2, b_2 that follow from (4.15). First, using the restriction for a_1 and (4.15) we find for $r < 2$ that

$$\frac{2(\beta+2)}{3} \leqslant a_2 \leqslant 2(\beta+2),$$

which implies

$$\max\Big(1, 1+\beta, \frac{2(\beta+2)}{3}\Big) \leqslant a_2 \leqslant 2(\beta+2).$$

Similarly, for $r \geqslant 2$, we get the restriction

$$\max\Big(1, 1+\beta, \frac{2(\beta+2)}{3}\Big) \leqslant a_2 < \frac{2(r-1)(\beta+2)}{2r-3}.$$

Having these ranges for a_2, we now choose it such that m_E is the largest possible. Thus, going back to (4.16), we see that for $r-2 > \beta$ we need to choose a_2 maximal. So for $r < 2$ we set $a_2 := 2(\beta+2)$, therefore (4.16) implies that

$$\frac{1}{m_E} = \frac{1}{r} + \frac{1}{2(\beta+2)} \Longleftrightarrow m_E = \frac{2r(\beta+2)}{2\beta+4+r} \tag{4.17}$$

and we see that to get that $m_E > 1$ we need to restrict β as

$$\beta > \frac{4-3r}{2(r-1)}. \tag{4.18}$$

Similarly for $r \geqslant 2$, setting the maximal $a_2 = 2(r-1)(\beta+2)/(2r-3)$, we get

$$m_E = \frac{2(r-1)(\beta+2)}{2r+\beta-1} \tag{4.19}$$

which is always strictly greater than 1 since $\beta > -1$.

For $\beta \geqslant r-2$ we choose a_2 minimal, i.e., we choose $a_2 := \max(\beta+1, \frac{2}{3}(\beta+2))$. This leads to

$$\begin{aligned} -\frac{1}{2} \leqslant \beta \leqslant 1 &\Longrightarrow m_E = \frac{2(\beta+2)}{3}, \\ 1 \leqslant \beta \qquad &\Longrightarrow m_E = \frac{2r(\beta+1)}{2r+\beta-1}. \end{aligned}$$

Note that in all cases we obtain $m_E > 1$.

5. Proof of the main theorem

In order to prove the main theorem, we need several approximations in what follows. First, since we do not know *a priori* that $e \geqslant e_{\min}$, we truncate $\mathbf{S}^*$ and $\boldsymbol{q}^*$ as

$$\tilde{\mathbf{S}}^*(e, \mathbf{D}) := \mathbf{S}^*(\max(e, e_{\min}), \mathbf{D}), \quad \tilde{\boldsymbol{q}}^*(e, |\mathbf{D}|^2, \nabla e) := \boldsymbol{q}^*(\max(e, e_{\min}), |\mathbf{D}|^2, \nabla e).$$

Note that once having the minimum principle for e, the above truncation is not needed.

Further, to be able to use the monotone operator theory and to pass from the Galerkin approximative scheme to the "continuous" problem we need to guarantee that $\boldsymbol{v}$ is a possible test function in (2.14). To achieve it, we mollify the convective term[3]. Thus, let z be a standard regularization kernel. For $\eta > 0$ we define $z_\eta(x) := z(x/\eta)/\eta^2$. Then we find some $\omega_\eta \in \mathcal{D}(\Omega)$ such that $\operatorname{dist}(\operatorname{supp} \omega_\eta, \partial\Omega) \geqslant \eta$ and $\omega_\eta(x) = 1$ for all $x \in \Omega$, $\operatorname{dist}(x, \partial\Omega) \geqslant 2\eta$. Finally, for arbitrary $\boldsymbol{v} \in W^{1,r}_{\boldsymbol{n}}$ we define $\boldsymbol{v}_\eta := ((\boldsymbol{v}\omega_\eta) * z_\eta)_{\mathrm{div}}$. Note that a direct consequence of this definition is that $\operatorname{div} \boldsymbol{v}_\eta = 0$ in Ω, $\boldsymbol{v}_\eta \cdot \boldsymbol{n} = 0$ on $\partial\Omega$. Moreover, using Lemma 3.1 it is not difficult to observe that for all $\boldsymbol{v} \in L^r(0, T; W^{1,r}_{\boldsymbol{n}}) \cap L^q(Q)$

$$\boldsymbol{v}_\eta \overset{\eta \to 0}{\to} \boldsymbol{v} \quad \text{strongly in } L^q(Q).$$

Finally, we replace the balance of the global energy (1.3) by the balance of the internal energy (1.15) for which the standard theory for parabolic equations can be easily used. Thus, our final approximative system takes the following form:

$$\operatorname{div} \boldsymbol{v} = 0, \tag{5.1}$$

$$\boldsymbol{v}_{,t} + \operatorname{div}(\boldsymbol{v}_\eta \otimes \boldsymbol{v}) - \operatorname{div} \tilde{\mathbf{S}}^*(e, \mathbf{D}(\boldsymbol{v})) = -\nabla p, \tag{5.2}$$

$$e_{,t} + \operatorname{div}(\boldsymbol{v}_\eta e) + \operatorname{div} \tilde{\boldsymbol{q}}^*(e, \mathbf{D}(v), \nabla e) = \tilde{\mathbf{S}}^*(e, \mathbf{D}(\boldsymbol{v})) \cdot \mathbf{D}(\boldsymbol{v}), \tag{5.3}$$

with the boundary conditions (1.5) and (1.7) and the initial conditions (1.4) where $E_0 := \frac{1}{2}|\boldsymbol{v}_0| + e_0$. The existence of a solution to (5.1)–(5.3) is established by using the "double" Galerkin approximation scheme and the monotone operator theory. Once having a solution to the approximative system (5.1)–(5.3), we introduce the equation for the global energy and let $\eta \to 0$. In order to be able to identify the limits of the nonlinear term we use the Aubin-Lions lemma, the Div-Curl lemma (see [19]), and the compactness of the velocity gradient that is achieved by using the so-called

[3] This mollification is not needed in case $r \geqslant 2$ since in this case one can use the velocity field as a test function in (2.14).

L^∞ truncation method. For more details we refer an interested reader to [6], where the three-dimensional case is treated.

