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ESSENTIAL NORM OF THE DIFFERENCE OF COMPOSITION OPERATORS ON BLOCH SPACE

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Abstract. Let φ and ψ be holomorphic self-maps of the unit disk, and denote by C_φ, C_ψ the induced composition operators. This paper gives some simple estimates of the essential norm for the difference of composition operators $C_\varphi - C_\psi$ from Bloch spaces to Bloch spaces in the unit disk. Compactness of the difference is also characterized.

Keywords: *Bloch* space, composition operator, essential norm, difference, compactness

MSC 2010: 47B38, 47B33, 32A10, 32A37, 32H05

1. Introduction

Let $\mathbb{D}$ be the unit disk in the complex plane. The algebra of all holomorphic functions with domain $\mathbb{D}$ will be denoted by $H(\mathbb{D})$. Let $S(\mathbb{D})$ be the set of analytic self-maps of $\mathbb{D}$. Every self-map φ induces the composition operator C_φ defined by $C_\varphi f = f \circ \varphi$ for $f \in H(\mathbb{D})$.

We recall that the Bloch space $\mathcal{B}$ consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{\mathcal{B}} = \sup_{z\in\mathbb{D}} (1-|z|^2)|f'(z)| < \infty;$$

then $\|\cdot\|_{\mathcal{B}}$ is a complete semi-norm on $\mathcal{B}$, which is Möbius invariant.

It is known that $\mathcal{B}$ is a Banach space under the norm

$$\|f\| = |f(0)| + \|f\|_{\mathcal{B}}.$$

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The essential norm of a continuous linear operator T is the distance from T to the compact operators, that is,

$$\|T\|_e = \inf\{\|T - K\|: \ K \text{ is compact}\}.$$

Notice that $\|T\|_e = 0$ if and only if T is compact, so the estimate on $\|T\|_e$ leads to conditions for T to be compact.

Much effort has been expended on characterizing those analytic maps which induce bounded or compact composition operators. Readers interested in this topic can be referred to the books [22] by Shapiro and [3] by Cowen and MacCluer, which are excellent sources for the development of the theory of composition operators up to the middle of the last decade, and the recent papers [4], [16], [26], [27], [28], [29], [30] and others.

In the past few years, many authors have been interested in studying the mapping properties of the difference of two composition operators, that is, an operator of the form

$$T = C_\varphi - C_\psi.$$

The primary motivation for this has been the desire to understand the topological structure of the whole set of composition operators acting on a given function space. When $\mathcal{X}$ is a Banach space of analytic functions, we write $C(\mathcal{X})$ for the space of composition operators on $\mathcal{X}$ under the operator norm topology. In 2005, Moorhouse [18] answered the question of compact difference for composition operators acting on the Bergman space A^2_α, $\alpha > -1$, and gave a partial answer to the component structure of $C(A^2_\alpha)$. Most papers in this area have focused on the classical reflexive spaces, however, some classical non-reflexive spaces have also been discussed lately in the unit disk $\mathbb{D}$ in the complex plane. Hosokawa and Ohno, [10] in 2006, and [11] in 2007, discussed the topological structures of the sets of composition operators and gave a characterization of compact difference on the Bloch space and little Bloch space in the unit disk.

In 2008, Fang and Zhou [6] also gave a characterization of compact difference between the Bloch space and the set of all bounded analytic functions on the unit polydisk. In 2001, MacCluer and co-authors [15] used the pseudo-hyperbolic metric to discuss the topological components of the set of composition operators acting on $H^\infty(\mathbb{D})$. They provided a geometric condition under which two composition operators with non-compact difference lie in the same component. In 2005, Hosokawa and co-authors [9] continued this investigation. They studied properties of the topological space of weighted composition operators on the space of bounded analytic functions on the open unit disk in the uniform operator topology. These results were extended to the setting of $H^\infty(B_N)$ by Toews [24] in 2004, and independently by Gorkin and co-authors [8] in 2003, and to the setting of $H^\infty(\mathbb{D}^N)$ by Fang and Zhou [5] in 2008, where B_N is the unit ball, $\mathbb{D}^N$ is the unit polydisk. In 2008, Bonet and co-authors [2] discussed the compact difference for the composition operator on weighted Banach spaces of holomorphic functions in the unit disk, which also were extended to the unit polydisk by Wolf in [25] in 2008. The case of weighted composition operators on weighted Banach spaces of holomorphic functions in the unit disk was treated by Lindström and Wolf in [13] in 2008.

Building on this foundation, the present paper continues this line of research, and gives some simple estimates of the essential norm for the difference of composition operators acting on the Bloch space in the unit disk. By way of application, a characterization of compact difference is given.

