

Irena Hinterleitner; Volodymyr Kiosak  
Special Einstein's equations on Kähler manifolds

*Archivum Mathematicum*, Vol. 46 (2010), No. 5, 333--337

Persistent URL: <http://dml.cz/dmlcz/141387>

## Terms of use:

© Masaryk University, 2010

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.



This paper has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* <http://project.dml.cz>

## SPECIAL EINSTEIN'S EQUATIONS ON KÄHLER MANIFOLDS

IRENA HINTERLEITNER AND VOLODYMYR KIOSAK

ABSTRACT. This work is devoted to the study of Einstein equations with a special shape of the energy-momentum tensor. Our results continue Stepanov's classification of Riemannian manifolds according to special properties of the energy-momentum tensor to Kähler manifolds. We show that in this case the number of classes reduces.

### 1. INTRODUCTION

The geometric properties of (pseudo-) Riemannian manifolds  $V_n$ , depending on the form the Einstein equations acquire in them, were studied by many authors. A large number of papers is devoted to the study of Einstein's equations with certain restrictions on the energy-momentum tensor and its first covariant derivatives [2, 5, 6, 7, 9].

S. E. Stepanov [8, 10] classified space-time manifolds according to certain relations among the first covariant derivatives of the energy-momentum tensor. He found three fundamental classes, related to geometrical assumptions about space-time. By combinations of the conditions determining the three fundamental classes he found three further classes. A seventh class is characterized by the vanishing of the covariant derivative of the energy-momentum tensor.

In the present paper we partially take over Stepanov's classification to Kähler spaces and investigate analogous, generalized classifying conditions. We show that for two out of the three fundamental classes space-time is Ricci symmetric and the energy-momentum tensor is covariantly constant.

In consequence, the energy-momentum tensor is covariantly constant also for the three classes derived from the fundamental ones. Thus for Kähler spaces the number of classes of Einstein equations reduces to one with covariantly constant and one with non-constant energy-momentum tensor. We study some of their properties and generalizations.

All geometric objects are formulated locally under the assumption of sufficient smoothness. Whereas S. E. Stepanov formulated his classifications by making use of bundles, for our purpose it is sufficient to write down the classifying relations in form of tensor equations.

---

2000 *Mathematics Subject Classification*: primary 53B20; secondary 53B30, 53B35, 53B50, 32Q15, 35Q76.

*Key words and phrases*: Einstein's equations, Kähler manifolds, pseudo-Riemannian spaces, Riemannian spaces.

2. EINSTEIN'S EQUATIONS

The equation  $\text{Ric} - \frac{1}{2} Rg = T$  is called *Einstein's equation*. Here Ric is the Ricci tensor on the manifold  $V_n$ ,  $g$  is the metric tensor,  $R$  is the scalar curvature, and  $T$  is the energy-momentum tensor. In components this reads as follows:

$$(1) \quad R_{ij} - \frac{1}{2} R g_{ij} = T_{ij},$$

where  $R_{ij}$ ,  $g_{ij}$  and  $T_{ij}$  are components of Ric,  $g$  and  $T$ .

From the Bianchi identities of the Ricci tensor follows  $T_{\alpha i, \beta} g^{\beta\alpha} = 0$ , (where the comma denotes the covariant derivative with respect to a connection on the manifold  $V_n$ ), and  $g^{ij}$  are elements of the inverse matrix to  $g_{ij}$ .

Stepanov distinguishes the following three fundamental types of manifolds in terms of covariant derivatives of the energy-momentum tensor:

$$(2) \quad \Omega_1 : \quad T_{ij,k} + T_{jk,i} + T_{ki,j} = 0,$$

$$(3) \quad \Omega_2 : \quad T_{ij,k} - T_{ik,j} = 0,$$

$$(4) \quad \Omega_3 : \quad T_{ij,k} = a_k g_{ij} + b_i g_{jk} + b_j g_{ik},$$

where  $a_k$  and  $b_i$  are arbitrary vectors.

In space-time manifolds of type  $\Omega_1$  the scalar curvature is covariantly constant and the Ricci tensor is a Killing tensor, i.e.  $R_{ij} \frac{dx^i}{ds} \frac{dx^j}{ds}$  is constant along geodesic curves with parameter  $s$ .

In the case  $\Omega_2$  the scalar curvature is constant, too, and the Levi-Civita connection of the metric, considered as a connection on the tangent bundle  $TM$ , satisfies the conditions of a Yang-Mills potential.

$\Omega_3$  is a slight generalization in comparison with the condition in [8, 10] on  $R_{ij,k}$ , reformulated in terms of  $T_{ij,k}$ . The original conditions of Stepanov characterise manifolds with non-constant curvature that admit non-trivial geodesic mappings.

In [8, 10] three further classes are derived by simultaneously imposing conditions  $\Omega_1$  and  $\Omega_2$ ,  $\Omega_2$  and  $\Omega_3$ , and  $\Omega_1$  and  $\Omega_3$ , respectively.