5.1. Galerkin approximation

Let $\{\boldsymbol{w}_j\}_{j=1}^\infty$ be a basis of $W^{1,r}_{\boldsymbol{n},\mathrm{div}}$ that is orthonormal in $L^2(\Omega)^2$ and such that $\boldsymbol{w}_j \in W^{1,q}_{\boldsymbol{n},\mathrm{div}}$ for all $j \in \mathbb{N}$ and all $q \in (1,\infty)$. Assume that $\{w_j\}_{j=1}^\infty$ is a basis of $W^{1,2}(\Omega)$ again orthonormal in $L^2(\Omega)$. Then the Galerkin approximation to (5.1)–(5.3) has the form:

For $k, l \in \mathbb{N}$ we look for the functions

$$\boldsymbol{v}^{k,l} := \sum_{i=1}^{k} c_i^{k,l}(t)\boldsymbol{w}_i \quad \text{and} \quad e^{k,l} := \sum_{i=1}^{l} d_i^{k,l}(t) w_i$$

that satisfy the system of ordinary differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{v}^{k,l}, \boldsymbol{w}_j) - (\boldsymbol{v}_\eta^{k,l} \otimes \boldsymbol{v}^{k,l}, \nabla \boldsymbol{w}_j) + (\mathbf{S}^{k,l}, \nabla \boldsymbol{w}_j) + \alpha(\boldsymbol{v}^{k,l}, \boldsymbol{w}_j)_{\partial\Omega} = 0 \tag{5.4}$$

for $j = 1, \ldots, k$ and

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(e^{k,l}, w_j) - (\boldsymbol{v}_\eta^{k,l} e^{k,l}, \nabla w_j) - (\boldsymbol{q}^{k,l}, \nabla w_j) \\ = (\mathbf{S}^{k,l} \cdot \mathbf{D}^{k,l}, w_j) + \alpha(|\boldsymbol{v}^{k,l}|^2, w_j)_{\partial\Omega} \end{aligned} \tag{5.5}$$

for $j = 1, \ldots, l$. Here, we set for simplicity

$$\mathbf{D}^{k,l} := \mathbf{D}(\boldsymbol{v}^{k,l}), \quad \mathbf{S}^{k,l} := \tilde{\mathbf{S}}^*(e^{k,l}, \mathbf{D}^{k,l}), \quad \boldsymbol{q}^{k,l} := \tilde{\boldsymbol{q}}^*(e^{k,l}, \mathbf{D}^{k,l}, \nabla e^{k,l}).$$

The system (5.4)–(5.5) is completed by the initial conditions

$$\begin{aligned} \boldsymbol{v}^{k,l}(0) = \boldsymbol{v}_0^{k,l} &:= \sum_{j=1}^{k} c_{0j}^{k,l} \boldsymbol{w}_j = P^k \boldsymbol{v}_0, \\ e^{k,l}(0) = e_0^{k,l} &:= \sum_{j=1}^{l} d_{0j}^{k,l} \boldsymbol{w}_j = P^l(r_{1/k} * e_0), \end{aligned}$$

where P^k and P^l denote the projections of the spaces $L^2_{\boldsymbol{n},\mathrm{div}}$, $L^2(\Omega)$ onto the $\mathrm{span}\{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_k\}$, $\mathrm{span}\{w_1, \ldots, w_l\}$, respectively. Note that here we also mollify the initial condition e_0, since we want to use the standard L^2 theory for parabolic equations for which the smooth initial data are needed, which is not our case since we assume only that $e_0 \in L^1(\Omega)$.

The assumptions on $\mathbf{S}^*$ and $\boldsymbol{q}^*$ and the Carathéodory theory for ODE enable us to establish the existence of solutions $c_i^{k,l}(t)$ $(i = 1, \dots, k)$ and $d_j^{k,l}(t)$ $(j = 1, \dots, l)$ to (5.4)–(5.5) defined on $[0, T_0)$ for some $T_0 > 0$, and by using the estimates established in the next subsection we can extend the solution to the whole time interval $(0, T)$.

5.1.1. Estimates uniform w.r.t. l and the limit for $l \to \infty$. First, we derive uniform estimates (that can however depend on k). Multiplying the jth equation in (5.4) by $c_j^{k,l}$, taking the sum over $j = 1, \dots, k$ and integrating the result over $(0, t)$ for $t \in (0, T)$ (that is nothing else than testing by $\boldsymbol{v}^{k,l}$), we get

$$\|\boldsymbol{v}^{k,l}(t)\|_2^2 + 2\int_0^t (\mathbf{S}^{k,l}, \mathbf{D}^{k,l}) + 2\alpha\|\boldsymbol{v}^{k,l}\|^2_{L^2(\partial\Omega)^2}\,\mathrm{d}\tau = \|\boldsymbol{v}_0^{k,l}\|_2^2. \tag{5.6}$$

Here, we have used the fact that $(\boldsymbol{v}_\eta^{k,l} \otimes \boldsymbol{v}^{k,l}, \nabla \boldsymbol{v}^{k,l}) = 0$ (see [6], [16] for details). Therefore, using (2.1) and the Korn inequality we observe that

$$\sup_{t\in(0,T)} \|\boldsymbol{v}^{k,l}(t)\|_2^2 + \int_0^T \|\boldsymbol{v}^{k,l}\|_{1,r}^r + \|\boldsymbol{v}^{k,l}\|^2_{L^2(\partial\Omega)^2}\,\mathrm{d}t \leqslant C, \tag{5.7}$$

where the constant C does not depend on k, l, η. Moreover, the same procedure as in Section 4 gives that

$$\int_0^T \|\boldsymbol{v}^{k,l}\|_{2r}^{2r}\,\mathrm{d}t \leqslant C. \tag{5.8}$$

It is also an easy consequence of (5.7) and the orthonormality of $\boldsymbol{w}_j$ that

$$\|c_i^{k,l}\|_\infty \leqslant C \quad \text{for all } i = 1, \dots, k. \tag{5.9}$$

Moreover, using (5.4)–(5.9) we immediately get that

$$\|(c_i^{k,l})_{,t}\|_\infty \leqslant C(k) \quad \text{for all } i = 1, \dots, k. \tag{5.10}$$

Having (5.9)–(5.10) and using the properties of the chosen basis, it is standard to show that there is a not relabeled subsequence such that

$$c^{k,l} \rightharpoonup^* c^k \quad \text{weakly}^* \text{ in } W^{1,\infty}(0,T), \tag{5.11}$$

$$\boldsymbol{v}^{k,l} \to \boldsymbol{v}^k \quad \text{strongly in } \mathcal{C}(0,T; W^{1,q}_{\boldsymbol{n}}) \text{ for all } q \in (1,\infty). \tag{5.12}$$

Next, using the convergence results (5.11)–(5.12) and the standard procedure for parabolic equations applied to (5.5) (see [6]), we can find a not relabeled subsequence such that

$$e^{k,l} \rightharpoonup e^k \quad \text{weakly in } L^2(0,T;W^{1,2}(\Omega)), \tag{5.13}$$

$$e^{k,l}_{,t} \rightharpoonup e^k_{,t} \quad \text{weakly in } L^2(0,T;W^{-1,2}(\Omega)), \tag{5.14}$$

$$\boldsymbol{q}^{k,l} \rightharpoonup \boldsymbol{q}^k \quad \text{weakly in } L^2(0,T;L^2(\Omega)), \tag{5.15}$$

$$e^{k,l} \to e^k \quad \text{strongly in } L^q(0,T;L^q(\Omega)) \text{ for all } q<4, \tag{5.16}$$

and such that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{v}^k,\boldsymbol{w}_j) - (\boldsymbol{v}^k_\eta \otimes \boldsymbol{v}^k, \nabla \boldsymbol{w}_j) + (\mathbf{S}^k, \nabla \boldsymbol{w}_j) + \alpha(\boldsymbol{v}^k,\boldsymbol{w}_j)_{\partial\Omega} = 0 \tag{5.17}$$

for all $j=1,\dots,k$ and

$$\langle e^k_{,t},\varphi\rangle - (\boldsymbol{v}^k_\eta e^k, \nabla\varphi)_Q - (\boldsymbol{q}^k,\nabla\varphi)_Q = (\mathbf{S}^k, \nabla \boldsymbol{v}^k \varphi)_Q + \alpha(|\boldsymbol{v}^k|^2,\varphi)_{\partial\Omega} \tag{5.18}$$