2. Notation and background

For $a \in \mathbb{D}$, the involution φ_a which interchanges the origin and the point a, is defined by

$$\varphi_a(z) = \frac{a-z}{1-\overline{a}z}.$$

For z, w in $\mathbb{D}$, the pseudo-hyperbolic distance between z and w is given by

$$\varrho(z,w) = |\varphi_z(w)| = \left|\frac{z-w}{1-\overline{z}w}\right|,$$

and the hyperbolic metric is given by

$$\beta(z,w) = \inf_\gamma \int_\gamma \frac{|\,\mathrm{d}\xi|}{1-|\xi|^2} = \frac{1}{2}\ln\frac{1+\varrho(z,w)}{1-\varrho(z,w)},$$

where γ is any piecewise smooth curve in $\mathbb{D}$ from z to w.

It is well known that

$$1-\varrho^2(z,w) = \frac{(1-|z|^2)(1-|w|^2)}{|1-\overline{z}w|^2}.$$

For $\varphi \in S(\mathbb{D})$, the Schwarz-Pick lemma shows that $\varrho(\varphi(z),\varphi(w)) \leqslant \varrho(z,w)$, and if equality holds for some $z \neq w$, then φ is an automorphism of the disk. It is also well known that for $\varphi \in S(\mathbb{D})$, C_φ is always bounded on $\mathcal{B}$.

The Schwarz-Pick type derivative $\varphi^\#$ of φ is defined by

$$\varphi^\#(z) = \frac{1-|z|^2}{1-|\varphi(z)|^2}\,\varphi'(z).$$

It follows from the Schwarz-Pick lemma that

$$|\varphi^{\#}(z)| \leqslant 1 \tag{2.1}$$

for all $z \in \mathbb{D}$.

For $z, w \in \mathbb{D}$, define

$$\flat(z, w) = \sup_{\|f\|_{\mathcal{B}} \leqslant 1} |(1 - |z|^2) f'(z) - (1 - |w|^2) f'(w)|.$$

It follows from Proposition 2.2 in [11] that

$$\varrho^2(z, w) \leqslant \flat(z, w) \leqslant 20\varrho(z, w) \tag{2.2}$$

for any $z, w \in \mathbb{D}$.

From the definition of Bloch space, it is easy to check the next lemma by adapting some integral estimates.

Lemma 2.1. *If $f \in \mathcal{B}$, then*

$$|f(z)| \leqslant \Big(1 + \frac{1}{2} \ln \frac{4}{1 - |z|^2}\Big) \|f\|$$

for any $z \in \mathbb{D}$.

The following lemma (i.e. Montel's theorem) will be important in the sequel. The proof is so elementary that we omit it here.

Lemma 2.2. *Suppose that $\{f_k\}$ is a bounded sequence in $\mathcal{B}$. Then there exists a subsequence $\{f_{k_s}\}$ of $\{f_k\}$ which converges uniformly on compact subsets of $\mathbb{D}$ to a holomorphic function $f \in \mathcal{B}$.*

Lemma 2.3. *Let $\varphi \in S(\mathbb{D})$ and $\|\varphi\|_\infty < 1$. Then C_φ is a compact operator from $\mathcal{B}$ to $\mathcal{B}$.*

Proof. For $\varphi \in S(\mathbb{D})$ and $\|\varphi\|_\infty < 1$ there exists an $r > 0$ such that $\|\varphi\|_\infty < r < 1$. So for any $z \in \mathbb{D}$ we have $|\varphi(z)| < r$. For a sequence $\{f_j\} \subset \mathcal{B}$ with $\|f_j\| \leqslant M$ it follows from (2.1) that

$$\begin{aligned} |(C_\varphi f_j)'(z)|(1 - |z|^2) &= |f_j'(\varphi(z))\varphi'(z)|(1 - |z|^2) \\ &= |f_j'(\varphi(z))|(1 - |\varphi(z)|^2|\varphi^{\#}(z)| \\ &\leqslant \|f_j\| \leqslant M. \end{aligned}$$

