Kybernetika

Paolo Mercorelli
Robust decoupling through algebraic output feedback in manipulation systems

ROBUST DECOUPLING THROUGH ALGEBRAIC OUTPUT FEEDBACK IN MANIPULATION SYSTEMS

PAOLO MERCORELLI

This paper investigates the geometric and structural characteristics involved in the control of general mechanisms and manipulation systems. These systems consist of multiple cooperating linkages that interact with a reference member of the mechanism (the "object") by means of contacts on any available part of their links. Grasp and manipulation of an object by the human hand is taken as a paradigmatic example for this class of manipulators. Special attention is devoted to the output specification and its controllability. An example design of a force controller using algebraic output feedback is presented at the end of this paper. In this example, a matrix representing a static output feedback is designed. The coefficients of this matrix are the weights for the sensed outputs. With the approach proposed in this paper, a robust decoupling is obtained between the output feedback and the contact forces and joint positions.

1. INTRODUCTION

In the past three decades, research on the geometric approach to dynamic systems theory and control has allowed this approach to become a powerful and a thorough tool for the analysis and synthesis of dynamic systems [1, 6, 17]. Over the same time period, mechanical systems used in industry and developed in research labs have also evolved rapidly. Robotics is a notable case of such evolution. The robotics community has developed sophisticated analysis and control techniques to meet increasing requirements on the control of mechanical systems. These increasing requirements are motivated by higher performance specifications, an increasing number of degrees-of-freedom, and the introduction of the interacting robotic limbs. General systems of multiple interacting robot limbs can be used to model arbitrary mechanisms. Typical robotics concepts and tools, such as manipulability analysis, can then be applied to such systems [3] and [4]. A unified control theory of mechanical systems is conceivable by drawing upon recent results in robotics and extending them to general classes of mechanisms. To achieve this goal, it is necessary to remove certain assumptions that limit the generality of cooperating robot limbs and to fully understand the corresponding theory. Generalizations of robotic models that must
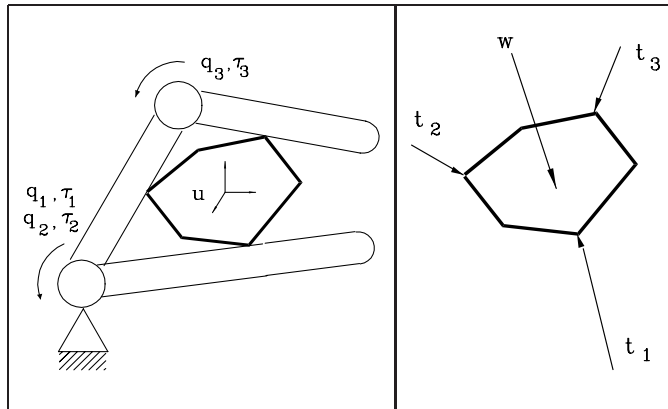
Fig. 1. Vector notation for general manipulation system analysis.

be considered to achieve this goal include:

- Each interacting limb can interact with the object and with any of its links.

- A limb's interaction with the object can be specified by several different models, including rigid attachment, rolling contact, and sliding contact between the bodies, see Figure 1.

These generalizations entail modifications in the theoretical approach. In fact, a survey of current literature shows recent advances towards the goal of a general, unified treatment of manipulation systems. In particular, recent results in [12, 15] and [16] gave important details concerning additional advances. References [12] and [16] mark progress in the analysis and synthesis of geometric controller for mechanical systems, and [10] proposes an explicit new matrix structure for grasping internal forces. Reference [11] reports the possibility of parameterizing input controlled subspaces to guarantee non-interaction. This paper enhances the results presented in [12]. In [12], a robust decoupling controller using an algebraic state input feedback is presented, while this paper presents a robust decoupling controller using an algebraic output-input feedback. In general, the design of output-input feedback controllers is a very ambitious task and not only in relation to the geometric approach. This approach allows observers, normally present in high-order systems, to be avoided. The force/motion control problem has attracted significant attention over last decade in the fields of robotic manipulation and mobile manipulators. Approaches exploiting input-output decoupling controllers are found, for instance, in the work [18].

The geometric approach allows very elegant solutions to control problems. Nevertheless, robustness analysis using a linear geometric control offers answers through rank conditions of matrices that are necessary conditions. These conditions are often not constructive conditions. Even though the rank conditions offer simple "on-off" conditions, it is also possible to measure the robustness. In [12], a robust decoupling controller is obtained using a state input feedback controller. The drawback of this

approach consists of a wide sensing structure. In fact, the whole state space should
be available and sensed. With the approach proposed in this paper, the robust de-
coupling is obtained with an output feedback from the contact forces and the joint
positions. The goal of this paper is to state structural conditions for achieving an
output-input feedback force/motion robust decoupling. An explicit expression of a
solution for a controlled and conditioned subspace is proposed. A complete con-
structive procedure for the design of the decoupling controller is presented. At the
end, a numerical example with computer simulations is shown.

1.1. Organization of the paper

The paper is organized as follows. Section 2 considers the kinematic and the dynamic
model. In Section 3 the general problem of internal forces is considered. Section 4
presents the structural properties required to achieve robust decoupling. The paper
closes with simulations and a conclusion.

2. KINEMATIC AND DYNAMIC MODEL

This section summarises notation and some results from the analysis of dynamics
for manipulation with general kinematics. The model of the general manipulation
system is comprised of a mechanism with an arbitrary number of actuated links and
of an object in contact, at one or more points, with some of the links. Let $\mathbf{q} \in \mathbb{R}^q$
be the vector of generalised coordinates that completely describes the configuration
of the manipulation system. Let $\tau \in \mathbb{R}^q$ be the vector of actuated (rotoidal) joint
torques and (prismatic) joint forces. Additionally, let $\mathbf{u} \in \mathbb{R}^d$ be the vector of the
generalised coordinates for the object ($d = 3$ for 2D cases while $d = 6$ for 3D cases),
and let $\mathbf{w} \in \mathbb{R}^\mathbf{d}$ be the vector of external disturbances acting on the object. To
clarify the vector notation, see Figure 1. The lumped parameter visco–elastic model
at the ith contact is described by introducing contact vectors $\mathbf{c}_i^m$ and $\mathbf{c}_i^o$. The d–
dimensional contact vector $\mathbf{c}_i^m$ ($\mathbf{c}_i^o$) represents the coordinates of a fixed frame at
the contact point on the link of the manipulator (on the object). The contact force
($\mathbf{f}_i$) and the moment ($\mathbf{m}_i$) exchanged at the contact are represented by the vector
$\mathbf{t}_i = [\mathbf{f}_i^T, \mathbf{m}_i^T]$ which, according to the visco–elastic model, can be written as follows:

$$\mathbf{t}_i = \mathbf{K}_i \mathbf{H}_i (\mathbf{c}_i^m - \mathbf{c}_i^o) + \mathbf{B}_i \mathbf{H}_i (\dot{\mathbf{c}}_i^m - \dot{\mathbf{c}}_i^o).$$

The parameters indicated with $\mathbf{K}_i$ and $\mathbf{B}_i$ are the stiffness and damping matrices,
respectively. $\mathbf{H}_i$ is a constant selection matrix describing several types of contact
models. In three-dimensional space and in the presence of hard-contact, matrix
$\mathbf{K}_i$ and $\mathbf{B}_i$ are matrices sized 3×3. If the contact is soft, these matrices have
dimension 4×3, see [8]. Matrix $\mathbf{H}_i$ is of dimension 3×6 for hard-contact and of
dimension 4×6 for soft contact. Notice that the presence of moment $\mathbf{m}_i$ in $\mathbf{t}_i$ depends
upon the contact interaction type. Now, let $\mathbf{t} = [\mathbf{f}_1, \cdots, \mathbf{f}_n, \mathbf{m}_1, \cdots, \mathbf{m}_n]$ be the
overall contact force vector built by grouping all the vectors $\mathbf{t}_i$ for all n contacts.
Accordingly, vector $\mathbf{t}$ is given by $\mathbf{t} = \mathbf{KH}(\mathbf{c}^m - \mathbf{c}^o) + \mathbf{BH}(\dot{\mathbf{c}}^m - \dot{\mathbf{c}}^o)$. The Jacobian
$\mathbf{J}$ and grasp matrix $\mathbf{G}$ are defined as $\mathbf{J} = \mathbf{H}\frac{\delta \mathbf{c}^m}{\delta \mathbf{q}}$ and $\mathbf{G}^T = \mathbf{H}\frac{\delta \mathbf{c}^o}{\delta \mathbf{u}}$. Thus, the local

approximation of the contact force vector $\mathbf{t}$ can be written as

$$\delta\mathbf{t} = \mathbf{K}\left(\mathbf{J}\delta\mathbf{q} - \mathbf{G}^T\delta\mathbf{u}\right) + \mathbf{B}\left(\mathbf{J}\delta\dot{\mathbf{q}} - \mathbf{G}^T\delta\dot{\mathbf{u}}\right). \tag{1}$$