for all $\varphi \in L^2(0,T;W^{1,2}(\Omega))$. Here we denoted $\mathbf{S}^k := \mathbf{S}^*(\max(e_{\min},e^k),\mathbf{D}(\boldsymbol{v}^k))$ and $\boldsymbol{q}^k = \boldsymbol{q}^*(\max(e_{\min},e),\mathbf{D}(\boldsymbol{v}^k),\nabla e^k)$. The attainment of the initial condition, i.e, the fact that $\boldsymbol{v}^k(0) = P^k(\boldsymbol{v}_0)$ and $e^k(0) = r_{1/k} * e_0$ is standard and we refer to [6]. Moreover, having (5.12) we can deduce that

$$\|e\|_{L^\infty(Q)} \leqslant C(k) \tag{5.19}$$

and then by using the minimum principle argument, it is also easy to observe that since $e_0 \geqslant e_{\min}$ and therefore $r_{1/k} * e_0 \geqslant e_{\min}$ as well, we have that $e^k \geqslant e_{\min}$ a.e. in Q we again refer to [6] for details.

5.1.2. Estimates independent of k. Similarly to the above, multiplying the jth equation in (5.17) by c^k_j, taking the sum over $j=1,\dots,k$ and integrating the result over $(0,t)$ for $t\in(0,T)$, we get

$$\|\boldsymbol{v}^k(t)\|_2^2 + 2\int_0^t (\mathbf{S}^k,\mathbf{D}(\boldsymbol{v}^k)) + 2\alpha\|\boldsymbol{v}^k\|^2_{L^2(\partial\Omega)^2}\,\mathrm{d}\tau = \|\boldsymbol{v}^k_0\|_2^2. \tag{5.20}$$

Hence, using (2.1) and the Korn inequality we observe that

$$\sup_{t\in(0,T)} \|\boldsymbol{v}^k(t)\|_2^2 + \int_0^T \|\boldsymbol{v}^k\|^r_{1,r} + \|\mathbf{S}^k\|^{r'}_{r'} + \|\boldsymbol{v}^k\|^2_{L^2(\partial\Omega)^2}\,\mathrm{d}t \leqslant C \tag{5.21}$$

and

$$\int_0^T \|\boldsymbol{v}^k\|_{2r}^{2r}\,\mathrm{d}t \leqslant C. \tag{5.22}$$

Next, we follow the procedure described in Section 4. First, setting $\varphi \equiv 1$ in (5.18), we get (using that $e \geqslant e_{\min} > 0$)

$$\sup_{t\in(0,T)} \|e^k(t)\|_1 \leqslant C. \tag{5.23}$$

Next, taking $\varphi = (e^k)^\lambda$ with $-1 < \lambda < 0$ in (5.18) (note that φ is bounded since $e^k \geqslant e_{\min} > 0$ and it can be deduced that it is a possible test function, see [6] for detailed explanation) and using (5.21), we get similarly to (4.5) that

$$\int_0^T \|(e^k)^{\frac{1}{2}(\beta+\lambda+1)}\|_{1,2}^2\,\mathrm{d}t \leqslant C(\lambda^{-1}). \tag{5.24}$$

Similarly, following (4.9), we obtain a uniform bound

$$\int_0^T \|e^k\|_{\beta+2+\lambda}^{\beta+2+\lambda} + \|\boldsymbol{q}^k\|_{1+\frac{2\lambda+1}{3+2\beta}}^{1+\frac{2\lambda+1}{3+2\beta}}\,\mathrm{d}t \leqslant C(\lambda^{-1}) \quad \text{for all } -\tfrac{1}{2} < \lambda < 0. \tag{5.25}$$

It remains to deduce estimates for time derivatives. Thus, using (5.21) and the fact that we have mollified the convective term, we can deduce that

$$\|\boldsymbol{v}^k_{,t}\|_{(X^{r,2r}_{\mathrm{div}})^*} \leqslant C(\eta^{-1}). \tag{5.26}$$

Similarly, using (5.21) and (5.25) we can deduce that for sufficiently large m we have

$$\|e^k_{,t}\|_{L^1(0,T;W^{-1,m'}(\Omega))} \leqslant C. \tag{5.27}$$

5.1.3. Limit for $k \to \infty$. Using (5.21) and (5.26), we see that we can find a not relabeled subsequence such that

$$\begin{aligned}
&\boldsymbol{v}^k_{,t} \rightharpoonup \boldsymbol{v}_{,t} && \text{weakly in } (X^{r,2r}_{\mathrm{div}})^*, && (5.28)\\
&\boldsymbol{v}^k \rightharpoonup^* \boldsymbol{v} && \text{weakly}^* \text{ in } L^\infty(0,T;L^2(\Omega)^2), && (5.29)\\
&\boldsymbol{v}^k \rightharpoonup \boldsymbol{v} && \text{weakly in } X^{r,2r}_{\mathrm{div}}, && (5.30)\\
&\mathbf{S}^k \rightharpoonup \mathbf{S} && \text{weakly in } L^{r'}(0,T;L^{r'}(\Omega)^{2x2}), && (5.31)
\end{aligned}$$

and therefore after using the Aubin-Lions lemma and Lemma 3.3

$$\begin{aligned}
&\boldsymbol{v}^k \to \boldsymbol{v} && \text{strongly in } L^q(0,T;L^q(\Omega)^2) \text{ for all } q \in [1,2r), && (5.32)\\
&\boldsymbol{v}^k \to \boldsymbol{v} && \text{strongly in } L^2(0,T;L^2(\partial\Omega)^2). && (5.33)
\end{aligned}$$

In the same way, using (5.24) and (5.25) we can find a subsequence that is again not relabeled such that

$$e^k \rightharpoonup e \qquad \text{weakly in } L^q(0,T;L^q(\Omega)) \text{ for all } q < 2+\beta, \tag{5.34}$$

$$\boldsymbol{q}^k \rightharpoonup \boldsymbol{q} \qquad \text{weakly in } L^q(0,T;L^q(\Omega)^2) \text{ for } q < \frac{4+2\beta}{3+2\beta}, \tag{5.35}$$

$$(e^k)^{\frac{\beta+\lambda+1}{2}} \rightharpoonup \overline{e^{\frac{\beta+\lambda+1}{2}}} \qquad \text{weakly in } L^2(0,T;W^{1,2}(\Omega)). \tag{5.36}$$

