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ON ANOTHER EXTENSION OF q-PFAFF-SAALSCHÜTZ FORMULA

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Abstract. In this paper we give an extension of q-Pfaff-Saalschütz formula by means of Andrews-Askey integral. Applications of the extension are also given, which include an extension of q-Chu-Vandermonde convolution formula and some other q-identities.

Keywords: Andrews-Askey integral, ${}_{r+1}\varphi_r$ basic hypergeometric series, q-Pfaff-Saalschütz formula, q-Chu-Vandermonde convolution formula

MSC 2010: 05A30, 33D15, 33D05

1. Introduction and statement of main result

The following is Andrews-Askey integral [1] which can be derived from Ramanujan's ${}_1\psi_1$ summation:

$$\int_c^d \frac{(qt/c,qt/d;q)_\infty}{(at,bt;q)_\infty}\,\mathrm{d}_qt=\frac{d(1-q)(q,dq/c,c/d,abcd;q)_\infty}{(ac,ad,bc,bd;q)_\infty} \tag{1.1}$$

provided that no zero factors occur in the denominators of the integral.

Andrews-Askey integral is an important formula in basic hypergeometric series. In [4], the author gives a more general q-integral: If $|q|<1$ and no zero factors occur in the denominators of the integral, then

$$\begin{aligned}\int_s^t &\frac{(q\omega/s,q\omega/t;q)_\infty P_n(\omega,c/a;q)P_m(\omega,d/b;q)}{(a\omega,b\omega;q)_\infty}\,\mathrm{d}_q\omega\\ &=\frac{t(1-q)(c;q)_n(d;q)_m(q,tq/s,s/t,abst;q)_\infty}{a^nb^m(as,at,bs,bt;q)_\infty}\\ &\quad\times\sum_{k=0}^n\frac{(q^{-n},as,at;q)_kq^k}{(q,c,abst;q)_k}\,{}_3\varphi_2\left(\begin{matrix}bs,bt,q^{-m}\\ d,abstq^k\end{matrix};q,q\right),\end{aligned} \tag{1.2}$$

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where

$$P_0(a,b;q)=1,\quad P_n(a,b;q)=(a-b)(a-bq)\dots(a-bq^{n-1}),\quad n\geqslant 1.$$

It is obvious that the case $m=n=0$ of (1.2) results in (1.1). In this paper we use (1.2) to derive an extension of the q-Pfaff-Saalschütz formula. The following theorem is the main result of this paper.

Theorem 1.1. *If $|q|<1$ and no zero factors occur in the denominators, then*

$$\sum_{k=0}^{m}\frac{(q^{-m},c/a,c/b;q)_k q^k}{(q,c,c/abq^{m-1};q)_k}\,{}_3\varphi_2\left(\begin{matrix}a,b,q^{-n}\\ cq^k,ab/cq^{n-1}\end{matrix};q,q\right)=\frac{(a,b;q)_m(c/a,c/b;q)_n}{(c;q)_{m+n}(ab/c;q)_m(c/ab;q)_n}. \tag{1.3}$$

Note that there are some important special cases of (1.3). For example, the case $m=0$ of (1.3) results in the q-Pfaff-Saalschütz formula:

$${}_3\varphi_2\left(\begin{matrix}a,b,q^{-n}\\ c,abc^{-1}q^{1-n}\end{matrix};q,q\right)=\frac{(c/a,c/b;q)_n}{(c,c/ab;q)_n}. \tag{1.4}$$

2. Notation and known results

We first recall some definitions, notation and known results from [2] which will be used for the proof of Theorem 1.1. Throughout this paper, it is supposed that $0<|q|<1$. The q-shifted factorials are defined as

$$(a;q)_0=1,\quad (a;q)_n=\prod_{k=0}^{n-1}(1-aq^k),\quad (a;q)_\infty=\prod_{k=0}^{\infty}(1-aq^k). \tag{2.1}$$

We also adopt the following compact notation for multiple q-shifted factorials:

$$(a_1,a_2,\dots,a_m;q)_n=(a_1;q)_n(a_2;q)_n\dots(a_m;q)_n, \tag{2.2}$$

where n is an integer or ∞. In 1846, Heine introduced the ${}_{r+1}\varphi_r$ basic hypergeometric series, which is defined by

$${}_{r+1}\varphi_r\left(\begin{matrix}a_1,a_2,\dots,a_{r+1}\\ b_1,b_2,\dots,b_r\end{matrix};q,x\right)=\sum_{n=0}^{\infty}\frac{(a_1,a_2,\dots,a_{r+1};q)_n x^n}{(q,b_1,b_2,\dots,b_r;q)_n}. \tag{2.3}$$