By Lemma 2.2 there exists a subsequence $\{f_{j_s}\}$ of $\{f_j\}$ which converges uniformly on compact subsets of $\mathbb{D}$ to a holomorphic function $f \in \mathcal{B}$, and $f'_{k_s}(z)$ also converge uniformly on compact subsets of $\mathbb{D}$ to the holomorphic function $f'(z)$. So if s is large enough, for any $\varepsilon > 0$ and $w \in E = \{rz\colon\ z \in \overline{\mathbb{D}}\} \subset \mathbb{D}$ we have

$$|f'_{j_s}(w) - f'(w)| < \varepsilon.$$

So

$$\begin{aligned}
\|C_\varphi f_{j_s} - C_\varphi f\| &= \sup_{z\in\mathbb{D}}(1-|z|^2)|f'_{j_s}(\varphi(z))\varphi'(z) - f'(\varphi(z))\varphi'(z)| \\
&\quad + |f_{j_s}(\varphi(0)) - f(\varphi(0))| \\
&= \sup_{z\in\mathbb{D}} \frac{1-|z|^2}{1-|\varphi(z)|^2}|\varphi'(z)|(1-|\varphi(z)|^2)|f'_{j_s}(\varphi(z)) - f'(\varphi(z))| \\
&\quad + |f_{j_s}(\varphi(0)) - f(\varphi(0))| \\
&\leqslant \sup_{z\in\mathbb{D}}|f'_{j_s}(\varphi(z)) - f'(\varphi(z))| + |f_{j_s}(\varphi(0)) - f(\varphi(0))| \\
&\leqslant \sup_{w\in E}|f'_{j_s}(w) - f'(w)| + |f_{j_s}(\varphi(0)) - f(\varphi(0))| \to 0,
\end{aligned}$$

as $s \to \infty$. This implies that C_φ is a compact operator on $\mathcal{B}$. □

The following lemma is due to [17].

Lemma 2.4. *Let $\varphi \in S(\mathbb{D})$. The essential norm of $C_\varphi\colon\ \mathcal{B} \to \mathcal{B}$ is*

$$\|C_\varphi\|_e = \lim_{s\to 1} \sup_{|\varphi(z)|>s} |\varphi^{\#}(z)|.$$

Remark 1. This lemma implies that C_φ is compact if and only if $\varphi^{\#}(z) \to 0$ whenever $|\varphi(z)| \to 1$.

Remark 2. From the definition of the essential norm it is clear that if C_φ or C_ψ is a compact operator, then $\|C_\varphi - C_\psi\|_e = \|C_\psi\|_e$ or $\|C_\varphi\|_e$. In this case, Lemma 2.4 can be used to give an estimate of $\|C_\varphi - C_\psi\|_e$. So, throughout the remainder of this paper, we assume that neither C_φ nor C_ψ is a compact operator. By Lemma 2.3 we may assume that $\|\varphi\|_\infty = 1$ and $\|\psi\|_\infty = 1$.

Lemma 2.5. *For $\varphi, \psi\colon\ \mathbb{D} \to \mathbb{D}$ we have*

$$|\varphi^{\#}(z) - \psi^{\#}(z)| \leqslant \frac{10}{r} \sup\{\varrho(\varphi(w), \psi(w))\colon\ \varrho(z, w) \leqslant r\} \tag{2.3}$$

for $0 < r < 1$.

Proof. Let $\varphi_w(z) = (w - z)/(1 - \overline{w}z)$ be an automorphism of the disk. For any $z, z', w, w' \in \mathbb{D}$, by the conformal invariance of pseudo-hyperbolic distance, we have

$$\Big|\frac{\varphi_w(z) - \varphi_w(z')}{1 - \overline{\varphi_w(z')}\varphi_w(z)}\Big| = \varrho(\varphi_w(z), \varphi_w(z')) = \varrho(z, z'),$$

so

$$|\varphi_w(z) - \varphi_w(z')| \leqslant 2\varrho(z, z'). \tag{2.4}$$

For fixed z, denote $f(w) = \varphi_w(z)$. Then $\partial f/\partial w(w) = 1/(1 - \overline{w}z)$ and

$$\frac{\partial f}{\partial \overline{w}}(w) = \frac{(w - z)z}{(1 - \overline{w}z)^2} = \frac{z\varphi_w(z)}{1 - \overline{w}z}.$$

Consequently,

$$|\nabla f(w)| = \sqrt{\Big|\frac{\partial f}{\partial w}(w)\Big|^2 + \Big|\frac{\partial f}{\partial \overline{w}}(w)\Big|^2} \leqslant \frac{2}{1 - |w|} \leqslant \frac{4}{1 - |w|^2}.$$

Let γ be any piecewise smooth curve in $\mathbb{D}$ from w to w', and $\gamma(0) = w'$, $\gamma(1) = w$. Then

$$\begin{aligned}
|\varphi_w(z) - \varphi_{w'}(z)| &= |f(w) - f(w')| = |f(\gamma(1)) - f(\gamma(0))| \\
&\leqslant \int_0^1 |(f \circ \gamma)'(t)|\,\mathrm{d}t = \int_0^1 |\nabla f(\gamma(t))||\gamma'(t)|\,\mathrm{d}t = \int_\gamma |\nabla f(\xi)|\,|\mathrm{d}\xi| \\
&\leqslant 4\int_\gamma \frac{|\,\mathrm{d}\xi|}{1 - |\xi|^2}.
\end{aligned}$$