In order to show the strong convergence of e^k, inspired by [13], we use the Div-Curl Lemma[4]. Indeed, defining $\boldsymbol{w}^k := (e^k, e^k v_1^k + q_1^k, e^k v_2^k + q_2^k)$ and $\boldsymbol{u}^k := ((e^k)^m, 0, 0)$ for some $0 < m < \min(1, \frac{1}{2}(\beta+2))$, we see with help of (5.21) that $\operatorname{div}_{t,x} \boldsymbol{w}^k$ is bounded in $L^1(Q)$ and with help of (5.24) we have that $\operatorname{Curl}_{t,x} \boldsymbol{u}^k$ is bounded in $L^2(Q)$. Therefore, applying the Div-Curl Lemma we observe, after using the bound (5.25) and the strong convergence result (5.32), that

$$\boldsymbol{w}^k \cdot \boldsymbol{u}^k \rightharpoonup \boldsymbol{w} \cdot \boldsymbol{u}$$

with $\boldsymbol{w} := (e, ev_1 + \overline{q_1}, e^k v_2) + \overline{q_2}$ and $\boldsymbol{u} := (\overline{e^m}, 0, 0)$. Consequently, we get

$$(e^k)^{m+1} \rightharpoonup e\overline{e^m}.$$

Thus, using the convexity of the $(m+1)$st power, we have that $e^{m+1} \leqslant e\overline{e^m}$ and consequently, since e is positive a.e. in Q we obtain $e^m \leqslant \overline{e^m}$. On the other hand, we have from the concavity (as $m \in (0,1)$) that $e^m \geqslant \overline{e^m}$ and therefore $e^m = \overline{e^m}$. Since the mth power is strictly concave, this relation implies (for a subsequence) that[5]

$$e^k \to e \qquad \text{a.e. in } Q. \tag{5.37}$$

Thus using (5.32), (5.34), the definition of m_E (2.5), (5.37), and the estimates established in Section 4 we observe that

$$e^k \to e \qquad \text{strongly in } L^q(0,T;L^q(\Omega)) \text{ for all } q < 2+\beta, \tag{5.38}$$

$$e^k \boldsymbol{v}^k \to e\boldsymbol{v} \qquad \text{strongly in } L^m(0,T;L^m(\Omega)^2) \text{ for all } m < m_E. \tag{5.39}$$

First, we identify the limit of (5.17). Having (5.28)–(5.33), it is easy to obtain (by using the fact that $\boldsymbol{w}_j$ is a basis of $W^{1,r}_{\boldsymbol{n}}$)

$$\langle \boldsymbol{v}_{,t}, \boldsymbol{w} \rangle - (\boldsymbol{v}_\eta \otimes \boldsymbol{v}, \nabla \boldsymbol{w})_Q + (\mathbf{S}, \nabla \boldsymbol{w})_Q + \alpha(\boldsymbol{v}, \boldsymbol{w})_\Gamma = 0 \qquad \text{for all } \boldsymbol{w} \in X^{r,2r}_{\operatorname{div}}. \tag{5.40}$$

[4] In fact, if β is sufficiently large one could simply use the Aubin-Lions lemma.
[5] See [19] for details.

Moreover, it is standard to show that the initial condition is attained in the following sense (we refer to [16] for details):

$$\lim_{t \to 0_+} \|\boldsymbol{v}(t) - \boldsymbol{v}_0\|_2^2 = 0, \tag{5.41}$$

and since $\boldsymbol{v}$ is a possible test function in (5.40), one can also obtain (by using (5.29)) that

$$\boldsymbol{v} \in \mathcal{C}(0, T; L^2(\Omega)^2).$$

Hence, to obtain the convergence result for (5.17), it remains to show that

$$\mathbf{S} = \mathbf{S}^*(e, \mathbf{D}(\boldsymbol{v})) \quad \text{a.a. in } Q. \tag{5.42}$$

However, this can be achieved by using the Minty method and the strong convergence results (5.32)–(5.33) and (5.38), here we refer again to [6] for a detailed proof. Moreover, since $\mathbf{S}^*$ is strictly monotone, see (2.2), we immediately deduce from (5.42) that (for a subsequence)

$$\mathbf{D}(\boldsymbol{v}^k) \to \mathbf{D}(\boldsymbol{v}) \quad \text{a.e. in } Q. \tag{5.43}$$

In addition, we can deduce that

$$[\mathbf{S}^k - \mathbf{S}^*(e^k, \mathbf{D}(\boldsymbol{v}))] \cdot \mathbf{D}(\boldsymbol{v}^k - \boldsymbol{v}) \to 0 \quad \text{strongly in } L^1(Q)$$

from which it follows that

$$\mathbf{S}^k \cdot \mathbf{D}(\boldsymbol{v}^k) \rightharpoonup \mathbf{S} \cdot \mathbf{D}(\boldsymbol{v}) \quad \text{weakly in } L^1(0, T; L^1(\Omega)). \tag{5.44}$$

Consequently, using (5.18) and (5.44), one can strengthen the information (5.27) and deduce that for a subsequence

$$e^k_{,t} \rightharpoonup e_{,t} \quad \text{weakly in } L^1(0, T; W^{-1,m'}(\Omega)) \tag{5.45}$$

for some sufficiently large m. Thus, using (5.34)–(5.39) and (5.44)–(5.45), we can let $k \to \infty$ in (5.18) to observe

$$\begin{aligned} &\langle e_{,t}, \varphi\rangle - (\boldsymbol{v}_\eta e, \nabla\varphi)_Q - (\boldsymbol{q}, \nabla\varphi)_Q = (\mathbf{S}, \mathbf{D}(\boldsymbol{v})\varphi)_Q + \alpha(|\boldsymbol{v}|^2, \varphi)_\Gamma \\ &\qquad \text{for all } \varphi \in L^\infty(0, T; L^\infty(\Omega)) \cap L^m(0, T; W^{1,m}(\Omega)) \text{ for some } m \gg 1. \end{aligned} \tag{5.46}$$

Hence, to establish the convergence $k \to \infty$ it remains to show that

$$\boldsymbol{q} = \boldsymbol{q}^*(e, \mathbf{D}(\boldsymbol{v}), \nabla e) \quad \text{a.e. in } Q. \tag{5.47}$$

Using (1.10), we see that it is enough to show that

$$\kappa^*(e^k, |\mathbf{D}(\boldsymbol{v}^k)|)\nabla e^k \rightharpoonup \kappa^*(e, |\mathbf{D}(\boldsymbol{v})|)\nabla e \quad \text{weakly in } L^1(0,T;L^1(\Omega)^2). \tag{5.48}$$

Since we can rewrite this term as

$$\kappa^*(e^k, |\mathbf{D}(\boldsymbol{v}^k)|)\nabla e^k = \frac{2}{\beta+\lambda+1}\nabla(e^k)^{\frac{1}{2}(\beta+\lambda+1)}\kappa^*(e^k, |\mathbf{D}(\boldsymbol{v}^k)|)(e^k)^{\frac{1}{2}(1-\beta-\lambda)},$$

we can use (5.36) and therefore to prove (5.48) it is enough to show that

$$\kappa^*(e^k, |\mathbf{D}(\boldsymbol{v}^k)|)(e^k)^{\frac{1}{2}(1-\beta-\lambda)} \to \kappa^*(e, |\mathbf{D}(\boldsymbol{v})|)e^{\frac{1}{2}(1-\beta-\lambda)} \quad \text{strongly in } L^2(Q). \tag{5.49}$$