The following dynamic model is the linearization of the nonlinear dynamics of manipulation systems derived in [14]. Consider a reference equilibrium configuration $(\mathbf{q}, \mathbf{u}, \dot{\mathbf{q}}, \dot{\mathbf{u}}, \tau, \mathbf{t}) = (\mathbf{q}_o, \mathbf{u}_o, \mathbf{0}, \mathbf{0}, \tau_o, \mathbf{t}_o)$, such that $\tau_o = \mathbf{J}^T\mathbf{t}_o$ and $\mathbf{w}_o = -\mathbf{G}\mathbf{t}_o$. In the neighborhood of such an equilibrium, linearized dynamics of the manipulation system can be written as follows:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}_\tau\tau' + \mathbf{B}_w\mathbf{w}', \tag{2}$$

such that state, input and disturbance vectors are defined as departures from the following reference equilibrium configuration:

$$\mathbf{x} = \left[(\mathbf{q} - \mathbf{q}_o)^T \ (\mathbf{u} - \mathbf{u}_o)^T \ \dot{\mathbf{q}}^T \ \dot{\mathbf{u}}^T\right]^T,$$
$$\tau' = \tau - \mathbf{J}^T\mathbf{t}_o, \ \mathbf{w}' = \mathbf{w} + \mathbf{G}\mathbf{t}_o \ \text{ and}$$
$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{L}_k & \mathbf{L}_b \end{bmatrix}; \ \mathbf{B}_\tau = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{M}_h^{-1} \\ \mathbf{0} \end{bmatrix}; \ \mathbf{B}_w = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{M}_o^{-1} \end{bmatrix},$$

to be more precise

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{I}_q & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_u \\ -\mathbf{M}_h^{-1}\mathbf{J}^T\mathbf{K}\mathbf{J} & \mathbf{M}_h^{-1}\mathbf{J}^T\mathbf{K}\mathbf{G}^T & -\mathbf{M}_h^{-1}\mathbf{J}^T\mathbf{B}\mathbf{J} & \mathbf{M}_h^{-1}\mathbf{J}^T\mathbf{B}\mathbf{G}^T \\ \mathbf{M}_o^{-1}\mathbf{G}\mathbf{K}\mathbf{J} & -\mathbf{M}_o^{-1}\mathbf{G}\mathbf{K}\mathbf{G}^T & \mathbf{M}_o^{-1}\mathbf{G}\mathbf{B}\mathbf{J} & -\mathbf{M}_o^{-1}\mathbf{G}\mathbf{B}\mathbf{G}^T \end{bmatrix}, \tag{3}$$

where $\mathbf{M}_h$ and $\mathbf{M}_o$ are the inertia matrices of the manipulator and the object, respectively. To simplify the notation we will henceforth omit the prime in τ' and $\mathbf{w}'$. Simple expressions are obtained for $\mathbf{L}_k$ and $\mathbf{L}_b$ by neglecting terms due to gravity variations, rolling phenomena at the contacts, and local variations of the Jacobian and grasp matrices and under the hypothesis that stiffness and damping matrices are proportional ($\mathbf{B} \propto \mathbf{K}$), see [9].

$$\mathbf{L}_k = -\mathbf{M}^{-1}\mathbf{P}_k; \qquad \mathbf{L}_b = -\mathbf{M}^{-1}\mathbf{P}_b$$

where

$$\mathbf{M} = \text{diag}(\mathbf{M}_h, \mathbf{M}_o);$$
$$\mathbf{P}_k = \mathbf{S}^T\mathbf{K}\mathbf{S};$$
$$\mathbf{P}_b = \mathbf{S}^T\mathbf{B}\mathbf{S};$$
$$\mathbf{S} = [\mathbf{J} \ -\mathbf{G}^T].$$

The remainder of this section provides results obtained in [14] on the control of internal forces, a problem of paramount importance in robotic manipulation. Let us define $\mathbf{t}'$ as the first order approximation of departures of contact force vector $\mathbf{t}$

from the reference equilibrium $\mathbf{t}_o$. According to equation (1), $\mathbf{t}'$ (henceforth $\mathbf{t}$) can be regarded as an output of the linearized model (2), $\mathbf{t} = \mathbf{C}_t \mathbf{x}$ where

$$\mathbf{C}_t = \begin{bmatrix} \mathbf{KJ} & -\mathbf{KG}^T & \mathbf{BJ} & -\mathbf{BG}^T \end{bmatrix}.$$

When manipulation systems with general kinematics are taken into account, [13] not all of the internal forces are controllable. In [5], the *reachable internal forces of subspace* $\mathcal{R}_{ti,\tau}$ for dynamic systems (2) are analyzed and the internal force output $\mathbf{e}_{ti}$ is defined as the projection of the force vector $\mathbf{t}$ onto subspace $\mathcal{R}_{ti,\tau}$

$$\mathbf{e}_{ti} = \mathbf{E}_{ti}\mathbf{x} \qquad\qquad \text{where}$$
$$\mathbf{E}_{ti} = [\mathbf{Q(K)}\ \mathbf{0}\ \mathbf{Q(K)}\ \mathbf{0}] \qquad\quad \text{and} \qquad\qquad (4)$$
$$\mathbf{Q(K)} = (\mathbf{I} - \mathbf{KG}^T(\mathbf{GKG}^T)^{-1}\mathbf{G})\mathbf{KJ}.$$

3. DECOUPLING OF INTERNAL FORCES

In many advanced robotics tasks, visco–elasticity at the contacts cannot be neglected, and decoupling control of the internal forces from the object's motions is needed. Rigid–body kinematics [3, 14] involves motions of the object and manipulator which do not include visco–elastic deformations. For this reason, they are regarded as the low–energy motions of the whole system. Rigid–body kinematics represents the easiest way to move the manipulated object, and therefore it is of particular interest in controlling manipulation. In [3], *coordinated* rigid–body motions of the mechanisms are defined as motions of the manipulator $\delta\mathbf{q}$ and of the object $\delta\mathbf{u}$ such that

$$\begin{bmatrix} \delta\mathbf{q} \\ \delta\mathbf{u} \end{bmatrix} \in \mathsf{im} \begin{bmatrix} \mathbf{\Gamma}_{qc} \\ \mathbf{\Gamma}_{uc} \end{bmatrix}$$

where

$$\mathbf{J}\mathbf{\Gamma}_{qc} = \mathbf{G}^T\mathbf{\Gamma}_{uc}. \qquad\qquad (5)$$

Thus, rigid–body object motions are those in the column space of $\mathbf{\Gamma}_{uc}$. The output $\mathbf{e}_{uc}$ is defined as the projection of object displacements $\mathbf{u}$ onto the column space of $\mathbf{\Gamma}_{uc}$,

$$\mathbf{e}_{uc} = \mathbf{E}_{uc}\mathbf{x} \qquad\qquad \text{where}$$
$$\mathbf{E}_{uc} = \mathbf{\Gamma}_{uc}^P [\mathbf{0}\ \mathbf{I}\ \mathbf{0}\ \mathbf{0}] \qquad \text{and} \qquad\qquad (6)$$
$$\mathbf{\Gamma}_{uc}^P = \mathbf{\Gamma}_{uc}(\mathbf{\Gamma}_{uc}^T\mathbf{\Gamma}_{uc})^{-1}\mathbf{\Gamma}_{uc}^T.$$

In this paper, we adopt the notion of *internal force decoupling control* that is formalized by the following definition.

Definition 1. Consider the couple $(\mathbf{A}, \mathbf{B}_\tau)$ in (2). The control law $\tau = \mathbf{Fx} + \mathbf{U}\tau_{ref}$ is an *internal force control decoupled from the object motion* if the state feedback and the input selection matrices are such that

a) $\mathcal{R}_{ti} = \mathsf{min}\ \mathcal{I}(\mathbf{A} + \mathbf{B}_\tau\mathbf{F}, \mathbf{B}_\tau\mathbf{U}) \subseteq \mathsf{ker}\,\mathbf{E}_{uc}$;
b) $\mathsf{im}\,\mathbf{E}_{ti} = \mathbf{E}_{ti}\,\mathsf{min}\ \mathcal{I}(\mathbf{A} + \mathbf{B}_\tau\mathbf{F}, \mathbf{B}_\tau\mathbf{U})$

where $\mathsf{min}\ \mathcal{I}(\mathbf{A}, \mathbf{B})$ is the minimal subspace $\mathbf{A}$–invariant containing the column space of $\mathbf{B}$.

Remark 1. Note that $\operatorname{im} \mathbf{B}_\tau \mathbf{U}$ satisfies the following relationship: $\operatorname{im} \mathbf{B}_\tau \mathbf{U} = \operatorname{im}(\mathbf{B}_\tau) \cap \mathcal{R}_{ti}$. Also, $\operatorname{im}(\mathbf{B}_\tau) \cap \mathcal{R}_{ti} \neq 0$. In other words, the system must be no-left invertible.

Observe that the decoupling control of internal forces does not affect the rigid–body object motion (claim a) and preserves the reachability of internal forces (claim b). In [5], it has been proven that for general manipulation systems with $\operatorname{ker}(\mathbf{G}^T) \neq \{0\}$, the problem of finding a decoupling internal force control always has a solution, and a decoupling feedback control law is proposed. The choice of matrices $\mathbf{F}$ and $\mathbf{U}$ is based on the geometric concept of the *controlled invariant*, see [1].