So

$$|\varphi_w(z) - \varphi_{w'}(z)| \leqslant 4 \inf_\gamma \int_\gamma \frac{|\,\mathrm{d}\xi|}{1 - |\xi|^2} = 2\ln\frac{1 + \varrho(w, w')}{1 - \varrho(w, w')}. \tag{2.5}$$

If $x = \varrho(w, w') < \frac{1}{2}$, it is clear that $F(x) = \ln{(1 + x)/(1 - x)} - 4x$ is a decreasing function and $F(0) = 0$. So we have $\ln{(1 + x)/(1 - x)} < 4x$. It follows from (2.5) that

$$|\varphi_w(z) - \varphi_{w'}(z)| \leqslant 8\varrho(w, w'). \tag{2.6}$$

If $x = \varrho(w, w') \geqslant \frac{1}{2}$, then we have

$$|\varphi_w(z) - \varphi_{w'}(z)| \leqslant 2 \leqslant 4\varrho(w, w'). \tag{2.7}$$

Combining (2.6) and (2.7), we obtain

$$|\varphi_w(z) - \varphi_{w'}(z)| \leqslant 8\varrho(w, w'). \tag{2.8}$$

Taken together (2.8) with (2.4) and using the triangle inequality, we have

$$\begin{aligned}|\varphi_w(z) - \varphi_{w'}(z')| &\leqslant |\varphi_w(z) - \varphi_w(z')| + |\varphi_w(z') - \varphi_{w'}(z')| \\ &\leqslant 2\varrho(z, z') + 8\varrho(w, w').\end{aligned} \tag{2.9}$$

Note that for $z \in \mathbb{D}$, the derivative of $\varphi_{\varphi(z)} \circ \varphi \circ \varphi_z$ at the origin equals $\varphi^{\#}(z)$. Therefore, if $r \in (0, 1)$, the Cauchy integral formula for derivatives yields the representation

$$\varphi^{\#}(z) = \frac{1}{2\pi\mathrm{i}} \int_{|\zeta|=r} \frac{\varphi_{\varphi(z)} \circ \varphi \circ \varphi_z(\zeta)}{\zeta^2}\, \mathrm{d}\zeta.$$

An analogous formula holds for $\psi^{\#}(z)$. Now we can apply (2.9) to get the estimates

$$\begin{aligned}&|(\varphi_{\varphi(z)} \circ \varphi \circ \varphi_z)(\zeta) - (\varphi_{\psi(z)} \circ \psi \circ \varphi_z)(\zeta)| \\ &\qquad \leqslant 2\varrho(\varphi(\varphi_z(\zeta)), \psi(\varphi_z(\zeta))) + 8\varrho(\varphi(z), \psi(z)).\end{aligned}$$

As ζ traverses the set $\{|\zeta| = r\}$, we can easily prove that the point $w = \varphi_z(\zeta)$ runs through the pseudo-hyperbolic circle $\varrho(z, w) = r$ by the Schwarz-Pick lemma. That is, for $z \in \mathbb{D}$, $\{w \in \mathbb{D}\colon\ \varrho(z, w) = r\} = \{w = \varphi_z(\zeta)\colon\ |\zeta| = r\}$. Let

$$S = \sup\{\varrho(\varphi(w), \psi(w))\colon\ \varrho(z, w) \leqslant r\}.$$

We arrive at the estimate

$$|\varphi^{\#}(z) - \psi^{\#}(z)| \leqslant \frac{10}{2\pi} \int_{|\zeta|=r} \frac{S}{r^2} |\,\mathrm{d}\zeta| = \frac{10S}{r},$$

which completes the proof of this lemma. $\square$

Lemma 2.6. *For fixed $r \in (0,1)$, take a positive number $\delta \in (r,1)$. Denote $E_\delta = \{z \in \mathbb{D}\colon |\varphi(z)| > \delta\}$, $T_{r,\delta,w} = \{z \in E_\delta\colon \varrho(z,w) \leqslant r\}$, and $C(\delta) = \inf\{|\varphi(w)|\colon z \in T_{r,\delta,w}\}$. If $z \in E_\delta$ and $z \in T_{r,\delta,w}$, then $w \in E_{C(\delta)}$ and $\lim\limits_{\delta\to 1} C(\delta) = 1$.*

Proof. For any $z \in E_\delta$ and $z \in T_{r,\delta,w}$, by the Schwarz-Pick lemma, we know that $\varrho(\varphi(z),\varphi(w)) \leqslant \varrho(z,w) \leqslant r$. So

$$\Big|\frac{\varphi(z)-\varphi(w)}{1-\overline{\varphi(z)}\varphi(w)}\Big|^2 \leqslant r^2.$$