Since we have already established pointwise convergence (at least for a subsequence) of $\mathbf{D}(\boldsymbol{v}^k)$ and e^k, see (5.38) and (5.43), it is enough to show that

$$\int_Q |\kappa^*(e^k, |\mathbf{D}(\boldsymbol{v}^k)|)(e^k)^{\frac{1}{2}(1-\beta-\lambda)}|^{2+\varepsilon}\,\mathrm{d}x\,\mathrm{d}t \leqslant C \tag{5.50}$$

for some $\varepsilon > 0$. Indeed, having (5.50) and the pointwise convergence, one can use the Vitali theorem to deduce (5.49). Thus, using (2.3), assuming that $\max(-\frac{1}{2}, -1-\beta) < \lambda < 0$ and setting $\varepsilon := 2(2\lambda+1)/(1+\beta-\lambda) > 0$, we get

$$\begin{aligned}&\int_Q |\kappa^*(e^k, |\mathbf{D}(\boldsymbol{v}^k)|)(e^k)^{\frac{1}{2}(1-\beta-\lambda)}|^{2+\varepsilon}\,\mathrm{d}x\,\mathrm{d}t \\ &\qquad\leqslant C\int_Q (e^k)^{\frac{1}{2}(1+\beta-\lambda)(2+\varepsilon)}\,\mathrm{d}x\,\mathrm{d}t = C\int_Q (e^k)^{\beta+2+\lambda}\,\mathrm{d}x\,\mathrm{d}t \overset{(5.25)}{\leqslant} C(\lambda^{-1}).\end{aligned}$$

Thus, the proof of (5.47) is complete. The proof of attainment of e_0 is postponed to the next subsection, where a more difficult case is treated.

5.1.4. Reconstruction of the pressure p. In this subsection we reconstruct the pressure p corresponding to the equation (5.40). We recall the theory developed in [8], [6] where the existence of the pressure $p \in L^{r'}(Q)$ is established and where the following identity is shown:

$$\begin{aligned}\langle \boldsymbol{v}_{,t}, \boldsymbol{w}\rangle - (\boldsymbol{v}_\eta \otimes \boldsymbol{v}, \nabla \boldsymbol{w})_Q + (\mathbf{S}, \nabla \boldsymbol{w})_Q + \alpha(\boldsymbol{v}, \boldsymbol{w})_\Gamma = (p, \operatorname{div} \boldsymbol{w})_Q \\ \text{for all } \boldsymbol{w} \in X^{r,2r}.\end{aligned} \tag{5.51}$$

Moreover, the pressure p is given as $p := p_1 + p_2 + p_3$ with $\fint_\Omega p_i = 0$ for $i = 1,2,3$ and the particular pressures solve for a.a. $t \in (0,T)$ and all $\varphi \in W^{2,r}(\Omega)$ such that

$\nabla\varphi \in W^{1,r}_{\boldsymbol{n}}$ the identities

$$(p_1, \triangle\varphi) = (\mathbf{S}, \nabla^2\varphi), \tag{5.52}$$

$$(p_2, \triangle\varphi) = -(\boldsymbol{v}_\eta \otimes \boldsymbol{v}, \nabla^2\varphi), \tag{5.53}$$

$$(p_3, \triangle\varphi) = (\boldsymbol{v}, \nabla\varphi)_{\partial\Omega}. \tag{5.54}$$

5.2. Limit $\eta \to 0$

In this subsection we complete the proof of Theorem 2.1. To do so, we first denote $(\boldsymbol{v}^\eta, e^\eta, p^\eta, \mathbf{S}^\eta, \boldsymbol{q}^\eta)$ a solution to (5.51) and (5.46).

5.2.1. Uniform estimates and weak convergence result. Using weak lower semicontinuity of norms, the Fatou lemma, and the uniform estimates (5.21)–(5.25), we have

$$\begin{aligned}\sup_{t\in(0,T)} & (\|\boldsymbol{v}^\eta(t)\|_2^2 + \|e^\eta\|_1) \\ & + \int_0^T \|\boldsymbol{v}^\eta\|_{1,r}^r + \|\boldsymbol{v}^\eta\|_{2r}^{2r} + \|\mathbf{S}^\eta\|_{r'}^{r'} + \|\boldsymbol{v}^\eta\|^2_{L^2(\partial\Omega)^2}\,\mathrm{d}t \\ & + \int_0^T \|(e^\eta)^{\frac{1}{2}(\beta+\lambda+1)}\|_{1,2}^2 + \|\boldsymbol{q}^\eta\|_{1+\frac{2\lambda+1}{3+2\beta}}^{1+\frac{2\lambda+1}{3+2\beta}} + \|e^\eta\|_{\beta+2+\lambda}^{\beta+2+\lambda}\,\mathrm{d}t \\ \leqslant & \; C(\lambda^{-1}).\end{aligned} \tag{5.55}$$

Next, we deduce a uniform bound also for the pressure. First, setting $\varphi := \mathcal{N}^{-1}(p_3^\eta)$ in (5.54), we get with help of Lemma 3.1

$$\|p_3^\eta\|_2^2 = (\boldsymbol{v}^\eta, \nabla\varphi)_{\partial\Omega} \leqslant C\|\boldsymbol{v}^\eta\|_{L^2(\partial\Omega)^2}\|\varphi\|_{2,2} \leqslant C\|\boldsymbol{v}^\eta\|_{L^2(\partial\Omega)^2}\|p_3^\eta\|_2.$$

Consequently, using (5.55) we have that

$$\int_0^T \|p_3^\eta\|_2^2\,\mathrm{d}t \leqslant C. \tag{5.56}$$

Similarly, setting $\varphi := \mathcal{N}^{-1}(|p_1^\eta|^{r'-2}p_1^\eta - \fint_\Omega |p_1^\eta|^{r'-2}p_1^\eta\,\mathrm{d}x)$, we get by using Lemma 3.1

$$\|p_1^\eta\|_{r'}^{r'} = (\mathbf{S}^\eta, \nabla^2\varphi) \leqslant \|\mathbf{S}^\eta\|_{r'}\|\nabla^2\varphi\|_r \leqslant C\|\mathbf{S}^\eta\|_{r'}\|p_1^\eta\|_{r'}^{r'-1}$$

and by using (5.55) we deduce that

$$\int_0^T \|p_1^\eta\|_{r'}^{r'}\,\mathrm{d}t \leqslant C. \tag{5.57}$$

Finally, setting $\varphi := \mathcal{N}^{-1}(|p_2^\eta|^{r-2}p_2^\eta - \fint_\Omega |p_2^\eta|^{r-2}p_2^\eta \,\mathrm{d}x)$, we again obtain with help of Lemma 3.1

$$\|p_2^\eta\|_r^r = -(\boldsymbol{v}_\eta^\eta \otimes \boldsymbol{v}^\eta, \nabla^2\varphi) \leqslant \|\boldsymbol{v}_\eta^\eta \otimes \boldsymbol{v}^\eta\|_r \|\nabla^2\varphi\|_{r'} \leqslant C\|\boldsymbol{v}^\eta\|_{2r}^2 \|p_2^\eta\|_r^{r-1}$$

and consequently (5.55) implies

$$\int_0^T \|p_2^\eta\|_r^r \,\mathrm{d}t \leqslant C. \tag{5.58}$$

Thus, combining (5.56)–(5.58) we see that the following uniform bound holds

$$\int_0^T \|p^\eta\|_{\min(r,r')}^{\min(r,r')} \,\mathrm{d}t \leqslant C. \tag{5.59}$$