4. ROBUST DECOUPLING CONTROL OF CONTACT FORCES

The control law in [5] is model based, so an accurate identification procedure of the model's parameters is needed. While techniques estimating the dynamic parameters of the object and of the manipulator are well established, the identification of the visco–elastic contact matrices $\mathbf{K}$ and $\mathbf{B}$ still remains a hard task. Also, a certain degree of uncertainty is present in the system model. This section is devoted to the analysis and design of a robust decoupling controller for manipulation systems with structured (visco–elastic) uncertainties.

We assume that a structured uncertainty is present in the visco–elastic contact behavior. In particular, we assume that the estimated stiffness and damping matrices have the following structures: $\mathbf{K}_s = k_s \mathbf{Z}$ and $\mathbf{B}_s = b_s \mathbf{Z}$. In these structures, matrix $\mathbf{Z}$ represents the a priori knowledge of the visco–elastic behavior, while k_s and b_s (real positive values) represent estimated stiffness and damping parameters. We assume that the measured values for k_s and b_s are corrupted by errors Δk and Δb, as described by $\Delta k \in [\underline{\Delta k}, \overline{\Delta k}]$ and $\Delta b \in [\underline{\Delta b}, \overline{\Delta b}]$, thus it holds

$$\begin{aligned} \mathbf{K} &= (k_s + \Delta k)\mathbf{Z}; \\ \mathbf{B} &= (b_s + \Delta b)\mathbf{Z}. \end{aligned} \tag{7}$$

The uncertainties on the stiffness and damping matrices reflect the linearized dynamics of equation (2) and the output matrix of equation (4), which becomes uncertain. This is denoted by:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}(\Delta k, \Delta b)\mathbf{x} + \mathbf{B}_\tau \tau; \\ \mathbf{e}_{ti} = \mathbf{E}_{ti}(\Delta k, \Delta b)\mathbf{x}; \\ \mathbf{e}_{uc} = \mathbf{E}_{uc}\mathbf{x}, \end{cases} \tag{8}$$

where

$$\begin{aligned} \mathbf{A}(\Delta k, \Delta b) &= \mathbf{A}_s + \Delta k \mathbf{A}_{ke} + \Delta b \mathbf{A}_{be}; \\ \mathbf{E}_{ti}(\Delta k, \Delta b) &= \mathbf{E}_s + \Delta k \mathbf{E}_{ke} + \Delta b \mathbf{E}_{be}. \end{aligned} \tag{9}$$

$\mathbf{A}_s$ and $\mathbf{E}_s$ represent the state and force output matrices from Section 2 that are calculated with nominal values $\mathbf{K} = k_s \mathbf{Z}$ and $\mathbf{B} = b_s \mathbf{Z}$, while

$$\mathbf{A}_{ke} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{S}^T \mathbf{Z} \mathbf{S} & \mathbf{0} \end{bmatrix}; \qquad \mathbf{A}_{be} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}^T \mathbf{Z} \mathbf{S} \end{bmatrix};$$

$$\mathbf{E}_{ke} = [\mathbf{Q}(\mathbf{Z}) \ \mathbf{0} \ \mathbf{0}]; \qquad \mathbf{E}_{be} = [\mathbf{0} \ \mathbf{0} \ \mathbf{Q}(\mathbf{Z}) \ \mathbf{0}],$$

specify the structure of the system uncertainty. After having characterized the structured uncertainties affecting the manipulation dynamics, we formalized the concept of a *robust internal force decoupling controller* [2] by extending the requirements on decoupling and reachability of Definition 1.

Definition 2. Consider the set of dynamic systems $(\mathbf{A}(\Delta k, \Delta b), \mathbf{B}_\tau)$ for all $(\Delta k, \Delta b)$, a subspace $\mathcal{V}$ is a *generalized controlled invariant*. If there exists a constant matrix $\mathbf{F}$, such that

$$(\mathbf{A}(\Delta k, \Delta b) + \mathbf{B}_\tau \mathbf{F})\mathcal{V} \subseteq \mathcal{V} \quad \forall(\Delta k, \Delta b).$$

Using definition (6), it is easy to extend the generalization of controlled invariants to self–bounded controlled invariants. Then, it is possible to define

$$\mathcal{V}^\star(\mathbf{A}(\Delta k, \Delta b), \mathbf{B}_\tau, \ker(\mathbf{E}_{uc}))$$

as the maximal generalized $(\mathbf{A}(\Delta k, \Delta b), \mathbf{B}_\tau)$–controlled invariant contained in the null space of the (object motion) output matrix $\mathbf{E}_{uc}$. The algorithm in [2] allows one to evaluate the subspace $\mathcal{V}^\star$. This computation is the first step for the synthesis of the robust-decoupling controller.

Algorithm for the computation of $\mathcal{V}^\star$

$\mathcal{V}_0 = \ker(\mathbf{E}_{uc})$;
$\mathcal{V}_{k+1} = \mathcal{V}_k \cap \mathbf{A}_s^{-1}(\operatorname{im} \mathbf{B}_\tau, \mathcal{V}_k) \cap \mathbf{A}_{ke}^{-1}\mathcal{V}_k \cap \mathbf{A}_{be}^{-1}\mathcal{V}_k$;
if $\mathcal{V}_n = \mathcal{V}_{n-1}$, then $\mathcal{V}_n = \mathcal{V}^\star(\mathbf{A}(\cdot), \mathbf{B}_\tau, \ker(\mathbf{E}_{uc}))$

Remark 1. It can be shown that the state feedback matrix $\mathbf{F}$, which makes the subspace $\mathcal{V}^\star$ invariant with respect to $(\mathbf{A}_s + \mathbf{B}_\tau \mathbf{F})$, fulfils the condition of Definition 6. Recall that $\mathbf{A}_s$ is equal to $\mathbf{A}(\Delta k, \Delta b)$ for $(\Delta k, \Delta b) = (0, 0)$. Therefore the subspace $\mathcal{V}^\star$ is controlled invariant with respect to the pair $(\mathbf{A}_s, \mathbf{B}_\tau)$.

Let us define the input selection matrix $\mathbf{U}$ as

$$\operatorname{im}(\mathbf{B}_\tau \mathbf{U}) = \mathcal{V}^\star \cap \operatorname{im} \mathbf{B}_\tau. \tag{10}$$

Then, the following proposition, whose proof comes easily from Remark 1, shows the state feedback matrices $\mathbf{F}$ and the input selectors $\mathbf{U}$ decoupling internal forces from object motions. This occurs despite the visco–elastic uncertainties.

Proposition 1. A necessary condition for claim b in Definition 3 to hold is

$$\mathbf{E}_{ti}\mathcal{V}^\star = \operatorname{im} \mathbf{E}_{ti}.$$

P r o o f . The proposition is simply proven by observing that for any choice of $\mathbf{F}$ and $\mathbf{U}$, the minimal invariant $\min \mathcal{I}\left((\mathbf{A}(\Delta k, \Delta b) + \mathbf{B}_\tau \mathbf{F}), \mathbf{B}_\tau \mathbf{U}\right)$ is a subspace of $\mathcal{V}^\star$ for all $(\Delta k, \Delta b)$. Notice that the existence of non–empty $\mathcal{V}^\star$ is necessary for the fulfillment of the decoupling condition (a). □

Proposition 2. Choose matrices $\mathbf{F}$ and $\mathbf{U}$ according to Remark 1 and to Proposition 3. Conditions

$$\dim(\mathcal{V}^\star) = \operatorname{rank}(\mathbf{B}_\tau \mathbf{U}),$$
$$\mathbf{E}_{ti} \mathcal{V}^\star = \operatorname{im} \mathbf{E}_{ti},$$

are sufficient for claim $b)$ in Definition 3 to hold.

P r o o f . Simply observe that under these conditions

$$\min \mathcal{I}\left((\mathbf{A}(\Delta k, \Delta b) + \mathbf{B}_\tau \mathbf{F}), \mathbf{B}_\tau \mathbf{U}\right)$$

is equal to $\mathcal{V}^\star$, which does not depend on corrupting errors $(\Delta k, \Delta b)$. □

Definition 3. Consider the class of linear systems

$$\left(\mathbf{A}(\Delta k, \Delta b), \mathbf{B}_\tau, \left[\begin{array}{c} \mathbf{E}_{ti}(\Delta k, \Delta b) \\ \mathbf{E}_{uc} \end{array}\right]\right) \qquad \forall (\Delta k, \Delta b).$$

The control law $\tau = \mathbf{L}\mathbf{C}\mathbf{x} + \mathbf{U}\tau_{ref}$, where matrix $\mathbf{C}$ is the matrix of the measured output, is a *decoupling control of internal forces that are robust with respect to the visco–elastic uncertainties* if

a) $\min \mathcal{I}\left((\mathbf{A}(\Delta k, \Delta b) + \mathbf{B}_\tau \mathbf{L}\mathbf{C}), \mathbf{B}_\tau \mathbf{U}\right) \subseteq \ker \mathbf{E}_{uc}$;

b) $\operatorname{im} \mathbf{E}_{ti}(\Delta k, \Delta b) = \mathbf{E}_{ti}(\Delta k, \Delta b) \min \mathcal{I}\left((\mathbf{A}(\Delta k, \Delta b) + \mathbf{B}_\tau \mathbf{L}\mathbf{C}), \mathbf{B}_\tau \mathbf{U}\right).$

Definition 4. A subspace $\mathcal{V}^\star$ is a *generalized-controlled invariant* if

$$\mathbf{A}(\Delta k, \Delta b)\mathcal{V}^\star \subseteq \mathcal{V}^\star \oplus \mathbf{B} \qquad \forall (\Delta k, \Delta b).$$

Definition 5. A subspace $\mathcal{V}^\star$ is a *generalized-conditioned invariant* if

$$\mathbf{A}(\Delta k, \Delta b)(\mathcal{V}^\star \cap \ker(\mathbf{C})) \subseteq \mathcal{V}^\star \qquad \forall (\Delta k, \Delta b).$$

For further details about these definitions, see [1].