Consequently,

$$|\varphi(z)|^2 + |\varphi(w)|^2 - 2\operatorname{Re}\varphi(z)\overline{\varphi(w)} \leqslant r^2(1 + |\varphi(z)|^2|\varphi(w)|^2 - 2\operatorname{Re}\varphi(z)\overline{\varphi(w)}).$$

By $1 > |\varphi(z)|^2 > \delta^2 > r^2$ we get

$$\delta^2 - r^2 + (1-r^2)|\varphi(w)|^2 - 2(1-r^2)|\varphi(w)| < 0.$$

So a direct calculation shows that

$$|\varphi(w)| > \frac{(1-r^2) - \sqrt{(1-r^2)(1-\delta^2)}}{1-r^2}.$$

It is easy to check that

$$C(\delta) = \frac{(1-r^2) - \sqrt{(1-r^2)(1-\delta^2)}}{1-r^2}$$

and $\lim\limits_{\delta\to 1} C(\delta) = 1$. It is obvious that $w \in E_{C(\delta)}$. This completes the proof of this lemma. $\square$

With a little calculation, we can get the following lemma.

Lemma 2.7. *Suppose $f \in \mathcal{B}$ and $\|f\|_{\mathcal{B}} \leqslant 1$. Then*

$$\|(C_\varphi - C_\psi)f\|_{\mathcal{B}} \leqslant \sup_{\|z\|\leqslant 1}\{|\varphi^{\#}(z) - \psi^{\#}(z)| + |\varphi^{\#}(z)|\flat(\varphi(z),\psi(z))\}.$$

The following lemma comes from [17].

Lemma 2.8. *Let $K_m f(z) = f((m-1)m^{-1}z)$, then K_m has the following properties:*

(a) *K_m is a compact operator on $\mathcal{B}$.*

(b) *For every $f \in \mathcal{B}$, $(I-K_m)f$ tends to 0 uniformly on compact subsets of the unit disk as $m \to \infty$, so $[(I-K_m)f]'(z)$ also tends to 0 on compact subsets of the unit disk.*

3. Main theorem

In 2007, Hosokawa and Ohno [11] showed that $C_\varphi - C_\psi$ is compact if and only if (i) $\varphi^{\#}(z)\varrho(\varphi(z),\psi(z)) \to 0$ whenever $|\varphi(z)| \to 1$, (ii) $\psi^{\#}(z)\varrho(\varphi(z),\psi(z)) \to 0$ whenever $|\psi(z)| \to 1$, (iii) $\varphi^{\#}(z) - \psi^{\#}(z) \to 0$ whenever $|\varphi(z)\psi(z)| \to 1$.

Before explaining our main result we need to fix some notation.

For $\delta > 0$, set

$$E_\delta^{(1)} = \{z \in \mathbb{D}\colon\ |\varphi(z)| > \delta\},$$
$$E_\delta^{(2)} = \{z \in \mathbb{D}\colon\ |\psi(z)| > \delta\}$$

and

$$E_\delta = \{z \in \mathbb{D}\colon\ \max(|\varphi(z)|, |\psi(z)|) > \delta\}.$$

It is clear that $E_\delta = E_\delta^{(1)} \cup E_\delta^{(2)}$.

Denote $F_\delta = \mathbb{D} - E_\delta$. $\chi_{E_\delta^{(i)}}(z)$ is a characteristic function depending on $E_\delta^{(i)}$ defined by

$$\chi_{E_\delta^{(i)}}(z) = \begin{cases} 1, & z \in E_\delta^{(i)}, \\ 0, & z \notin E_\delta^{(i)}, \end{cases} \qquad i = 1, 2.$$

Theorem 3.1. *Suppose $\varphi, \psi\colon \mathbb{D} \to \mathbb{D}$ and neither C_φ nor $C_\psi\colon \mathcal{B} \to \mathcal{B}$ is a compact operator, then*

$$\begin{aligned} &\frac{1}{2}\lim_{\delta\to 1}\sup_{z\in E_\delta}(1-\varrho^2(\varphi(z),\psi(z)))(\chi_{E_\delta^{(1)}}(z)|\psi^{\#}(z)| + \chi_{E_\delta^{(2)}}(z)|\varphi^{\#}(z)|)\varrho(\varphi(z),\psi(z)) \\ &\qquad \leqslant \|C_\varphi - C_\psi\|_e \leqslant 80\lim_{\delta\to 1}\sup_{z\in E_\delta}\varrho(\varphi(z),\psi(z)). \end{aligned}$$

