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A NOTE ON THE POWERS OF CESÀRO BOUNDED OPERATORS

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Abstract. In this note we give a negative answer to Zemánek's question (1994) of whether it always holds that a Cesàro bounded operator T on a Hilbert space with a single spectrum satisfies $\lim\limits_{n\to\infty} \|T^{n+1} - T^n\| = 0$.

Keywords: Volterra operator, stability of operators

MSC 2010: 47A35, 47B38, 47B37

1. Introduction

The famous Katznelson-Tzafriri theorem [7] states that if T is a power-bounded operator on a complex Banach space X, then the difference $T^{n+1} - T^n$ tends to 0 uniformly (when $n \to \infty$) if (and only if) the peripheral spectrum of T is a subset of the set $\{1\}$. We note that J. Esterle had proved the result when the spectrum is $\{1\}$ (see [5]). This theorem plays a key role in operator theory and operator ergodic theory. A systematic and comprehensive study emphasizing its connection with stability theory can be found in [3] and [4].

Taking the Esterle-Katznelson-Tzafriri theorem as a starting point, it seems natural to ask whether the power boundedness requirement of this theorem can be relaxed. First, Allan [1] asked whether the condition $n^{-1}\|T^n\| \to 0$, when $n \to \infty$, (with the above spectral property) might lead one to directly infer this theorem. One can easily see the necessity of this condition. A negative answer and many interesting examples related to this problem were provided by Tomilov and Zemánek in their paper [11] using an operator matrix construction. However, they left open the following problem: if T is a Cesàro bounded operator in a Hilbert space and its spectrum is just

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the set $\{1\}$, does it follow that $\lim\limits_{n\to\infty} \|T^n - T^{n+1}\| = 0$? We note that this question originally goes back to Zemánek's paper on the Gelfand-Hille theorems [13]. Here, applying the matrix construction outlined in [11], we will prove that the answer to Zemánek's question is also negative. In our counterexample, convergence holds in the strong operator topology instead of the norm topology, which can be proved by exploiting D. Tsedenbayar's earlier result [12] on the classical Volterra operator.

2. Preliminary results

Let X be a complex Banach space. We say that a bounded, linear operator T on X is Cesàro bounded if

$$\sup_{n\geqslant 1}\left\|\frac{1}{n}\sum_{k=1}^{n} T^k\right\| < \infty.$$

We can construct Cesàro bounded operators from power-bounded operators by building on the following matrix construction that was studied in [11]. Let us choose a power-bounded operator T on X, i.e. $\sup\limits_{n\geqslant 1}\|T^n\| < \infty$, and let us consider the operator

$$\mathcal{T} := \begin{pmatrix} T & T-I \\ O & T \end{pmatrix}$$

on $X \oplus X$. The powers of $\mathcal{T}$ can be derived by a simple computation:

$$\mathcal{T}^n = \begin{pmatrix} T^n & nT^{n-1}(T-I) \\ O & T^n \end{pmatrix}.$$

In the following lemma we summarize some of the basic properties of the $\mathcal{T}$ operator, which were described in [11].

Lemma 2.1. *The operator $\mathcal{T}$ has the following properties:*

(i) $\sigma(\mathcal{T}) = \sigma(T)$;

(ii) $\lim\limits_{n\to\infty} n^{-1}\|\mathcal{T}^n\| = 0$ *if and only if* $\lim\limits_{n\to\infty} \|T^{n+1} - T^n\| = 0$;

(iii) $\mathcal{T}$ *is Cesàro bounded if and only if* T *is power-bounded;*

(iv) *for a fixed* $m \in \mathbb{N}$, *we have that* $\lim\limits_{n\to\infty} \|\mathcal{T}^n(\mathcal{T} - I)^m\| = 0$ *if and only if* $\lim\limits_{n\to\infty} \|T^n(T-I)^m\| = 0$ *and* $\lim\limits_{n\to\infty} n\|T^n(T-I)^{m+1}\| = 0$.