4.1. Algebraic output decoupling of internal forces

In this section, we focus on the decoupling control of internal forces. This is a control goal that is less ambitious than the non-interaction of internal forces and rigid–body object motions.

Definition 6. Consider the set of dynamic systems $(\mathbf{A}(\Delta k, \Delta b), \mathbf{B}_\tau)$ for all $(\Delta k, \Delta b)$. A subspace $\mathcal{V}^\star$ is a *generalized-conditioned invariant* if there exists a constant matrix $\mathbf{L}$ such that

$$(\mathbf{A}(\Delta k, \Delta b) + \mathbf{B}_\tau \mathbf{L} \mathbf{C})\mathcal{V}^\star \subseteq \mathcal{V}^\star \quad \forall (\Delta k, \Delta b).$$

Let us define the input selection matrix $\mathbf{U}$ as

$$\mathsf{im}(\mathbf{B}_\tau \mathbf{U}) = \mathcal{V}^\star \cap \mathsf{im} \mathbf{B}_\tau. \tag{11}$$

Proposition 3. The decoupling condition (a) of Definition 3 is satisfied iff the maximal controlled invariant $\mathcal{V}^\star$ is not empty, and the input selection matrix $\mathbf{U}$ is not null.

The P r o o f of this Proposition is straightforward. □

Proposition 4. Choose matrices $\mathbf{L}$ and $\mathbf{U}$ according to Definition 6 and Proposition 3. Condition

$$\mathsf{rank}(\mathbf{Q}(\mathbf{Z})) = \mathsf{rank}(\mathbf{E}_{ti} \mathbf{B}_\tau \mathbf{U})$$

is sufficient for claim $b)$ of Definition 3 to hold.

P r o o f. Since $\mathbf{K} \propto \mathbf{Z}$, from the definition of matrix $\mathbf{Q}(\cdot)$ in (4), it ensures that the column spaces of $\mathbf{E}_{ti}$ coincides with the column space of $\mathbf{Q}(\mathbf{Z})$. Thus, $\mathsf{rank}(\mathbf{Q}(\mathbf{Z})) = \mathsf{rank}(\mathbf{E}_{ti} \mathbf{B}_\tau \mathbf{U})$ implies that $\mathsf{im}(\mathbf{E}_{ti}) = \mathsf{im}(\mathbf{E}_{ti} \mathbf{B}_\tau \mathbf{U})$, and the proof ends. □

From Definition 6 and Proposition 4, it is possible to question whether matrix $\mathbf{L}$ exists. The matrix $\mathbf{L}$ represents a static-output feedback. The sensed outputs are weighted by the coefficients of the matrix $\mathbf{L}$. It will be shown that the decoupling control of the internal forces can be obtained by means of an algebraic output feedback control from the sensed output consisting of contact forces $\mathbf{t}$ and of manipulator joint positions $\mathbf{q}$. These have an output relationship for the linearized model denoted by the following:

$$\mathbf{y}_m = \mathbf{C} \mathbf{x}$$
$$\mathbf{C} = \begin{bmatrix} \mathbf{I}_{q \times q} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{K} \mathbf{J} & -\mathbf{K} \mathbf{G}^T & \mathbf{B} \mathbf{J} & -\mathbf{B} \mathbf{G}^T \end{bmatrix}. \tag{12}$$

Under the assumptions of non-indeterminacy $\ker(\mathbf{G}^T) = \mathbf{0}$ and isotropy $\mathbf{K} = \alpha\mathbf{B}$ and from (5), the null space of $\mathbf{C}$ is easily computed as

$$\ker(\mathbf{C}) = \mathrm{im} \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{\Gamma}_{qc} & \mathbf{0} & \ker(\mathbf{J}) \\ \mathbf{\Gamma}_{uc} & -\alpha\mathbf{I} & \mathbf{0} \end{bmatrix}.$$

Proposition 5. Given basis $\mathbf{\Gamma}_{\mathbf{uc}}$ defined in (6), then $\ker(\mathbf{\Gamma}_{uc}^T)$ is $\mathbf{M}_o^{-1}\mathbf{GKG}^{T}$–invariant.

P r o o f. Considering that $\mathrm{im}(\mathbf{\Gamma}_{\mathbf{uc}}) \oplus \mathrm{im}(\mathbf{\Gamma}_{\mathbf{uc}}^{\perp}) = \mathrm{im}(\mathbf{I}_{\mathbf{u}})$ and $\mathbf{M}_o^{-1}\mathbf{GKG}^{T}(\mathbf{\Gamma}_{\mathbf{uc}}) \oplus \mathbf{M}_o^{-1}\mathbf{GKG}^{T}(\mathbf{\Gamma}_{\mathbf{uc}}^{\perp}) = \mathrm{im}(\mathbf{I}_{\mathbf{u}})$ and that $\mathbf{M}_o^{-1}\mathbf{GKG}^{T}(\mathbf{\Gamma}_{\mathbf{uc}}) \cup \mathbf{M}_o^{-1}\mathbf{GKG}^{T}(\mathbf{\Gamma}_{\mathbf{uc}}^{\perp}) = \mathbf{0}$. □

Remark 2. The subspace null of $\mathbf{Q}$ can be calculated very easily using $\ker(\mathbf{Q}) = \ker(\mathbf{J}) + \mathcal{S}$, where $\mathcal{S} = \{\mathbf{s}|\mathbf{KJs} \in \ker(\mathbf{I} - \mathbf{KG}^T(\mathbf{GKG}^T)^{-1}\mathbf{G}) = \mathrm{im}(\mathbf{KG}^T), \mathbf{s} \notin \ker(\mathbf{J})\}$. From (6), it is easy to show that: $\mathcal{S} = \mathrm{im}(\mathbf{\Gamma}_{qc})$ and thus:

$$\ker(\mathbf{Q}) = \mathrm{im}(\mathbf{\Gamma}_r) + \mathrm{im}(\mathbf{\Gamma}_{qc}). \tag{13}$$

If no redundant movements are present, then $\mathrm{im}(\mathbf{\Gamma}_r) = 0$. This yields $\ker(\mathbf{Q}) = \mathrm{im}(\mathbf{\Gamma}_{qc})$ and $\mathrm{rank}(\ker\mathbf{Q}) = r + c$, where r is the dimension of the redundant movement subspace and c is dimension of the movement coordinate subspace. To show $\mathrm{im}(\mathbf{E}_{ti}) = \mathrm{im}(\mathbf{E}_{ti}(\mathcal{V}^\star \cap \mathrm{im}\mathbf{B}_\tau))~\forall~\mathbf{K}$ and $\forall~\mathbf{B}$, it will be enough to show the following:

$$\mathbf{E}_{ti}\mathcal{V}^\star = \mathrm{im}(\mathbf{E}_{ti}), \tag{14}$$

To prove (14), it is shown that

$$\ker(\mathbf{Q}) \cap \mathrm{im} \begin{bmatrix} \mathbf{\Gamma}_h & \mathbf{S}_q\mathbf{Z} \end{bmatrix} = \mathbf{0}, \tag{15}$$

and

$$\mathrm{rank}(\begin{bmatrix} \mathbf{\Gamma}_h & \mathbf{S}_q\mathbf{Z} \end{bmatrix}) = \mathrm{rank}(\mathbf{Q}). \tag{16}$$

Lemma 1.

$$\ker(\mathbf{Q}) \cap \mathrm{im} \begin{bmatrix} \mathbf{\Gamma}_h & \mathbf{S}_q\mathbf{Z} \end{bmatrix} = \mathbf{0}. \tag{17}$$

P r o o f. From the previous remark, (17) can be verified by checking if vectors $\mathbf{x}$, $\mathbf{y}$, $\mathbf{v}$, and $\mathbf{w}$ in that the equation

$$\mathbf{\Gamma}_r\mathbf{x} + \mathbf{\Gamma}_{qc}\mathbf{y} = \mathbf{\Gamma}_h\mathbf{v} + \mathbf{S}_q\mathbf{Zw}$$

exist. In fact, from (5) $\mathrm{im}(\mathbf{\Gamma}_{qc})$ does not belong to the $\ker(\mathbf{J})$, thus $\mathrm{im}(\mathbf{\Gamma}_{qc}) \subseteq \mathrm{im}(\mathbf{M}_h^{-1}\mathbf{J}^T)$. From (25) and (26), $\mathrm{im}(\mathbf{\Gamma}_h)$ and $\mathrm{im}(\mathbf{S}_q)$ are also included in $\mathrm{im}(\mathbf{M}_h^{-1}\mathbf{J}^T)$. $\mathrm{im}(\mathbf{\Gamma}_r)$ is not included in $\mathrm{im}(\mathbf{M}_h^{-1}\mathbf{J}^T)$ because this is included in