Using a generalized form of the introduced dependencies, we are going to study manifolds characterized by the following conditions:

$$(5) \quad \Omega_1^* : \quad T_{ij,k} + T_{jk,i} + T_{ki,j} = \lambda_k T_{ij} + \lambda_i T_{jk} + \lambda_j T_{ki} + \mu_k g_{ij} + \mu_i g_{jk} + \mu_j g_{ik},$$

$$(6) \quad \Omega_2^* : \quad T_{ij,k} - T_{ik,j} = \rho_k T_{ij} - \rho_j T_{ik} + \sigma_k g_{ij} - \sigma_j g_{ik},$$

$$(7) \quad \Omega_3^* : \quad T_{ij,k} = \phi_k T_{ij} + \gamma_i T_{jk} + \gamma_j T_{ki} + \eta_k g_{ij} + \chi_i g_{jk} + \chi_j g_{ik},$$

where  $\phi_i$ ,  $\lambda_i$ ,  $\mu_i$ ,  $\rho_i$ ,  $\gamma_i$ ,  $\eta_i$ ,  $\sigma_i$  and  $\chi_i$  are arbitrary vectors.

3. KÄHLER SPACES

An  $n$ -dimensional (pseudo-)Riemannian manifold  $(M_n, g)$  is called a *Kähler space*  $K_n$  if besides the metric tensor  $g$ , a *structure*  $F$ , which is an affinor (i.e. a tensor field of type  $(1, 1)$ ), is given on  $M_n$  such that the following holds [3, 4, 11]:

$F^2 = -\text{Id}$ ,  $g(FX, X) = 0$  ( $\forall X \in TK_n$ ),  $\nabla F = 0$ , i.e. in coordinate form

$$(8) \quad F_i^h F_i^\alpha = -\delta_i^h; \quad F_i^\alpha g_{\alpha j} + F_j^\alpha g_{\alpha i} = 0; \quad F_{i,j}^h = 0,$$

where  $\delta_j^i$  is the Kronecker symbol.

Making use of this we can show that

$$(9) \quad g_{ij} = g_{\alpha\beta} F_i^\alpha F_j^\beta; \quad R_{ij} = R_{\alpha\beta} F_i^\alpha F_j^\beta.$$

Then due to (1), for the energy-momentum tensor the following relation holds

$$(10) \quad T_{ij} = T_{\alpha\beta} F_i^\alpha F_j^\beta; \quad F_i^\alpha T_{\alpha j} + F_j^\alpha T_{\alpha i} = 0.$$

We prove the following theorem.

**Theorem 1.** *If in a Kähler space the condition  $\Omega_2^*$  or  $\Omega_3^*$  holds, then the energy-momentum tensor satisfies*

$$(11) \quad T_{ij,k} = \rho_k T_{ij} + \sigma_k g_{ij}.$$

**Proof.** Assume that in a Kähler space  $K_n$  the condition (6) holds, multiply it by  $F_l^i F_h^j$ , contract with respect to  $i$  and  $j$  and exchange  $l$  for  $i$  and  $h$  for  $j$ . We obtain

$$(12) \quad T_{\alpha\beta,k} F_i^\alpha F_j^\beta - T_{\alpha k,\beta} F_i^\alpha F_j^\beta = \rho_k T_{\alpha\beta} F_i^\alpha F_j^\beta - \rho_\beta T_{\alpha k} F_i^\alpha F_j^\beta + \sigma_k g_{\alpha\beta} F_i^\alpha F_j^\beta - \sigma_\beta g_{\alpha k} F_i^\alpha F_j^\beta.$$

With the aid of (9) and (10) we can rewrite the last equation in the form

$$(13) \quad T_{ij,k} - T_{\alpha k,\beta} F_i^\alpha F_j^\beta = \rho_k T_{ij} - \rho_\beta T_{\alpha k} F_i^\alpha F_j^\beta + \sigma_k g_{ij} - \sigma_\beta g_{\alpha k} F_i^\alpha F_j^\beta.$$

After symmetrization of the indices  $i$  and  $k$  we get

$$(14) \quad T_{ij,k} + T_{jk,i} = \rho_k T_{ij} + \rho_i T_{jk} + \sigma_k g_{ij} - \sigma_i g_{jk}.$$

Exchanging the indices  $i$  and  $j$  we obtain

$$(15) \quad T_{ij,k} + T_{ik,j} = \rho_k T_{ij} + \rho_j T_{ik} + \sigma_k g_{ij} + \sigma_j g_{ik}.$$

Addition of (15) and (13) gives (11). Note that spaces satisfying  $\Omega_3^*$  satisfy also the condition  $\Omega_2^*$  as can be seen, when

$$(16) \quad \rho_i = \phi_i - \gamma_i; \quad \sigma_i = \eta_i - \chi_i$$

holds. □

By analyzing this result it is not difficult to prove

**Theorem 2.** *Kähler spaces  $K_n$  belonging to class  $\Omega_2$  or  $\Omega_3$  are characterized by the following conditions*

$$(17) \quad T_{ij,k} = 0, \quad R_{ij,k} = 0.$$

From this theorem it follows immediately that for Kähler spaces also in the derived cases ( $\Omega_1$  and  $\Omega_2$ ,  $\Omega_2$  and  $\Omega_3$ ,  $\Omega_1$  and  $\Omega_3$ ) the energy-momentum tensor is covariantly constant. So all the classes of Einstein equations, with the exception of  $\Omega_1$ , can be summarized under the characterization  $T_{ij,k} = 0$ . From this follows that for Kähler spaces  $K_n$  of class  $\Omega_i$  (respectively  $\Omega_i^*$ ) only those fulfilling condition (2) (resp. (5)) are relevant.