Thus, using (5.55), (5.59), and the identity (5.51) we obtain

$$\int_0^T \|\boldsymbol{v}_{,t}^\eta\|_{W_{\boldsymbol{n}}^{-1,\min(r,r')}}^{\min(r,r')} \,\mathrm{d}t \leqslant C, \tag{5.60}$$

and also using (5.55) we have that, see [8], [6] for details,

$$\|\boldsymbol{v}_{,t}^\eta\|_{(X_{\mathrm{div}}^{r,2r/(2r-3)})^*} \leqslant C. \tag{5.61}$$

Moreover, we can observe that

$$\|e_{,t}^\eta\|_{L^1(0,T;W^{-1,\sigma'}(\Omega))} \leqslant C \quad \text{for sufficiently large } \sigma. \tag{5.62}$$

Therefore, using (5.55), (5.59)–(5.61), Lemma 3.3, the Aubin-Lions lemma, and the Div-Curl lemma, we can extract a not relabeled subsequence and find $(\boldsymbol{v}, p, e, \boldsymbol{q}, \mathbf{S})$ such that[6]

$$\boldsymbol{v}_{,t}^\eta \rightharpoonup \boldsymbol{v}_{,t} \quad \text{weakly in } L^{\min(r,r')}(0,T;W_{\boldsymbol{n}}^{-1,\min(r,r')}) \cap (X_{\mathrm{div}}^{r,2r/(2r-3)})^*, \tag{5.63}$$

$$\boldsymbol{v}^\eta \rightharpoonup^* \boldsymbol{v} \quad \text{weakly}^* \text{ in } L^r(0,T;W_{\boldsymbol{n}}^{1,r}) \cap L^\infty(0,T;L^2(\Omega)^2), \tag{5.64}$$

$$\boldsymbol{v}^\eta \to \boldsymbol{v} \quad \text{strongly in } L^q(0,T;L^q(\Omega)^2) \text{ for all } q < 2r, \tag{5.65}$$

$$\boldsymbol{v}^\eta \to \boldsymbol{v} \quad \text{strongly in } L^2(0,T;L^2(\partial\Omega)^2), \tag{5.66}$$

$$e^\eta \to e \quad \text{strongly in } L^n(0,T;L^n(\Omega)) \text{ for all } n < 2+\beta, \tag{5.67}$$

$$(e^\eta)^{\frac{\beta+\lambda+1}{2}} \rightharpoonup e^{\frac{\beta+\lambda+1}{2}} \quad \text{weakly in } L^2(0,T;W^{1,2}(\Omega)) \quad \text{for all } -1<\lambda<0, \tag{5.68}$$

$$p^\eta \rightharpoonup p \quad \text{weakly in } L^{\min(r,r')}(0,T;L^{\min(r,r')}(\Omega)), \tag{5.69}$$

$$\mathbf{S}^\eta \rightharpoonup \mathbf{S} \quad \text{weakly in } L^{r'}(0,T;L^{r'}(\Omega)^{2\times2}), \tag{5.70}$$

$$\boldsymbol{q}^\eta \rightharpoonup \boldsymbol{q} \quad \text{weakly in } L^m(0,T;L^m(\Omega)^2) \text{ for all } m < \frac{4+2\beta}{3+2\beta}. \tag{5.71}$$

[6] Here we use the same procedure as in the previous subsection to show pointwise convergence of e^η.

Having these convergence results, one can easily let $\eta \to 0$ in (5.51) to obtain (2.14). It is also standard to deduce the first part of (2.17). Moreover, defining $E^\eta := \frac{1}{2}|\boldsymbol{v}^\eta|^2 + e^\eta$, setting $\boldsymbol{w} := \boldsymbol{v}^\eta \varphi$ in (5.40), adding the result to (5.46) and letting $\eta \to 0$ it is not difficult to obtain (2.15), we refer to [6] for details. Hence, to complete the proof, it is enough to show the second part of (2.17), i.e., the attainment of e_0, and to show the pointwise convergence of $\mathbf{D}(\boldsymbol{v}^\eta)$. Indeed, once having this convergence result, we can easily show that $\mathbf{S} = \mathbf{S}^*(e, \mathbf{D}(\boldsymbol{v}))$ and, similarly to the preceding subsection, that $\boldsymbol{q} = \boldsymbol{q}^*(e, \mathbf{D}(\boldsymbol{v}), \nabla e)$ a.e. in Q. Moreover, one can also use the Fatou lemma and to let $\eta \to 0$ in (5.46) to obtain (2.10) and (2.16).

5.3. Pointwise convergence of $\mathbf{D}(\boldsymbol{v}^\eta)$

In this subsection we use the so-called L^∞ truncation method (applied first to the scalar parabolic equation in [4], see also [3], [2], and generalized to the Navier-Stokes-like system in [14]) to obtain the pointwise convergence of the velocity gradient. In fact, we slightly improve the method in such a way that no diagonal procedure for extracting a subsequence is needed. First, we define

$$g^\eta := |\nabla \boldsymbol{v}^\eta|^r + |\nabla \boldsymbol{v}|^r + (|\mathbf{S}^\eta| + |\mathbf{S}^*(e^\eta, \mathbf{D}(\boldsymbol{v}))|)(|\mathbf{D}(\boldsymbol{v}^\eta)| + |\mathbf{D}(\boldsymbol{v})|) \tag{5.72}$$

and from (2.1) and (5.55) it follows that

$$\int_Q g^\eta \,\mathrm{d}x\,\mathrm{d}t \leqslant C \tag{5.73}$$

independently of η. Next, we prove the essential observation that will be used in the sequel.

Lemma 5.1. *Let g^η be defined through* (5.72). *Then for all $j \in \mathbb{N}$ and all $\eta > 0$ there exist a constant $L_{j,\eta} \in (2^{-2^{2^{j+1}}}, 2^{-2^{2^j}})$ and a set $E_{j,\eta} \subset Q$ defined as*

$$E_{j,\eta} := \{(t,x) \in Q;\ L_{j,\eta}^2 \leqslant |\boldsymbol{v}^\eta(t,x) - \boldsymbol{v}(t,x)| < L_{j,\eta}\}$$

such that

$$\int_{E_{j,\eta}} g^\eta \,\mathrm{d}x\,\mathrm{d}t \leqslant 2^{-j}. \tag{5.74}$$

Proof. First, for fixed j, η we define recurrently for $k = 0, \dots, 2^j$

$$\begin{aligned} L_0 &:= 2^{-2^{2^j}}, \\ L_k &:= L_{k-1}^2 \Longrightarrow L_k = 2^{-2^{2^j} 2^k} \end{aligned}$$

and introduce the sets

$$E_k := \{(t,x) \in Q;\ L_k^2 \leqslant |\boldsymbol{v}^\eta - \boldsymbol{v}| < L_k\}.$$

Since E_k are disjoint, we find by using (5.73) that

$$\sum_{k=1}^{2^j} \int_{E_k} g^\eta \,\mathrm{d}x\,\mathrm{d}t = \int_{\bigcup_k E_k} g^\eta \,\mathrm{d}x\,\mathrm{d}t \leqslant \int_Q g^\eta \,\mathrm{d}x\,\mathrm{d}t \leqslant C.$$

Therefore, there exists $k^* \in \{1, \dots, 2^j\}$ such that

$$\int_{E_{k^*}} g^\eta \,\mathrm{d}x\,\mathrm{d}t \leqslant 2^{-j}. \tag{5.75}$$