$\ker(\mathbf{J})^1$. Thus, the above mentioned equation could be written in the following way:

$$\mathbf{\Gamma}_{qc}\mathbf{y} = \mathbf{\Gamma}_h\mathbf{v} + \mathbf{S}_q\mathbf{Z}\mathbf{w}. \tag{18}$$

If this equation is true, then multiplying by $\mathbf{M}_o^{-1}\mathbf{GKJ}$ equation (18) it follows

$$\mathbf{M}_o^{-1}\mathbf{GKJ}\mathbf{\Gamma}_{qc}\mathbf{y} = \mathbf{M}_o^{-1}\mathbf{GKJ}\mathbf{\Gamma}_h\mathbf{v} + \mathbf{M}_o^{-1}\mathbf{GKJS}_q\mathbf{Z}\mathbf{w}.$$

From $\mathbf{\Gamma}_h \subseteq \ker(\mathbf{GKJ})$, then

$$\mathbf{M}_o^{-1}\mathbf{GKG}^T\mathbf{\Gamma}_{uc}\mathbf{y} = \mathbf{M}_o^{-1}\mathbf{GKJS}_q\mathbf{Z}\mathbf{w}.$$

However, this is never verified. $\mathbf{M}_o^{-1}\mathbf{GKJS}_q\mathbf{Z} \subseteq \ker(\mathbf{\Gamma}_{uc}^T)$ because of the choice of $\mathbf{Z}$. While it will be very easy to show that if $\mathbf{M}_o^{-1}\mathbf{GKG}^T\mathbf{\Gamma}_{uc} \subseteq \ker(\mathbf{\Gamma}_{uc}^T)$ then matrix $\mathbf{M}_o^{-1}\mathbf{GKG}^T$ would be an orthogonal projector. However, this is not true because this is not a projector form.[2] This shows that (17) is proven. □

Lemma 2.

$$\mathsf{rank}\begin{bmatrix} \mathbf{\Gamma}_h & \mathbf{S}_q\mathbf{Z} \end{bmatrix} = \mathsf{rank}(\mathbf{\Gamma}_h) + \mathsf{rank}(\mathbf{S}_q\mathbf{Z}) = q - r - c.$$

P r o o f. The first equality comes from the null intersection between $\mathsf{im}(\mathbf{\Gamma}_h)$ and $\mathsf{im}(\mathbf{S}_q\mathbf{Z})$. In fact, from (25) $\mathsf{im}(\mathbf{\Gamma}_h)$ is a subspace of $\max\mathcal{I}(\mathbf{M}_h^{-1}\mathbf{J}^T\mathbf{KJ}, \ker(\mathbf{GKJ}))$ which, from (26), is orthogonal to $\mathsf{im}(\mathbf{M}_h^{-1}\mathbf{S}_q)$. The proof of the second equality of the lemma begins with the following relation.

$$\max\mathcal{I}(\mathbf{M}_h^{-1}\mathbf{J}^T\mathbf{KJ}, \ker(\mathbf{GKJ})) = \mathsf{im}(\mathbf{M}_h^{-1}\mathbf{S}_q)^\perp,$$

and it follows that

$$\mathsf{im}(\mathbf{M}_h^{-1}\mathbf{J}^T) \subseteq \max\mathcal{I}(\mathbf{M}_h^{-1}\mathbf{J}^T\mathbf{KJ}, \ker(\mathbf{GKJ})) \oplus \mathsf{im}(\mathbf{M}_h^{-1}\mathbf{S}_q).$$

Now, from (26) $\mathsf{im}(\mathbf{M}_h^{-1}\mathbf{S}_q) \subseteq \mathsf{im}(\mathbf{M}_h^{-1}\mathbf{J}^T)$. From the above mentioned inclusion and from definition (25) of $\mathbf{\Gamma}_h$ it follows that

$$\begin{aligned} \mathsf{im}(\mathbf{M}_h^{-1}\mathbf{J}^T) &= \mathbf{M}_h^{-1}\mathbf{J}^T \cap \left(\max\mathcal{I}(\mathbf{M}_h^{-1}\mathbf{J}^T\mathbf{KJ}, \ker(\mathbf{GKJ})) \oplus \mathsf{im}(\mathbf{M}_h^{-1}\mathbf{S}_q)\right) \\ &= \left(\mathbf{M}_h^{-1}\mathbf{J}^T \cap \max\mathcal{I}(\mathbf{M}_h^{-1}\mathbf{J}^T\mathbf{KJ}, \ker(\mathbf{GKJ}))\right) \oplus \mathsf{im}(\mathbf{M}_h^{-1}\mathbf{S}_q) \\ &= \mathsf{im}(\mathbf{\Gamma}_h) \oplus \mathsf{im}(\mathbf{M}_h^{-1}\mathbf{S}_q). \end{aligned}$$

It follows that

$$\mathsf{rank}(\mathbf{\Gamma}_h) + \mathsf{rank}(\mathbf{S}_q) = \mathsf{rank}(\mathbf{M}_h^{-1}\mathbf{J}^T) = \mathsf{rank}(\mathbf{J}) = q - r$$

and

$$\mathsf{rank}(\mathbf{\Gamma}_h) = q - r - \mathsf{rank}(\mathbf{S}_q). \tag{19}$$

---

[1] In general for a linear application $\mathbf{L}$ the following relationship holds: $\mathsf{im}(\mathbf{L}^T) + \ker(\mathbf{L}) = \mathbf{I}$.

[2] It is useful to recall that given a subspace $\mathcal{L}$ which the basis matrix is $\mathbf{L}$ then $\ker(\mathbf{L}^T) = (\mathsf{im}(\mathbf{L}))^\perp$ and the orthogonal projector is $(\mathbf{I} - \mathbf{L}(\mathbf{L}^T\mathbf{L})^{-1}\mathbf{L}^T)$.

It remains to calculate $\mathsf{rank}(\mathbf{S}_q\mathbf{Z})$. Recalling that $\mathbf{S}_q$ and $\mathbf{Z}$ are basis matrices and from (24) $\mathsf{rank}(\mathbf{Z}) \leq \mathsf{rank}(\mathbf{S}_q)$, then

$$\mathsf{rank}(\mathbf{S}_q\mathbf{Z}) = \mathsf{rank}(\mathbf{Z}). \tag{20}$$

From the definition of $\mathbf{Z}$ in (24) it follows that

$$\mathsf{rank}(\mathbf{Z}) = \mathsf{rank}(\mathbf{S}_q) - \mathsf{rank}(\mathbf{Z}^{\perp}), \tag{21}$$

where $\mathsf{rank}(\mathbf{S}_q)$ is the number of components of $\mathbf{z} \in \mathbf{Z}$. The last part of this demonstration consists of estimating $\mathsf{rank}(\mathbf{Z}^{\perp})$, which from (24) is

$$\mathsf{rank}(\mathbf{Z}^{\perp}) = \mathsf{rank}(\mathbf{S}_q^T\mathbf{J}^T\mathbf{K}\mathbf{G}^T\mathbf{M}_o^{-1}\mathbf{\Gamma}_{uc}).$$

From (26), it is easy to show that $\mathsf{ker}(\mathbf{S}_q^T) \subseteq \mathsf{ker}(\mathbf{GKJ})$, and thus $\mathsf{ker}(\mathbf{S}_q^T) \cap \mathsf{im}(\mathbf{J}^T\mathbf{K}\mathbf{G}^T) = \mathbf{0}$ and

$$\mathsf{rank}(\mathbf{Z}^{\perp}) = \mathsf{rank}(\mathbf{J}^T\mathbf{K}\mathbf{G}^T\mathbf{M}_o^{-1}\mathbf{\Gamma}_{uc}). \tag{22}$$

Now

$$\mathsf{rank}(\mathbf{Z}^{\perp}) = \mathsf{rank}(\mathbf{J}^T\mathbf{K}\mathbf{G}^T\mathbf{M}_o^{-1}\mathbf{\Gamma}_{uc}) = \mathsf{rank}(\mathbf{\Gamma}_{uc}) = c. \tag{23}$$

If (22) is transposed, then

$$\mathsf{rank}(\mathbf{Z}^{\perp}) = \mathsf{rank}(\mathbf{\Gamma}_{uc}^T\mathbf{M}_o^{-1}\mathbf{GKJ}),$$

and from (5)

$$\mathsf{rank}(\mathbf{Z}^{\perp}) = \mathsf{rank}(\mathbf{\Gamma}_{uc}^T\mathbf{M}_o^{-1}\mathbf{GKG}^T\mathbf{\Gamma}_{uc}) = \mathsf{rank}(\mathbf{\Gamma}_{uc}),$$

where the last equality follows because matrix $\mathbf{\Gamma}_{uc}^T\mathbf{M}_o^{-1}\mathbf{GKG}^T\mathbf{\Gamma}_{uc}$ has full rank. Finally, from (20), (21) and (23), it can be concluded:

$$\mathsf{rank}(\mathbf{S}_q\mathbf{Z}) = \mathsf{rank}(\mathbf{S}_q) - c.$$

Now, if this last result with (19) is compared, it follows:

$$\mathsf{rank} \begin{bmatrix} \mathbf{\Gamma}_h & \mathbf{S}_q\mathbf{Z} \end{bmatrix} = q - r - c.$$