P r o o f. We begin by proving the upper estimate. For fixed m, we know $C_\varphi K_m$ and $C_\psi K_m$ are compact operators on $\mathcal{B}$ by Lemma 2.8. Therefore

$$\|C_\varphi - C_\psi\|_e \leqslant \|C_\varphi - C_\psi - C_\varphi K_m + C_\psi K_m\|_{\mathcal{B}}.$$

Hence for $0 < \delta < 1$,

$$
\begin{aligned}
&\|C_\varphi - C_\psi - C_\varphi K_m + C_\psi K_m\|_{\mathcal{B}} \\
&\quad = \sup_{\|f\|\leqslant 1} \|(C_\varphi - C_\psi - C_\varphi K_m + C_\psi K_m) f\|_{\mathcal{B}} \\
&\quad \leqslant \sup_{\|f\|\leqslant 1} \sup_{z\in E_\delta} (1-|z|^2)|[(I-K_m)f]'(\varphi(z))\varphi'(z) - [(I-K_m)f]'(\psi(z))\psi'(z)| \\
&\qquad + \sup_{\|f\|\leqslant 1} \sup_{z\in E_\delta} (1-|z|^2)|f'(\varphi(z))\varphi'(z) - f'(\psi(z))\psi'(z) \\
&\qquad - \Bigl(1-\frac{1}{m}\Bigr) f'\Bigl(\Bigl(1-\frac{1}{m}\Bigr)\varphi(z)\Bigr)\varphi'(z) + \Bigl(1-\frac{1}{m}\Bigr) f'\Bigl(\Bigl(1-\frac{1}{m}\Bigr)\psi(z)\Bigr)\psi'(z)|.
\end{aligned}
$$

By virtue of (2.1) and (b) in Lemma 2.8 we can choose m large enough such that the first term on the right hand side is less than any given ε, and denoting the second term by I we have

$$
\begin{aligned}
I &\leqslant \sup_{\|f\|\leqslant 1} \sup_{z\in E_\delta} \Bigl\{(1-|z|^2)|f'(\varphi(z))\varphi'(z) - f'(\psi(z))\psi'(z)| \\
&\quad + (1-|z|^2)\Bigl|\Bigl(1-\frac{1}{m}\Bigr) f'\Bigl(\Bigl(1-\frac{1}{m}\Bigr)\varphi(z)\Bigr)\varphi'(z) \\
&\quad - \Bigl(1-\frac{1}{m}\Bigr) f'\Bigl((1-\frac{1}{m})\psi(z)\Bigr)\psi'(z)\Bigr|\Bigr\} \\
&\leqslant \sup_{\|f\|\leqslant 1} \sup_{z\in E_\delta} \{|\varphi^{\#}(z) - \psi^{\#}(z)| + |\varphi^{\#}(z)|\flat(\varphi(z),\psi(z)) \\
&\quad + \Bigl|\Bigl[\Bigl(1-\frac{1}{m})\varphi\Bigr]^{\#}(z) - \Bigl[\Bigl(1-\frac{1}{m}\Bigr)\psi\Bigr]^{\#}(z)\Bigr| \\
&\quad + \Bigl|\Bigl[\Bigl(1-\frac{1}{m}\Bigr)\varphi\Bigr]^{\#}(z)\Bigr|\flat\Bigl(\Bigl(1-\frac{1}{m}\Bigr)\varphi(z), \Bigl(1-\frac{1}{m}\Bigr)\psi(z)\Bigr)\Bigr\} \\
&\leqslant \sup_{z\in E_\delta} \Bigl\{\ 20 \sup_{\varrho(z,w)\leqslant\frac{1}{2}} \varrho(\varphi(w),\psi(w)) + 20\varrho(\varphi(z),\psi(z)) \\
&\quad + 20 \sup_{\varrho(z,w)\leqslant\frac{1}{2}} \varrho\Bigl(\Bigl(1-\frac{1}{m}\Bigr)\varphi(w), \Bigl(1-\frac{1}{m}\Bigr)\psi(w)\Bigr) \\
&\quad + 20\varrho\Bigl(\Bigl(1-\frac{1}{m}\Bigr)\varphi(z), \Bigl(1-\frac{1}{m}\Bigr)\psi(z)\Bigr)\Bigr\} \\
&\leqslant \sup_{w\in E_{C(\delta)}} 40\varrho(\varphi(w),\psi(w)) + \sup_{z\in E_\delta} 40\varrho(\varphi(z),\psi(z)) \\
&\leqslant 40 \sup_{w\in E_{C(\delta)}} \varrho(\varphi(w),\psi(w)) + 40 \sup_{w\in E_{C(\delta)}} \varrho(\varphi(w),\psi(w) \\
&= 80 \sup_{w\in E_{C(\delta)}} \varrho(\varphi(w),\psi(w)).
\end{aligned}
$$