The q-Chu-Vandermonde sums are

$$
{}_2\varphi_1\left(\begin{matrix} a, q^{-n} \\ c \end{matrix}; q, q\right) = \frac{a^n (c/a;q)_n}{(c;q)_n} \tag{2.4}
$$

and, reversing the order of summation, we have

$$
{}_2\varphi_1\left(\begin{matrix} a, q^{-n} \\ c \end{matrix}; q, cq^n/a\right) = \frac{(c/a;q)_n}{(c;q)_n}. \tag{2.5}
$$

F. H. Jackson defined the q-integral by [3]

$$
\int_0^d f(t)\,\mathrm{d}_q t = d(1-q)\sum_{n=0}^{\infty} f(dq^n) q^n \tag{2.6}
$$

and

$$
\int_c^d f(t)\,\mathrm{d}_q t = \int_0^d f(t)\,\mathrm{d}_q t - \int_0^c f(t)\,\mathrm{d}_q t. \tag{2.7}
$$

3. The proof of theorem 1.1

In this section we use the generalized Andrews-Askey integral (1.2) to prove Theorem 1.1.

Proof. Using the Andrews-Askey integral (1.1) we arrive at

$$
\int_c^d \frac{(qt/c, qt/d;q)_\infty}{(atq^n, btq^m;q)_\infty}\,\mathrm{d}_q t = \frac{d(1-q)(q, dq/c, c/d, abcdq^{m+n};q)_\infty}{(acq^n, adq^n, bcq^m, bdq^m;q)_\infty}. \tag{3.1}
$$

On the other hand, if we employ the formulas

$$
(at;q)_n = (-1)^n a^n q^{\binom{n}{2}} P_n(t, 1/aq^{n-1};q), \tag{3.2}
$$

$$
(bt;q)_m = (-1)^m b^m q^{\binom{m}{2}} P_m(t, 1/bq^{m-1};q) \tag{3.3}
$$

and use the generalized Andrews-Askey integral (1.2), we obtain

$$
\begin{aligned}
(3.4)\qquad \int_c^d \frac{(qt/c,qt/d;q)_\infty}{(atq^n,btq^m;q)_\infty}\,\mathrm{d}_q t &= \int_c^d \frac{(qt/c,qt/d;q)_\infty (at;q)_n (bt;q)_m}{(at,bt;q)_\infty}\,\mathrm{d}_q t \\
&= (-1)^{n+m} a^n b^m q^{\binom{n}{2}+\binom{m}{2}} \\
&\quad \times \int_c^d \frac{(qt/c,qt/d;q)_\infty P_m(t,a/abq^{m-1};q) P_n(t,b/baq^{n-1};q)}{(at,bt;q)_\infty}\,\mathrm{d}_q t \\
&= (-1)^{n+m} a^n b^m q^{\binom{n}{2}+\binom{m}{2}} \\
&\quad \times \frac{d(1-q)(a/bq^{m-1};q)_m (b/aq^{n-1};q)_n (q,dq/c,c/d,abcd;q)_\infty}{a^m b^n (ac,ad,bc,bd;q)_\infty} \\
&\quad \times \sum_{k=0}^{m} \frac{(q^{-m},ac,ad;q)_k q^k}{(q,a/bq^{m-1},abcd;q)_k} {}_3\varphi_2\left(\begin{matrix} bc,bd,q^{-n} \\ b/aq^{n-1},abcdq^k \end{matrix};q,q\right).
\end{aligned}
$$

Substituting the relations

$$(-1)^m (b/a)^m q^{\binom{m}{2}} (a/bq^{m-1};q)_m = (b/a;q)_m,$$

and

$$(-1)^n (a/b)^n q^{\binom{n}{2}} (b/aq^{n-1};q)_n = (a/b;q)_n$$

into (3.4) we obtain

$$
\begin{aligned}
(3.5)\qquad & \int_c^d \frac{(qt/c,qt/d;q)_\infty}{(atq^n,btq^m;q)_\infty}\,\mathrm{d}_q t \\
&= \frac{d(1-q)(b/a;q)_m (a/b;q)_n (q,dq/c,c/d,abcd;q)_\infty}{(ac,ad,bc,bd;q)_\infty} \\
&\quad \times \sum_{k=0}^{m} \frac{(q^{-m},ac,ad;q)_k q^k}{(q,a/bq^{m-1},abcd;q)_k} {}_3\varphi_2\left(\begin{matrix} bc,bd,q^{-n} \\ b/aq^{n-1},abcdq^k \end{matrix};q,q\right).
\end{aligned}
$$