$\square$

Remark 3. Equation (16) is proven only if in case of kinematic defectivity $(\mathsf{ker}(\mathbf{J}^T)) \neq \mathbf{0})$, i.e., being $\mathbf{J} \in \Re^{(t \times q)}$, thus only in case of $t > q$. It is easy to prove that in case of $t \leq q$ only a trivial extension exists. Let r e c be the ranks of matrices $\mathbf{\Gamma}_r$ and $\mathbf{\Gamma}_{uc}$, respectively. Then, $\mathsf{rank}(\mathbf{J}) = q - r$. From Lemma 2, we have that $\mathsf{rank} \begin{bmatrix} \mathbf{\Gamma}_h & \mathbf{S}_q\mathbf{Z} \end{bmatrix} = \mathsf{rank}(\mathbf{\Gamma}_h) + \mathsf{rank}(\mathbf{S}_q\mathbf{Z}) = q - r - c$. In conclusion, (16) shows that

$$\mathsf{rank}(\mathbf{Q}) = q - r - c.$$

This is derived from (13). In fact, $\mathsf{rank}(\mathbf{Q}) = \mathsf{rank}(\mathbf{Q}^T) = q - \mathsf{rank}(\mathsf{ker}(\mathbf{Q})) = q - (r + c)$.

Notion of decoupling control of internal forces is formalized in the next definition.

Definition 7. Consider the dynamic system (2). A control law of internal forces $\mathbf{e}_{ti}$ is **decoupled** with respect to the coordinate rigid–body motions $\mathbf{e}_{uc}$, if there exists a linear combination $\mathbf{B}_\tau \mathbf{U}$ of the input–matrix columns and an algebraic output feedback $\mathbf{L}$ of the measured output $\mathbf{y}_m$. For zero initial condition, the input $\tau = \mathbf{U}\tau_n$ affects only the internal forces $\mathbf{e}_{ti}$, while $\mathbf{e}_{uc}$ is equal to zero.

The following theorem shows that Definition 7 is considered as a structural property of general (non–indeterminate) manipulation systems, and decoupling control can be obtained through an algebraic feedback of the sensed output $\mathbf{y}_m$.

Theorem 1. *Robust decoupling*
Consider the linearized manipulation system (2), then there exists an algebraic output feedback $\mathbf{L}$ of $\mathbf{y}_m$ and a matrix $\mathbf{U}$ such that

a) the decoupling condition of Definition 7 holds and

b) $\mathsf{im}\,(\mathbf{E}_{ti}\,\mathsf{min}\ \mathcal{I}\,(\mathbf{A} + \mathbf{B}_\tau\mathbf{LC}, \mathbf{B}_\tau\mathbf{U})) = \mathsf{im}(\mathbf{E}_{ti}),\ \forall\ \mathbf{K}$ and $\mathbf{B}$.

P r o o f. *Robust decoupling:*
Reference [7] shows that to guarantee the existence of a decoupling-output feedback law, it is necessary that the trajectories lie in a subspace that must be controlled and conditioned invariant. Let us choose an $(\mathbf{A}, \mathbf{B}_\tau)$–controlled and conditioned candidate invariant subspace $\mathcal{V}^\star$ as

$$\mathcal{V}^\star = \mathsf{im}\begin{bmatrix} \boldsymbol{\Gamma}_h & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{S}_q\mathbf{Z} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathsf{ker}(\boldsymbol{\Gamma}_{uc}^T) & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Gamma}_h & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{S}_q\mathbf{Z} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathsf{ker}(\boldsymbol{\Gamma}_{uc}^T) & \mathbf{0} & \mathbf{0} \end{bmatrix},$$

where $\mathbf{Z}$ is such that

$$\mathsf{im}(\mathbf{M}_o^{-1}\mathbf{GKJS}_q\mathbf{Z}) = \mathsf{im}(\mathbf{M}_o^{-1}\mathbf{GKJS}_q) \cap \mathsf{ker}(\boldsymbol{\Gamma}_{uc}^T). \qquad (24)$$

$\boldsymbol{\Gamma}_h$ is the identical internal force as explained in [14]

$$\boldsymbol{\Gamma}_h\ :\ \ \mathbf{M}_o^{-1}\mathbf{J}^T \cap \mathsf{max}\mathcal{I}(\mathbf{M}_h^{-1}\mathbf{J}^T\mathbf{KJ}, \mathsf{ker}(\mathbf{GKJ})); \qquad (25)$$

and

$$\mathbf{S}_q = \mathsf{min}\mathcal{I}(\mathbf{M}_h^{-1}\mathbf{J}^T\mathbf{KJ}, \mathbf{M}_h^{-1}\mathbf{J}^T\mathbf{KG}^T) \qquad (26)$$

and $\mathbf{M}_h^{-1}$ is positive defined. The notation indicates matrices and column spaces as well. The proof consists of showing the following:

1) $\mathcal{V}^\star$ is a controlled and conditioned invariant subspace $\forall\ \mathbf{K}$ and $\forall\ \mathbf{B}$ (generalized controlled and conditioned invariant subspace).

2) $\mathcal{V}^\star \subseteq \mathsf{ker}\,(\mathbf{E}_{uc})\ \forall\ \mathbf{K}$ and $\forall\ \mathbf{B}$.

3) Show that Proposition 3 and 4 hold for the presented case.

To prove point 1), it should be shown that $\mathbf{A}\mathcal{V}^\star \subseteq \mathcal{V}^\star \oplus \mathbf{B}_\tau \; \forall \; \mathbf{K}$ and $\forall \; \mathbf{B}$, where

$$
\mathbf{A} = \begin{bmatrix}
\mathbf{0} & \mathbf{0} & \mathbf{I}_q & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_u \\
-\mathbf{M}_h^{-1}\mathbf{J}^T\mathbf{KJ} & \mathbf{M}_h^{-1}\mathbf{J}^T\mathbf{KG}^T & -\mathbf{M}_h^{-1}\mathbf{J}^T\mathbf{BJ} & \mathbf{M}_h^{-1}\mathbf{J}^T\mathbf{BG}^T \\
\mathbf{M}_o^{-1}\mathbf{GKJ} & -\mathbf{M}_o^{-1}\mathbf{GKG}^T & \mathbf{M}_o^{-1}\mathbf{GBJ} & -\mathbf{M}_o^{-1}\mathbf{GBG}^T
\end{bmatrix}.
$$

According to the physical hypothesis that $\ker(\mathbf{\Gamma}_{uc}^T)$ is $\mathbf{GKG}^T$ invariant then,

$$
\mathbf{A}\mathcal{V}^\star \subseteq \operatorname{im} \begin{bmatrix}
\mathbf{0} & \mathbf{\Gamma}_h & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{S}_q\mathbf{Z} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \ker(\mathbf{\Gamma}_{uc}^T) & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \ker(\mathbf{\Gamma}_{uc}^T) & \mathbf{0} & \mathbf{0}
\end{bmatrix}.
$$

Subspace $\mathcal{V}^\star$ is a generalized-controlled invariant because of the following reasons:

- invoking (6) and remembering that $\mathbf{\Gamma}_{uc}$ is not dependent on $\mathbf{K}$ and $\mathbf{B}$.

- Considering that $\ker(\mathbf{\Gamma}_{uc})^T$ is $\mathbf{GKG}^T$ invariant and independent of parameter $\mathbf{K}$.

- Recalling the structure of $\mathbf{B}_\tau$.

Recalling (12), $\mathcal{V}^\star$ is also a generalized-conditioned invariant in $(\mathbf{A}, \ker(\mathbf{C}))$, which is verified because

$$
\mathbf{A}\left(\ker(\mathbf{C}) \cap \mathcal{V}^\star\right) = \mathbf{A}\operatorname{im} \begin{bmatrix}
\mathbf{0} \\
\ker(\mathbf{\Gamma}_{uc}^T) \\
\mathbf{0} \\
-\alpha\ker(\mathbf{\Gamma}_{uc}^T)
\end{bmatrix} \subseteq \mathcal{V}^\star.
$$

$$
\mathbf{A}\left(\ker(\mathbf{C}) \cap \mathcal{V}^\star\right) = \operatorname{im} \begin{bmatrix}
\mathbf{0} \\
-\alpha\ker(\mathbf{\Gamma}_{uc}^T) \\
\mathbf{M}_h^{-1}\mathbf{J}^T\mathbf{KG}^T\ker(\mathbf{\Gamma}_{uc}^T) - \mathbf{M}_h^{-1}\mathbf{J}^T\mathbf{BG}^T\ker(\mathbf{\Gamma}_{uc}^T) \\
-\mathbf{M}_o^{-1}\mathbf{GKG}^T\ker(\mathbf{\Gamma}_{uc}^T) - \mathbf{M}_o^{-1}\mathbf{GBG}^T\alpha\ker(\mathbf{\Gamma}_{uc}^T))
\end{bmatrix} \subseteq \mathcal{V}^\star.
$$

To show the last inclusion, it is enough to notice the following:

- $\mathbf{\Gamma}_{uc}$ is not dependent on $\mathbf{K}$ and $\mathbf{B}$, see(6).