The second inequality follows by Lemma 2.7; the third inequality follows by Lemma 2.5 and (2); the fourth inequality follows by Lemma 2.6 and

$$
\varrho\Bigl(\Bigl(1-\frac{1}{m}\Bigr)\varphi(w), \Bigl(1-\frac{1}{m}\Bigr)\psi(w)\Bigr) \leqslant \varrho(\varphi(w),\psi(w)),
$$

where $C(\delta)=1-\frac{2}{3}\sqrt{3}\sqrt{1-\delta^2}$ and $\lim\limits_{\delta\to 1}C(\delta)=1$. So letting $m\to\infty$, $\delta\to 1$, we get the upper estimate.

Now we turn to the lower estimate. Define

$$\begin{aligned}a=&\frac{1}{2}\lim_{\delta\to 1}\sup_{z\in E_\delta}(1-\varrho^2(\varphi(z),\psi(z)))\\&\times(\chi_{E_\delta^{(1)}}(z)|\psi^{\#}(z)|+\chi_{E_\delta^{(2)}}(z)|\varphi^{\#}(z)|)\varrho(\varphi(z),\psi(z)).\end{aligned}$$

Let $\delta_m=1-1/m$, then $\delta_m\to 1$ as $m\to\infty$.

Recall that in the conditions of the theorem we assume that neither C_φ or C_ψ is a compact operator. So $\|\varphi\|_\infty=1$ and $\|\psi\|_\infty=1$ by Lemma 2.3, hence for every m, $E_{\delta_m}\neq\emptyset$. So there exists $z_m\in E_{\delta_m}$ such that

$$\begin{aligned}a=&\frac{1}{2}\lim_{m\to\infty}(1-\varrho^2(\varphi(z_m),\psi(z_m)))\\&\times(\chi_{E_\delta^{(1)}}(z_m)|\psi^{\#}(z_m)|+\chi_{E_\delta^{(2)}}(z_m)|\varphi^{\#}(z_m)|)\varrho(\varphi(z_m),\psi(z_m).\end{aligned}$$

Since $z_m\in E_{\delta_m}$ implies $|\varphi(z_m)|>\delta_m$ or $|\psi(z_m)|>\delta_m$, without loss of generality we assume $|\varphi(z_m)|\to 1$. Setting

$$f_m(z)=\varphi^2_{\varphi(z_m)}(z)-\varphi^2(z_m),$$

it is obvious that $\{f_m\}$ converges to zero uniformly on compact subsets of $\mathbb{D}$ as $m\to\infty$ and $\|f_m\|\leqslant 2$ for any $m=1,2,\ldots$. So the compactness of K implies that $\|Kf_m\|\to 0$ as $m\to\infty$, and it follows that

$$\begin{aligned}\|C_\varphi-C_\psi-K\|_{\mathcal{B}}&\geqslant\frac{1}{2}\limsup_{m\to\infty}\|(C_\varphi-C_\psi-K)f_m\|_{\mathcal{B}}\\&\geqslant\frac{1}{2}\limsup_{m\to\infty}(\|(C_\varphi-C_\psi)f_m\|_{\mathcal{B}}-\|Kf_m\|_{\mathcal{B}})\\&=\frac{1}{2}\limsup_{m\to\infty}\|(C_\varphi-C_\psi)f_m\|_{\mathcal{B}}\\&=\frac{1}{2}\limsup_{m\to\infty}\sup_{z\in\mathbb{D}}(1-|z|^2)|f_m'(\varphi(z))\varphi'(z)-f_m'(\psi(z))\psi'(z)|\\&\geqslant\frac{1}{2}\limsup_{m\to\infty}(1-|z_m|^2)|f_m'(\varphi(z_m))\varphi'(z_m)-f_m'(\psi(z_m))\psi'(z_m)|\\&=\limsup_{m\to\infty}(1-|z_m|^2)|\varphi_{\varphi(z_m)}(\varphi(z_m))\varphi'_{\varphi(z_m)}(\varphi(z_m))\varphi'(z_m)\\&\quad-\varphi_{\varphi(z_m)}(\psi(z_m))\varphi'_{\varphi(z_m)}(\psi(z_m))\psi'(z_m)|\\&=\limsup_{m\to\infty}(1-|z_m|^2)\left|\frac{\varphi(z_m)-\psi(z_m)}{1-\overline{\varphi(z_m)}\psi(z_m)}\right|\frac{1-|\varphi(z_m)|^2}{|1-\overline{\varphi(z_m)}\psi(z_m)|^2}|\psi'(z_m)|\\&=\limsup_{m\to\infty}(1-\varrho^2(\varphi(z_m),\psi(z_m))|\psi^{\#}(z_m)|\varrho(\varphi(z_m),\psi(z_m)).\end{aligned}$$