Combining (3.1) and (3.5) yields

$$
\begin{aligned}
(3.6)\qquad & \frac{d(1-q)(b/a;q)_m (a/b;q)_n (q,dq/c,c/d,abcd;q)_\infty}{(ac,ad,bc,bd;q)_\infty} \\
&\quad \times \sum_{k=0}^{m} \frac{(q^{-m},ac,ad;q)_k q^k}{(q,a/bq^{m-1},abcd;q)_k} {}_3\varphi_2\left(\begin{matrix} bc,bd,q^{-n} \\ b/aq^{n-1},abcdq^k \end{matrix};q,q\right) \\
&= \frac{d(1-q)(q,dq/c,c/d,abcdq^{m+n};q)_\infty}{(acq^n,adq^n,bcq^m,bdq^m;q)_\infty}.
\end{aligned}
$$

Replacing bc, bd and $abcd$ by a, b and c, respectively, and making simple rearrangements, we have (1.3). □

Letting $a \to \infty$, $a \to 0$ in (1.3), respectively, we obtain the following extensions of the q-Chu-Vandermonde convolution formula.

Corollary 3.1. *We have*

$$\sum_{k=0}^{m}\frac{(q^{-m},c/b;q)_kq^k}{(q,c,;q)_k}{}_2\varphi_1\left(\begin{matrix}b,q^{-n}\\cq^k\end{matrix};q,\frac{cq^n}{b}\right)=\left(\frac{c}{b}\right)^m\frac{(b;q)_m(c/b;q)_n}{(c;q)_{m+n}} \tag{3.7}$$

and

$$\sum_{k=0}^{m}\frac{(q^{-m},c/b;q)_k}{(q,c,;q)_k}(bq^m)^k{}_2\varphi_1\left(\begin{matrix}b,q^{-n}\\cq^k\end{matrix};q,q\right)=b^n\frac{(b;q)_m(c/b;q)_n}{(c;q)_{m+n}}. \tag{3.8}$$

It is easy to see that the case $m=0$ or $n=0$ in (3.7) or (3.8) results in the q-Chu-Vandermonde convolution formula.

4. Some applications

In this section we give some q-identities as applications of (1.3). First we give the following q-identity.

Theorem 4.1. *For any integer $n\geqslant 1$ we have*

$$ {}_3\varphi_2\left(\begin{matrix}a,b,q^{-n}\\c,abc^{-1}q^{2-n}\end{matrix};q,q\right)=\left\{1-\frac{(1-a)(1-b)}{(1-cq^{n-1})(1-abq/c)}\right\}\frac{(c/a,c/b;q)_{n-1}}{(c,c/ab;q)_{n-1}}. \tag{4.1}$$

Proof. Let $m=1$ in (1.3) to get

(4.2)

$$ {}_3\varphi_2\left(\begin{matrix}a,b,q^{-n}\\c,ab/cq^{n-1}\end{matrix};q,q\right)+\frac{q(1-q^{-1})(1-c/a)(1-c/b)}{(1-q)(1-c)(1-c/ab)}{}_3\varphi_2\left(\begin{matrix}a,b,q^{-n}\\cq,ab/cq^{n-1}\end{matrix};q,q\right)$$
$$=\frac{(1-a)(1-b)(c/a,c/b;q)_n}{(1-ab/c)(c;q)_{n+1}(c/ab;q)_n}.$$

Substituting the q-Pfaff-Saalschütz formula (1.4) on the left-hand side of (4.2) and making some simple rearrangements, we have

$$ {}_3\varphi_2\left(\begin{matrix}a,b,q^{-n}\\cq,abc^{-1}q^{1-n}\end{matrix};q,q\right)=\left\{1-\frac{(1-a)(1-b)}{(1-cq^{n})(1-ab/c)}\right\}\frac{(cq/a,cq/b;q)_{n-1}}{(cq,cq/ab;q)_{n-1}}. \tag{4.3}$$

After letting $cq=c$ in (4.3), we get (4.1). □

Corollary 4.2. *For any integer $n \geqslant 1$ we have*

$$ {}_2\varphi_1\left(\begin{matrix} b, q^{-n} \\ c \end{matrix}; q, \frac{cq^{n-1}}{b}\right) = \left(1 - \frac{c - bc}{bq - bcq^n}\right)\frac{(c/b;q)_{n-1}}{(c;q)_{n-1}}. \tag{4.4} $$

Proof. Letting $a \to \infty$ in (4.1), we obtain (4.4). □

Similarly, if we let $m = 2, 3, \ldots$, in (1.3), we can get some more identities like (4.1). Then we give another kind of a q-identity.