Thus, we can finally define $L_{j,\eta} := L_{k^*}$ and we see that (5.74) is satisfied, so the proof is complete. $\square$

We use Lemma 5.1 in the following way. First, we fix an arbitrary $j \in \mathbb{N}$. For such fixed j we find $\{L_{j,\eta}\}_{\eta>0}$ such that (5.74) holds. Further, we define $\boldsymbol{u}^{j,\eta}$ and sets $Q_{j,\eta}$ as

$$\begin{aligned} \boldsymbol{u}^{j,\eta} &:= (\boldsymbol{v}^\eta - \boldsymbol{v})\Big(1 - \min\Big\{\frac{|\boldsymbol{v} - \boldsymbol{v}^\eta|}{L_{j,\eta}}, 1\Big\}\Big), \\ Q_{j,\eta} &:= \{(t,x) \in Q;\ |\boldsymbol{v} - \boldsymbol{v}^\eta| < L_{j,\eta}\}. \end{aligned} \tag{5.76}$$

Since $|\boldsymbol{u}^{j,\eta}| \leqslant 1$ in Q, we deduce from (5.64)–(5.66) that

$$\boldsymbol{u}^{j,\eta} \rightharpoonup \boldsymbol{0} \quad \text{weakly in } L^r(0,T;W^{1,r}_{\boldsymbol{n}}), \tag{5.77}$$

$$\boldsymbol{u}^{j,\eta} \to \boldsymbol{0} \quad \text{strongly in } L^s(0,T;L^s(\Omega)^2) \text{ for all } s < \infty, \tag{5.78}$$

$$\boldsymbol{u}^{j,\eta} \to \boldsymbol{0} \quad \text{strongly in } L^2(0,T;L^2(\partial\Omega)^2) \tag{5.79}$$

as $\eta \to 0_+$. Next, we can observe that

$$\begin{aligned} &\int_Q |\operatorname{div} \boldsymbol{u}^{j,\eta}|^r \,\mathrm{d}x\,\mathrm{d}t \\ &\quad = \int_{Q_{j,\eta}} \Big|(\boldsymbol{v}^\eta - \boldsymbol{v}) \cdot \nabla \frac{|\boldsymbol{v}^\eta - \boldsymbol{v}|}{L_{j,\eta}}\Big|^r \,\mathrm{d}x\,\mathrm{d}t \\ &\quad \leqslant C L_{j,\eta}^r \int_{Q_{j,\eta}\setminus E_{j,\eta}} g^\eta \,\mathrm{d}x\,\mathrm{d}t + C \int_{E_{j,\eta}} g^\eta \,\mathrm{d}x\,\mathrm{d}t \overset{(5.74)}{\leqslant} C 2^{-j}. \end{aligned} \tag{5.80}$$

Consequently, using Lemma 3.1 we deduce

$$\int_0^T \|\boldsymbol{u}^{j,\eta} - \boldsymbol{u}^{j,\eta}_{\mathrm{div}}\|^r_{1,r}\,\mathrm{d}t = \int_0^T \|\nabla\mathcal{N}^{-1}(\operatorname{div}\boldsymbol{u}^{j,\eta})\|^r_{1,r}\,\mathrm{d}t \leqslant C2^{-j}. \tag{5.81}$$

Moreover, using (5.78) and (3.3), we get

$$\boldsymbol{u}^{j,\eta}_{\mathrm{div}} \to \mathbf{0} \quad \text{strongly in } L^s(0,T;L^s(\Omega)^2) \text{ for all } s<\infty. \tag{5.82}$$

Finally, we set $\boldsymbol{w} := \boldsymbol{u}^{j,\eta}_{\mathrm{div}}$ in (5.51) and let $\eta \to 0_+$. First, we discuss the term with the time derivative. Since $\operatorname{div}\boldsymbol{v}^\eta = \operatorname{div}\boldsymbol{v} = 0$, we have

$$\langle \boldsymbol{v}^\eta_{,t}, \boldsymbol{u}^{j,\eta}_{\mathrm{div}}\rangle = \langle \boldsymbol{v}^\eta_{,t} - \boldsymbol{v}_{,t}, \boldsymbol{u}^{j,\eta}\rangle + \langle \boldsymbol{v}_{,t}, \boldsymbol{u}^{j,\eta}_{\mathrm{div}}\rangle.$$

Now, having (5.77)–(5.79) and (5.82) we obtain that

$$\boldsymbol{u}^{j,\eta}_{\mathrm{div}} \rightharpoonup \mathbf{0} \quad \text{weakly in } X^{r,2r/(2r-3)}_{\mathrm{div}} \tag{5.83}$$

as $\eta \to 0$. Consequently, (5.63) implies that

$$\lim_{\eta\to 0_+} \langle \boldsymbol{v}_{,t}, \boldsymbol{u}^{j,\eta}_{\mathrm{div}}\rangle = 0. \tag{5.84}$$

For the remaining term we can observe (defining $\boldsymbol{w}^\eta := \boldsymbol{v}^\eta - \boldsymbol{v}$)

$$\langle \boldsymbol{v}^\eta_{,t} - \boldsymbol{v}_{,t}, \boldsymbol{u}^{j,\eta}\rangle = \Big\langle \boldsymbol{w}^\eta_{,t}, \boldsymbol{w}^\eta\Big(1-\min\Big(\frac{|\boldsymbol{w}^\eta|}{L_{j,\eta}},1\Big)\Big)\Big\rangle = G(|\boldsymbol{w}^\eta(T)|) - G(|\boldsymbol{w}^\eta(0)|) \geqslant 0,$$

where the function G is defined as

$$G(x) := \begin{cases} x^2\Big(\dfrac{1}{2} - \dfrac{x}{3L_{j,\eta}}\Big) & \text{for } x \leqslant L_{j,\eta}, \\ \dfrac{1}{6}L^2_{j,\eta} & \text{for } x > L_{j,\eta}, \end{cases}$$

and where we have also used the fact that $|\boldsymbol{w}^\eta(0)| = 0$. Thus, we observe that

$$\liminf_{\eta\to 0_+} \langle \boldsymbol{v}^\eta_{,t}, \boldsymbol{u}^{j,\eta}_{\mathrm{div}}\rangle \geqslant 0. \tag{5.85}$$

Similarly, using (5.64), (5.82), and integration by parts we find that

$$\lim_{\eta\to 0_+} -(\boldsymbol{v}^\eta_\eta \otimes \boldsymbol{v}, \nabla\boldsymbol{u}^{j,\eta}_{\mathrm{div}})_Q = \lim_{\eta\to 0_+} (\boldsymbol{v}^\eta_\eta \otimes \boldsymbol{u}^{j,\eta}_{\mathrm{div}}, \nabla\boldsymbol{v})_Q = 0. \tag{5.86}$$