- Invoking that $\ker(\mathbf{\Gamma}_{uc}^T)$ is $\mathbf{GKG}^T$ invariant and independent of parameter $\mathbf{K}$.

- $-\mathbf{M}_o^{-1}\mathbf{GKG}^T\ker(\mathbf{\Gamma}_{uc}^T) - \mathbf{M}_o^{-1}\mathbf{GBG}^T\alpha\ker(\mathbf{\Gamma}_{uc}^T) = \ker(\mathbf{\Gamma}_{uc}^T)$.

Point 2) is straightforward to prove. In fact, it is enough to notice that

$$
\ker(\mathbf{E}_{uc}) = \operatorname{im} \begin{bmatrix}
\mathbf{I}_q & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \ker(\mathbf{\Gamma}_{uc}^T) & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{I}_q & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_u
\end{bmatrix}.
$$

Point 3) could be shown through the following considerations. Proposition 3 is straightforward to prove that in this case it holds. To prove that Proposition 4 holds in the present case, it is enough to prove that $\mathsf{im}(\mathbf{E}_{ti}) = \mathsf{im}(\mathbf{E}_{ti}(\mathcal{V}^\star \cap \mathsf{im}\mathbf{B}_\tau))$ $\forall \mathbf{K}$ and $\forall \mathbf{B}$. The results of Lemma 1 and Lemma 2 were devoted to show that. $\square$

A procedure for designing the robust-decoupling controller of internal forces for a given manipulation system is reported in the sequel. The procedure is based on propositions and remarks from this Section.

Procedure:

step 1 Compute $\mathcal{V}^\star$.

step 2 If $\mathcal{V}^\star \neq \{\mathbf{0}\}$, choose $\mathbf{F}$ and $\mathbf{U}$ according to Remark 1 and to Proposition 3. If $\mathbf{U} \neq \mathbf{0}$, Proposition 3 holds and claim $a)$ of Definition 3 is satisfied. Otherwise, the robust decoupling controller does not exist and the procedure ends.

step 3 Check the sufficient conditions of Proposition 2. If they are satisfied, then stop.

step 4 Check the sufficient conditions of Proposition 4. If these are satisfied, then stop.

step 5 If step 3 and 4 fail, check the necessary condition of Proposition 1.

step 6 If the necessary condition of Proposition 1 is satisfied, check condition $b)$ of Definition 3.

step 7 Calculation of matrix $\mathbf{L}$:
according to step 1 until step 6. If there exists matrix $\mathbf{F}$, then consider equation (27) of definition (3) denoted by:

$$\min \, \mathcal{I}\left((\mathbf{A}(\Delta k, \Delta b) + \mathbf{B}_\tau \mathbf{L}\mathbf{C}), \mathbf{B}_\tau \mathbf{U}\right) \subseteq \ker \mathbf{E}_{uc}, \qquad (27)$$

then

$$\mathbf{F} = \mathbf{L} \begin{bmatrix} \mathbf{E}_{ti} \\ \mathbf{E}_{uc} \end{bmatrix},$$

this yields

$$\begin{bmatrix} \mathbf{E}_{ti} \\ \mathbf{E}_{uc} \end{bmatrix}^T \mathbf{L}^T = \mathbf{F}^T,$$

and

$$\mathsf{im}(\mathbf{F}^T) \subseteq \mathsf{im} \begin{bmatrix} \mathbf{E}_{ti} \\ \mathbf{E}_{uc} \end{bmatrix}^T,$$

finally

$$\ker \begin{bmatrix} \mathbf{E}_{ti} \\ \mathbf{E}_{uc} \end{bmatrix} \subseteq \ker(\mathbf{F}).$$
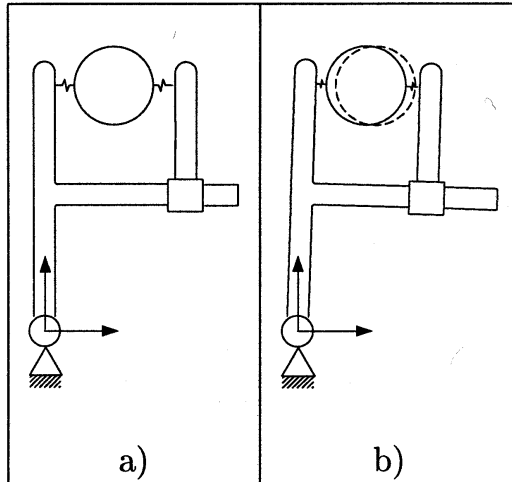
Fig. 2. Cross-section of manipulator.

Remark 4. Steps 1 and 2 refer to the decoupling property (claim a) of the robust controller. Steps 3 to 6 check that $\mathbf{F}$ and $\mathbf{U}$ fulfill the reachability condition (claim b).

Note that if the procedure does not end at step 3 or 4, a different choice of the state–feedback matrix $\mathbf{F}$ and of the input selection matrix may be needed to prove the robustness of the proposed-control law.

5. AN APPLICATION EXAMPLE

In this section, numerical results are reported for the gripper described in Figure 2. This system is a planar device with two degrees of freedom, a prismatic and a rotoidal joint. Joint variables are positive when links move left. In the reference frame, the contacts are $\mathbf{c}_1 = (0, 2)$, $\mathbf{c}_2 = (1, 2)$, and the object center of mass is $\mathbf{c}_b = (0.5, 2)$. As previously explained, $\mathbf{J} = \mathbf{H}\frac{\delta \mathbf{c}^m}{\delta \mathbf{q}}$ and $\mathbf{G}^T = \mathbf{H}\frac{\delta \mathbf{c}^o}{\delta \mathbf{u}}$, the identity matrix is assumed in the presented case matrix $\mathbf{H}$. The inertia matrices of the object and manipulator are assumed to be normalized to the identity matrix. The contact behavior is assumed isotropic at the contacts. Given that $\mathbf{q} = [q_1, q_2]^T$. In general $\mathbf{c}_1^m = (2\cos q_1, 2 - 2 sin q_1)$, $\mathbf{c}_2^m = (2\cos q_1 - q_2, 2)$, the Jacobian matrix, and its linearization around the point $q_1 = \frac{\pi}{2}$ assume the following values:

$$\mathbf{J} = \begin{bmatrix} -2\sin q_1 & 0 \\ -2\cos q_1 & 0 \\ -2\sin q_1 & -1 \\ -2\sin q_1 & 0 \end{bmatrix} ; \mathbf{J}_l = \begin{bmatrix} -2 & 0 \\ 0 & 0 \\ -2 & -1 \\ -2 & 0 \end{bmatrix} .$$

The grasp matrix was once assumed $\mathbf{u} = [x, y, \theta]^T$ to be the vector of the generalized coordinates for the object. Then, the contact points could be represented as follows $\mathbf{c}_1^o = (x + \cos\theta, 1 + y + \sin\theta)$, $\mathbf{c}_2^o = (1 + x - \cos\theta, 1 + y - \sin\theta)$. The grasp matrix and its linearization around $\theta = 0$ have the following form:

$$\mathbf{G} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ \sin\theta & -\cos\theta & \sin\theta & -\cos\theta \end{bmatrix} ; \mathbf{G}_l = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & -1 \end{bmatrix}.$$

If $\ker\begin{bmatrix} \mathbf{J} & -\mathbf{G}^T \end{bmatrix} = \text{im}\,(\mathbf{\Gamma})$ is calculated, then it follows:

$$\mathbf{\Gamma} = \begin{bmatrix} 0.0000 \\ 0.0000 \\ 0.0000 \\ 0.3780 \\ 0.3780 \end{bmatrix} ;$$

where

$$\mathbf{\Gamma}_{qc} = \begin{bmatrix} 0.0000 \\ 0.0000 \end{bmatrix} ; \mathbf{\Gamma}_{uc} = \begin{bmatrix} 0.0000 \\ 0.3780 \\ 0.3780 \end{bmatrix}.$$

Condition (6) is also satisfied. In fact, $\mathbf{G}^T\mathbf{\Gamma}_{uc} = 0$. It is possible to calculate

$$\ker\,(\mathbf{\Gamma}_{uc})^T = \text{im} \begin{bmatrix} -1.0000 & -0.0000 \\ 0.0000 & -0.7071 \\ 0.0000 & 0.7071 \end{bmatrix} ; \mathbf{S}_q = \begin{bmatrix} 0.9487 & -0.3162 \\ 0.3162 & 0.9487 \end{bmatrix}.$$