Theorem 4.3. *For any integer $m \geqslant 1$, we have*

$$ \begin{aligned} &\sum_{k=0}^{m} \frac{(q^{-m}, a, b; q)_k}{(q, c, ab/cq^{m-1}; q)_k} \cdot \frac{q^k}{1 - cq^k} \\ &\qquad = \left\{1 - \frac{(1-a)(1-b)}{(1-cq^m)(1-ab/c)}\right\}\frac{(c/a, c/b; q)_{m-1}}{(c;q)_m (c/ab;q)_{m-1}}. \end{aligned} \tag{4.5} $$

Proof. Let $n = 1$ in (1.3) to get

$$ \begin{aligned} &\sum_{k=0}^{m} \frac{(q^{-m}, c/a, c/b; q)_k q^k}{(q, c, c/abq^{m-1}; q)_k} {}_3\varphi_2\left(\begin{matrix} a, b, q^{-1} \\ cq^k, ab/c \end{matrix}; q, q\right) \\ &= \sum_{k=0}^{m} \frac{(q^{-m}, c/a, c/b; q)_k q^k}{(q, c, c/abq^{m-1}; q)_k}\left\{1 + \frac{q(1-q^{-1})(1-a)(1-b)}{(1-q)(1-ab/c)(1-cq^k)}\right\} \\ &= \sum_{k=0}^{m} \frac{(q^{-m}, c/a, c/b; q)_k q^k}{(q, c, c/abq^{m-1}; q)_k} - \frac{(1-a)(1-b)}{(1-ab/c)} \sum_{k=0}^{m} \frac{(q^{-m}, c/a, c/b; q)_k q^k}{(q, c, c/abq^{m-1}; q)_k} \cdot \frac{q^k}{1-cq^k} \\ &= \frac{(1-c/a)(1-c/b)(a,b;q)_m}{(1-c/ab)(c;q)_{m+1}(ab/c;q)_m}. \end{aligned} $$

Hence, we have

$$ \begin{aligned} &\frac{(1-a)(1-b)}{(1-ab/c)} \sum_{k=0}^{m} \frac{(q^{-m}, c/a, c/b; q)_k q^k}{(q, c, c/abq^{m-1}; q)_k} \cdot \frac{q^k}{1-cq^k} \\ &\qquad = \sum_{k=0}^{m} \frac{(q^{-m}, c/a, c/b; q)_k q^k}{(q, c, c/abq^{m-1}; q)_k} - \frac{(1-c/a)(1-c/b)(a,b;q)_m}{(1-c/ab)(c;q)_{m+1}(ab/c;q)_m}. \end{aligned} \tag{4.6} $$

We use the q-Pfaff-Saalschütz formula (1.4) in (4.6) with $n = m$, $a = c/a$ and $b = c/b$. After simple rearrangements, we have

$$ \begin{aligned} &\sum_{k=0}^{m} \frac{(q^{-m}, c/a, c/b; q)_k}{(q, c, c/abq^{m-1}; q)_k} \cdot \frac{q^k}{1 - cq^k} \\ &\qquad = \left\{1 - \frac{(1-c/a)(1-c/b)}{(1-cq^m)(1-c/ab)}\right\}\frac{(a, b; q)_{m-1}}{(c;q)_m (ab/c;q)_{m-1}}, \end{aligned} \tag{4.7} $$

which is equivalent to (4.5). □

Corollary 4.4. *For any integer $m \geqslant 1$ we have*

$$\sum_{k=0}^{m} \frac{(q^{-m}, b; q)_k}{(q;q)_k (c;q)_{k+1}} \Big(\frac{cq^m}{b}\Big)^k = \Big(1 - \frac{c - bc}{b - bcq^m}\Big) \frac{(c/b;q)_{m-1}}{(c;q)_m}, \tag{4.8}$$

and

$$\sum_{k=0}^{m} \frac{(q^{-m}, b; q)_k}{(q,c;q)_k} \cdot \frac{q^k}{1 - cq^k} = \Big(1 - \frac{1 - b}{1 - cq^m}\Big) \frac{(c/b;q)_{m-1}}{(c;q)_m}. \tag{4.9}$$

Proof. Letting $a \to \infty$, or $a \to 0$ in (4.5), we obtain, respectively, (4.8) and (4.9). □

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