In a further step of generalization we consider Kähler spaces characterized by the following conditions

$$(18) \quad \begin{aligned} \Omega_4^* : \quad T_{ij,k} - T_{ik,j} &= \rho_k T_{ij} - \rho_j T_{ik} + \sigma_k g_{ij} - \sigma_j g_{ik} + \rho_\alpha T_{i\beta} F_k^\alpha F_j^\beta \\ &- \rho_\beta T_{i\alpha} F_k^\alpha F_j^\beta + \sigma_\alpha g_{i\beta} F_k^\alpha F_j^\beta - \sigma_\beta g_{i\alpha} F_k^\alpha F_j^\beta. \end{aligned}$$

$$(19) \quad \begin{aligned} \Omega_5^* : \quad T_{ij,k} &= \phi_k T_{ij} + \gamma_i T_{jk} + \gamma_j T_{ki} + \eta_k g_{ij} + \chi_i g_{jk} + \chi_j g_{ik} \\ &+ \gamma_\alpha T_{\beta k} F_i^\alpha F_j^\beta + \gamma_\beta T_{k\alpha} F_i^\alpha F_j^\beta + \chi_\alpha g_{\beta k} F_i^\alpha F_j^\beta + \chi_\beta g_{\alpha k} F_i^\alpha F_j^\beta. \end{aligned}$$

Applying the methods used in the proof of Theorem 1 to (18) and taking into account (8), (9), (10) we convince ourselves that (18) acquires the form (19), this proves the next theorem.

**Theorem 3.** *There are no Kähler spaces  $K_n$  in the class  $\Omega_4^*$  other than spaces belonging to  $\Omega_5^*$ .*

In this way the Kähler spaces with non-constant energy-momentum tensor, considered in this paper, are divided into two essential classes:  $\Omega_1^*$  and  $\Omega_5^*$ .

**Acknowledgement.** This work was partially supported by the Ministry of Education, Youth and Sports of the Czech Republic, Research & Development, project No. 0021630511.

#### REFERENCES

- [1] Eisenhart, L. P., *Non-Riemannian Geometry*, Princeton University Press, 1926, Amer. Math. Soc. Colloquium Publications **8** (2000).
- [2] Hall, G. S., *Lorentz manifolds and general relativity theory*, Differential geometry (Warsaw, 1979), Banach Center Publ. **12** (1984), 47–52.
- [3] Mikeš, J., *Holomorphically projective mappings and their generalizations*, J. Math. Sci. **89** (3) (1998), 1334–1353.
- [4] Mikeš, J., Vanžurová, A., Hinterleitner, I., *Geodesic mappings and some generalizations*, Palacky University Press, Olomouc, 2009.
- [5] Petrov, A. Z., *New methods in the general theory of relativity*, Nauka, Moscow, 1966.
- [6] Reboucas, M. J., Santos, J., Teixeira, A. F. F., *Classification of energy momentum tensors in  $n \geq 5$  dimensional space-time: a review*, Brazil. J. Phys. **34** (2A) (2004), 535–543.
- [7] Schouten, J. A., Struik, D. J., *Einführung in die neueren Methoden der Differentialgeometrie*, Groningen, P. Noordhoff, 1935.
- [8] Stepanov, S. E., *The seven classes of almost symplectic structures*, Webs and quasigroups, Tver. Gos. Univ., Tver', 1992, pp. 93–96.
- [9] Stepanov, S. E., *On a group approach to studying the Einstein and Maxwell equations*, Theoret. and Math. Phys. **111** (1) (1997), 419–427.
- [10] Stepanov, S. E., Tsyganok, I. I., *The seven classes of the Einstein equations*, arXiv:1001.4673v1.
- [11] Yano, K., *Differential geometry of complex and almost complex spaces*, Pergamon Press, 1965.

BRNO UNIVERSITY OF TECHNOLOGY, FACULTY OF CIVIL ENGINEERING,  
DEPARTMENT OF MATHEMATICS,  
ŽIŽKOVA 17, 602 00 BRNO, CZECH REPUBLIC  
*E-mail: hinterleitner.irena@seznam.cz*

ODESSA NATIONAL POLYTECHNICAL UNIVERSITY,  
DEPARTMENT OF INFORMATICS AND MATHEMATICAL METHODS  
OF INFORMATION SYSTEMS PROTECTION,  
PROSP. SHEVCHENKO 1, 65044 ODESSA, UKRAINE  
*E-mail: vkiosak@ukr.net*