$\ker(\mathbf{GKJ}) = 0$. This yields that $\mathbf{\Gamma}_h = 0$. In the analyzed case, $\text{im}(\mathbf{M}_o^{-1}\mathbf{GKJS}_q) \subseteq \ker\,(\mathbf{\Gamma}_{uc})^T$, then $\mathbf{Z} = \mathbf{I}$. At the end it follows:

$$\mathcal{V}^\star = \text{im} \begin{bmatrix} 0 & 0 & 0 & 0 & 0.9487 & -0.3162 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.3162 & 0.9487 & 0 & 0 \\ -1.0000 & -0.0000 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.0000 & -0.7071 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.0000 & 0.7071 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.9487 & -0.3162 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.3162 & 0.9487 \\ 0 & 0 & -1.0000 & -0.0000 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.0000 & -0.7071 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.0000 & 0.7071 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Figure 3 shows the proposed control scheme structure. From this scheme the concept of the conditioned and controlled-invariant subspace as an algebraic feedback is visible. In fact, considering the theorem shown above,

a) $\min \, \mathcal{I}\,((\mathbf{A}(\Delta k, \Delta b) + \mathbf{B}_\tau \mathbf{LC}), \mathbf{B}_\tau \mathbf{U}) \subseteq \ker \mathbf{E}_{uc};$

(28)

b) $\text{im}\,\mathbf{E}_{ti}(\Delta k, \Delta b) = \mathbf{E}_{ti}(\Delta k, \Delta b)\min \, \mathcal{I}\,((\mathbf{A}(\Delta k, \Delta b) + \mathbf{B}_\tau \mathbf{LC}), \mathbf{B}_\tau \mathbf{U})$

a robust decoupling controller is active through the matrix $\mathbf{L}$. Equation (28) is interpreted as, all the contact forces $\mathbf{E}_{ti}$, remain in the expression $\text{im}(\mathbf{E}_{ti}) \,\forall\, \mathbf{K}$ and $\mathbf{B}$. This means that the contact forces do not influence the subspace of the movements. From Figures 4, it is visible how the contact forces "compensate". Essentially, no
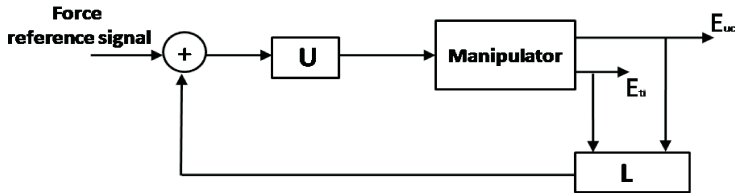
Fig. 3. Control Scheme.

movements are allowed, and the desired force on the object is obtained. The dynamics of the forces, represented on the down (left and right) part of Figure 4 are due to the particular choice of eigenvalues that characterise the force answer of the system. The robustness, with respect to the variations of **K** and **B**, is widely explained in the description of the proposed procedure. The robust decoupling controller exits if some structural conditions are satisfied. These structural conditions are satisfied by those mechanisms that present some symmetry in their geometric structure. In fact, a symmetric structure always uses symmetric contact forces to guarantee the existence of the compensation mentioned above.

6. CONCLUSIONS AND FUTURE WORK

In this paper, the dynamics of general mechanical systems for manipulation and their structural properties and geometric control are considered. Attention has been devoted to the control of internal forces by means of an algebraic feedback of measured outputs. Moreover, a particular effort is devoted to show a robust design analysis. The achieved robustness is limited to particular mechanisms that present physical and geometric symmetric structures. Future work will be oriented to developing such kinds of procedures for partial rank conditions inclusions. The goal will be to develop a measure for robustness and to develop procedures not only limited to an "on-off" robustness concept.

REFERENCES

[1] G. Basile and G. Marro: Controlled and Conditioned Invariants in Linear System Theory. Prentice Hall, New Jersey, 1992.

[2] S. P. Bhattacharyya: Generalized controllability, controlled invariant subspace and parameter invariant control. SIAM J. Algebraic Discrete Methods *4* (1983), 4, 529–533.
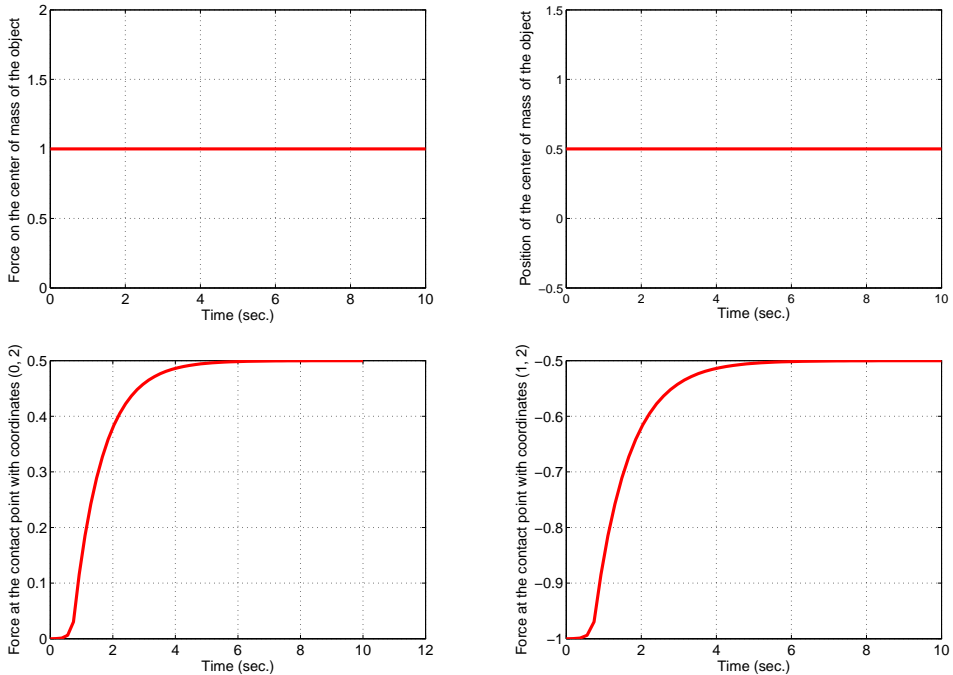
Fig. 4. Up-Left: Resulting horizontal squeezing force on the center of mass of the object. Up-Right: Resulting horizontal position of the center of mass of the object. Down-Left: Force acting at the contact point with coordinates (0, 2). Down-Right: Force acting at the contact point with coordinates (1, 2).

[3] A. Bicchi, C. Melchiorri, and D. Balluchi: On the mobility and manipulability of general multiple limb robots. IEEE Trans. Automat. Control *11* (1995), 2, 215–228.

[4] A. Bicchi and D. Prattichizzo: Manipulability of cooperating robots with unactuated joints and closed-chain mechanisms. IEEE Trans. Robotics and Automation *16* (2000), 4, 336–345.

[5] A. Bicchi, D. Prattichizzo, P. Mercorelli, and A. Vicino: Noninteracting force/motion control in general manipulation systems. In: Proc. 35th IEEE Conf. on Decision Control, CDC '96, Kobe 1996.

[6] A. Isidori: Nonlinear Control Systems: An Introduction. Springler-Verlag, Berlin 1989.

[7] G. Marro and F. Barbagli: The algebraic output feedback in the light of dual lattice structures. Kybernetika *35* (1999), 6, 693–706.

[8] M. T. Mason and J. K. Salisbury: Robot Hands and the Mechanics of Manipulation. The MIT Press, Cambridge 1985.

[9] L. Meirovitch: Analytical Methods in Vibrations. Macmillan Pub. Co., Inc., New York 1967.

[10] P. Mercorelli: A subspace to describe grasping internal forces in robotic manipulation systems. J. Math. Control Sci. Appl. *1* (2007), 1, 209-216.

[11] P. Mercorelli: Geometric structures for the parameterization of non-interacting dynamics for multi-body mechanisms. Internat. J. Pure Appl. Math. *59* (2010), 3, 257–273.

[12] P. Mercorelli and D. Prattichizzo: A geometric procedure for robust decoupling control of contact forces in robotic manipulation. Kybernetika *39* (2003), 4, 433-445.

[13] D. Prattichizzo and A. Bicchi: Consistent task specification for manipulation systems with general kinematics. Amer. Soc. Mech. Engrg. *119* (1997), 760–767.

[14] D. Prattichizzo and A. Bicchi: Dynamic analysis of mobility and graspability of general manipulation systems. Trans. Robotic Automat. *14* (1998), 2, 251–218.

[15] D. Prattichizzo and P. Mercorelli: Motion-decoupled internal force control in grasping with visco-elastic contacts. In: Proc. IEEE Conf. in Robotic and Automation, ICRA 2000, San Francisco 2000.

[16] D. Prattichizzo and P. Mercorelli: On some geometric control properties of active suspension systems. Kybernetika *36* (2000), 5, 549–570.

[17] W. M. Wonham: Linear Multivariable Control: A Geometric Approach. Springer Verlag, New York 1979.

[18] Y. Yamamoto and X. Yun: Effect of the dynamic interaction on coordinated control of mobile manipulators. IEEE Trans. Robotics Automat. *12* (1996), 5, 816–824.

Paolo Mercorelli, Ostfalia University of Applied Sciences, Robert Koch Platz 12, Faculty of Automotive Engineering, D-38440 Wolfsburg. Germany.
  e-mail: p.mercorelli@ostfalia.de