Next, let V denote the classical Volterra operator on $L^2(0,1)$; that is,

$$Vf(x) := \int_0^x f(s)\,\mathrm{d}s \quad \text{if } 0 < x < 1 \text{ and } f \in L^2(0,1).$$

Recall that V is compact and $\sigma(V) = \{0\}$. Let us now define an operator

$$\mathcal{T} := \begin{pmatrix} I - V & -V \\ O & I - V \end{pmatrix}.$$

Our main result here is the following theorem which addresses the question presented in [11, Problem 3] of whether it always holds that a Cesàro bounded Hilbert space operator T with a single spectrum $\{1\}$ satisfies $\|T^{n+1} - T^n\| \to 0$ $(n \to \infty)$. Our counterexample is a further study of the examples from [11].

Theorem 2.2. *The operator $\mathcal{T}$ is a Cesàro bounded operator with $\|\mathcal{T}^n\|/n \to 0$, $n \to \infty$, on $L^2(0,1) \oplus L^2(0,1)$ and with $\sigma(\mathcal{T}) = \{1\}$, but $\lim_{n\to\infty} \|\mathcal{T}^{n+1} - \mathcal{T}^n\| = 0$ does not hold.*

To prove this theorem, we need an estimate of values of certain Laguerre polynomials and a preliminary lemma that we will present here.

First, recall the definition of the nth generalized Laguerre polynomials with α parameter $(\alpha > -1)$:

$$L_n^{\alpha}(x) = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{x^k}{\Gamma(k+\alpha+1)}.$$

The classical formula presented below is the so-called Fejér's formula, which tells us what the asymptotic behaviour of the Laguerre polynomials is (for a proof, see [10]).

Theorem 2.3. *For $L_n^{(\alpha)}(x)$ with a real parameter $\alpha > -1$ and $\varepsilon > 0$, we have*

$$L_n^{(\alpha)}(x) = \pi^{-1/2} \mathrm{e}^{x/2} x^{-\alpha/2-1/4} n^{\alpha/2-1/4} \cos(2\sqrt{nx} - \alpha\pi/2 - \pi/4) + O(n^{\alpha/2-3/4})$$

if $x > 0$, where the bound for the remainder holds uniformly in $[\varepsilon, 1]$.

The next lemma is a technical one that we will require in order to give an asymptotic form of the terms appearing in our proof.

Lemma 2.4. *Let β be a real number, and $-\infty \leqslant a < b \leqslant \infty$. If $\{a_n\}$ is a sequence of positive numbers tending to ∞ as $n \to \infty$ and $f \in L^1[a,b]$, then*

$$\lim_{n\to\infty} \int_a^b f(x) \cos^2(a_n x + \beta)\,\mathrm{d}x = \left(\int_a^b f(x)\,\mathrm{d}x \right) \left(\frac{1}{\pi} \int_0^{\pi} \sin^2(x)\,\mathrm{d}x \right).$$

We should mention here that this lemma can be proved in a much more general context, and we refer the reader to [8] for its proof.

3. The counterexample

First, it is a well-known fact that $I-V$ is a power-bounded operator on $L^2(0,1)$ because it is similar to $(I+V)^{-1}$, which was noted by T. V. Pedersen and reported by Allan [2]. Actually, the resolvent $(I+V)^{-1}$ has norm 1 (see [6]). Lemma 2.1 now readily implies that

$$\mathcal{T}=\begin{pmatrix} I-V & -V \\ O & I-V \end{pmatrix}$$

is Cesàro bounded with a single spectrum $\{1\}$. A comprehensive study on the powers of the operator $I-V$ in general L^p spaces can be found in [8].

To demonstrate that $\|\mathcal{T}^{n+1}-\mathcal{T}^n\|$ does not converge to 0, it is sufficient to show by Lemma 2.1 (iv) that the following proposition holds. At the same time, it is worth mentioning here that this result answers another question from the paper [11, Problem 1].