Hence, (5.85)–(5.86) and (5.79) imply that

$$\limsup_{\eta\to 0_+}(\mathbf{S}^\eta,\mathbf{D}(\boldsymbol{u}^{j,\eta}_{\mathrm{div}}))_Q \leqslant 0, \tag{5.87}$$

and by using (5.81), (5.70) and the same procedure as in the preceding subsection we derive

$$\limsup_{\eta\to 0_+}(\mathbf{S}^*(e^\eta,\mathbf{D}(\boldsymbol{v}^\eta)) - \mathbf{S}^*(e^\eta,\mathbf{D}(\boldsymbol{v})),\mathbf{D}(\boldsymbol{u}^{j,\eta}))_Q \leqslant 2^{-j}. \tag{5.88}$$

Therefore, using the definition of $\boldsymbol{u}^{j,\eta}$ we can rewrite the last inequality as

$$\begin{aligned}
0 &\leqslant \limsup_{\eta\to 0_+}(\mathbf{S}^*(e^\eta,\mathbf{D}(\boldsymbol{v}^\eta)) - \mathbf{S}^*(e^\eta,\mathbf{D}(\boldsymbol{v})),\mathbf{D}(\boldsymbol{v}^\eta-\boldsymbol{v}))_{Q_{j,\eta}} \\
&\leqslant 2^{-j} + C\int_{Q_{j,\eta}} (|\mathbf{S}^\eta| + |\mathbf{S}^*(e^\eta,\mathbf{D}(\boldsymbol{v}))|)(|\nabla \boldsymbol{v}^\eta| + |\nabla \boldsymbol{v}|)\frac{|\boldsymbol{v}^\eta-\boldsymbol{v}|}{L_{j,\eta}}\,\mathrm{d}x\,\mathrm{d}t \\
&\leqslant 2^{-j} + CL_{j,\eta}\int_{Q_{j,\eta}\setminus E_{j,\eta}} g^\eta\,\mathrm{d}x\,\mathrm{d}t + C\int_{E_{j,\eta}} g^\eta\,\mathrm{d}x\,\mathrm{d}t \overset{(5.74)}{\leqslant} C2^{-j}.
\end{aligned} \tag{5.89}$$

Finally, using (5.65) we deduce that

$$|Q\setminus Q_{j,\eta}| \to 0 \quad \text{as } \eta\to 0,$$

and therefore by using the Hölder inequality and (5.89), we have

$$\begin{aligned}
\limsup_{\eta\to 0_+} &\int_Q |(\mathbf{S}^*(e^\eta,\mathbf{D}(\boldsymbol{v}^\eta)) - \mathbf{S}^*(e^\eta,\mathbf{D}(\boldsymbol{v})))\cdot\mathbf{D}(\boldsymbol{v}^\eta-\boldsymbol{v})|^{1/2}\,\mathrm{d}x\,\mathrm{d}t \\
&= \limsup_{\eta\to 0_+}\int_{Q\setminus Q_{j,\eta}} \ldots + \int_{Q_{j,\eta}} \ldots \\
&\overset{(5.73)}{\leqslant} C(\limsup_{\eta\to 0_+}|Q\setminus Q_{j,\eta}| + (\mathbf{S}^*(e^\eta,\mathbf{D}(\boldsymbol{v}^\eta)) \\
&\qquad - \mathbf{S}^*(e^\eta,\mathbf{D}(\boldsymbol{v})),\mathbf{D}(\boldsymbol{v}^\eta-\boldsymbol{v}))_{Q_{j,\eta}})^{1/2} \\
&\overset{(5.89)}{\leqslant} C2^{-j/2}.
\end{aligned}$$

Since j can be chosen arbitrarily, we see that the last limit is zero and by using the strict monotonicity of $\mathbf{S}^*$, i.e., the assumption (2.2) and the strong convergence of e^η (5.67), we can conclude that

$$\mathbf{D}(\boldsymbol{v}^\eta) \to \mathbf{D}(\boldsymbol{v}) \quad \text{a.e. in } Q.$$

5.4. Initial conditions

This subsection is devoted to the proof of the second part of (2.17). Here we assume that the first part of (2.17) holds (see [16] for a detailed proof). First, we set $\varphi := \chi_{(0,t)}$ in (5.18) and $\boldsymbol{w} := \boldsymbol{v}^\eta \chi_{(0,t)}$ and sum the resulting identities to obtain

$$\frac{1}{2}\|\boldsymbol{v}^\eta(t)\|_2^2 + \|e^\eta(t)\|_1 = \frac{1}{2}\|\boldsymbol{v}_0\|_2^2 + \|e_0\|_1. \tag{5.90}$$

Thus, letting $\eta \to 0_+$ and using (5.65), (5.67), and the Fatou lemma we obtain for a.a. $t \in (0,T)$

$$\frac{1}{2}\|\boldsymbol{v}(t)\|_2^2 + \|e(t)\|_1 \leqslant \frac{1}{2}\|\boldsymbol{v}_0\|_2^2 + \|e_0\|_1. \tag{5.91}$$

Moreover, one can redefine $\boldsymbol{v}$ and e on the zero measure subset of $(0,T)$ so that (5.91) holds for all $t \in (0,T)$. Therefore, using the first part of (2.17) and letting $t \to 0_+$ in (5.91) we deduce that

$$\limsup_{t\to 0_+} \|e(t)\|_1 \leqslant \|e_0\|_1. \tag{5.92}$$

Next, setting $\varphi := \chi_{(0,t)}(\psi/\sqrt{e^\eta})$ in (5.46) with arbitrary $\psi \geqslant 0$ such that $\psi \in W^{1,\infty}(\Omega)$ (this is possible since all terms are meaningful) we deduce (using the non-negativity of ψ) that

$$\frac{1}{2}(\sqrt{e^\eta(t)}, \psi) - \int_0^t (\boldsymbol{q}^\eta, (e^\eta)^{-1/2}\nabla\psi)\,\mathrm{d}\tau \geqslant \frac{1}{2}(\sqrt{e_0}, \psi). \tag{5.93}$$

Hence, using (5.55), (5.67), and (5.71) we can let $\eta \to 0_+$ in (5.93) to deduce for a.a. $t \in (0,T)$

$$\frac{1}{2}(\sqrt{e(t)}, \psi) - \int_0^t (\boldsymbol{q}, e^{-1/2}\nabla\psi)\,\mathrm{d}\tau \geqslant \frac{1}{2}(\sqrt{e_0}, \psi), \tag{5.94}$$

which can be again extended to the whole time interval $(0,T)$. Thus letting $t \to 0_+$, we have

$$\liminf_{t\to 0_+}(\sqrt{e(t)}, \psi) \geqslant (\sqrt{e_0}, \psi) \tag{5.95}$$

for all smooth nonnegative ψ by using the density argument and (2.10) for all non-negative $\psi \in L^2(\Omega)$. Therefore, we have

$$\begin{aligned}\limsup_{t\to 0_+} \|\sqrt{e(t)} - \sqrt{e_0}\|_2^2 &= \limsup_{t\to 0_+} \|e(t)\|_1 + \|e_0\|_1 - 2(\sqrt{e(t)}, \sqrt{e_0}) \\ &\overset{(5.92),(5.95)}{\leqslant} 2\|e_0\|_1 - 2\|e_0\|_1 = 0\end{aligned}$$

and (2.17) follows. Thus the proof of Theorem 2.1 is complete. The proof of Corollary 2.1 is then a consequence of the standard theory for the heat equation with an L^1-right-hand side and of the fact that for $r \geqslant 2$ the velocity is a possible test function in (2.14).

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