Similarly, if $|\psi(z_m)| \to 1$, then let

$$g_m(z) = \varphi^2_{\psi(z_m)}(z) - \psi^2(z_m).$$

We get

$$\|C_\varphi - C_\psi - K\|_{\mathcal{B}} \geqslant \limsup_{m\to\infty}(1 - \varrho^2(\varphi(z_m), \psi(z_m))|\varphi^{\#}(z_m)|\varrho(\varphi(z_m), \psi(z_m)).$$

So by the above argument we have the lower estimate:

$$\begin{aligned}
&\|C_\varphi - C_\psi - K\|_{\mathcal{B}} \\
&\quad\geqslant \frac{1}{2}\lim_{m\to\infty}(1 - \varrho^2(\varphi(z_m), \psi(z_m))) \\
&\qquad\times \big(\chi_{E_\delta^{(1)}}(z_m)|\psi^{\#}(z_m)| + \chi_{E_\delta^{(2)}}(z_m)|\varphi^{\#}(z_m)|\big)\varrho(\varphi(z_m), \psi(z_m)) = \\
&\quad= \lim_{\delta\to 0}\sup_{z\in E_\delta}(1 - \varrho^2(\varphi(z), \psi(z))) \\
&\qquad\times \big(\chi_{E_\delta^{(1)}}(z)|\psi^{\#}(z)| + \chi_{E_\delta^{(2)}}(z)|\varphi^{\#}(z)|\big)\varrho(\varphi(z), \psi(z)).
\end{aligned}$$

The proof is complete. $\square$

Remark 3. Concerning the lower estimate, take a sequence $\{z_m\} \subset E_\delta^{(1)} \setminus E_\delta^{(2)}$, that is, $|\varphi(z_m)| \to 1|$ but $|\psi(z_m)| \to t < 1$. Then $\varrho(\varphi(z_m), \psi(z_m)) \to 1$, and we have

$$(1 - \varrho^2(\varphi(z_m), \psi(z_m)))\varrho(\varphi(z_m), \psi(z_m))|\varphi^{\#}(z_m)| \to 0.$$

Similarly, if $\{z_m\} \subset E_\delta^{(2)} \setminus E_\delta^{(1)}$, then

$$(1 - \varrho^2(\varphi(z_m), \psi(z_m)))\varrho(\varphi(z_m), \psi(z_m))|\varphi^{\#}(z_m)| \to 0.$$

Thus the supremum of the lower estimate of Theorem 3.1 should be taken over $E_\delta^{(1)} \cap E_\delta^{(2)}$, hence $\chi_{E_\delta^{(i)}}$ can be removed. Consequently, we have a better estimate:

$$\begin{aligned}
&\lim_{\delta\to 1}\sup_{z\in E_\delta^{(1)}\cap E_\delta^{(2)}}(1 - \varrho^2(\varphi(z), \psi(z)))\max\{|\varphi^{\#}(z)|, |\psi^{\#}(z)|\} \\
&\qquad\leqslant \|C_\varphi - C_\psi\|_e \leqslant 80\lim_{\delta\to 1}\sup_{z\in E_\delta}\varrho(\varphi(z), \psi(z)).
\end{aligned}$$

Corollary 3.2. *For $\varphi, \psi \in S(\mathbb{D})$, and neither C_φ nor C_ψ being a compact operator we have*

(a) *if* $\lim_{\delta\to 1} \sup_{z\in E_\delta} \varrho(\varphi(z), \psi(z)) = 0$, *then* $C_\varphi - C_\psi$ *is a compact operator on* $\mathcal{B}$;

(b) *if* $C_\varphi - C_\psi$ *is a compact operator on* $\mathcal{B}$, *then*

$$\lim_{\delta\to 1} \sup_{z\in E_\delta} (1 - \varrho^2(\varphi(z), \psi(z)))(\chi_{E_\delta^{(1)}}(z)|\psi^{\#}(z)| + \chi_{E_\delta^{(2)}}(z)|\varphi^{\#}(z)|)\varrho(\varphi(z), \psi(z)) = 0;$$

(c) *if* $C_\varphi - C_\psi$ *is a compact operator on* $\mathcal{B}$, *then*

$$\lim_{\delta\to 1} \sup_{z\in E_\delta^{(1)}\cap E_\delta^{(2)}} (1 - \varrho^2(\varphi(z), \psi(z))) \max\{|\varphi^{\#}(z)|, |\psi^{\#}(z)|\} = 0.$$

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