Proposition 3.1. *For the Volterra operator V on $L^2(0,1)$, we have that*

$$\liminf_{n\to\infty}\|n(I-V)^nV^2\|>0.$$

Proof. Let us define the function series

$$f_n(x):=nx\chi_{(0,1/\sqrt{n})}(x),\quad 0<x<1,\ n=1,2,\ldots,$$

where $\chi_{(0,1/\sqrt{n})}$ denotes the characteristic function of the interval $(0,1/\sqrt{n})$. We shall check that $n\|(I-V)^nV^2f_n\|_2\|f_n\|_2^{-1}$ does not converge to 0 $(n\to\infty)$, which will prove the proposition.

First, we can readily see that

$$Vf_n(x)=\int_0^x f_n(s)\,\mathrm{d}s=\begin{cases}\frac{1}{2}nx^2 & \text{if } 0<x\leqslant 1/\sqrt{n},\\ \frac{1}{2} & \text{if } 1/\sqrt{n}\leqslant x<1,\end{cases}$$

and we recall that the well-known formula holds if $f\in L^2(0,1)$:

$$V^nf(x)=\int_0^x\frac{(x-s)^{n-1}}{(n-1)!}f(s)\,\mathrm{d}s\qquad\text{for } n\in\mathbb{N}.$$

Evidently, $\|f_n\|_2=n^{1/4}/\sqrt{3}$. Then a simple calculation gives

$$\begin{aligned}&n(I-V)^nV^2n^{-1/4}f_n(x)\\&=n^{3/4}\sum_{k=0}^n\binom{n}{k}(-1)^kV^{k+2}f_n(x)=n^{3/4}\sum_{k=0}^n\binom{n}{k}(-1)^k\int_0^x\frac{(x-s)^k}{k!}Vf_n(s)\,\mathrm{d}s\\&=n^{3/4}\int_0^x L_n^{(0)}(x-s)Vf_n(s)\,\mathrm{d}s,\end{aligned}$$

using the definition of the nth Laguerre polynomial with $\alpha=0$.

Next, pick an $\varepsilon \in (0, 1/5)$. Then, if $x > 1/\sqrt{n} + \varepsilon$, we may compute the integral

$$\begin{aligned}
&\int_0^x L_n^{(0)}(x-s) V f_n(s)\,\mathrm{d}s \\
&\qquad = \frac{1}{2}\int_0^{1/\sqrt{n}} L_n^{(0)}(x-s) n s^2\,\mathrm{d}s + \frac{1}{2}\int_{1/\sqrt{n}}^x L_n^{(0)}(x-s)\,\mathrm{d}s.
\end{aligned}$$

First, we can calculate the second term of the sum. In fact,

$$\begin{aligned}
\int_{1/\sqrt{n}}^x L_n^{(0)}(x-s)\,\mathrm{d}s &= \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{1}{k!} \int_{1/\sqrt{n}}^x (x-s)^k\,\mathrm{d}s \\
&= \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{1}{k!} \frac{1}{k+1} (x - 1/\sqrt{n})^{k+1} \\
&= \frac{1}{n+1} (x - 1/\sqrt{n}) L_n^{(1)}(x - 1/\sqrt{n}).
\end{aligned} \tag{1}$$

We now turn to the first integral in the above sum. Integrating twice by parts, we get

$$\begin{aligned}
\int_0^{1/\sqrt{n}} (x-s)^k s^2\,\mathrm{d}s = &-\frac{(x-1/\sqrt{n})^{k+1}}{n(k+1)} - \frac{2(x-1/\sqrt{n})^{k+2}}{\sqrt{n}(k+1)(k+2)} \\
&- \frac{2(x-1/\sqrt{n})^{k+3}}{(k+1)(k+2)(k+3)} + \frac{2x^{k+3}}{(k+1)(k+2)(k+3)}.
\end{aligned}$$

From the definition of the generalized Laguerre polynomials, it is a straightforward matter to show that

$$\begin{aligned}
\int_0^{1/\sqrt{n}} L_n^{(0)}(x-s) n s^2\,\mathrm{d}s &= \sum_{k=0}^n n(-1)^k \binom{n}{k} \frac{1}{k!} \int_0^{1/\sqrt{n}} (x-s)^k s^2\,\mathrm{d}s \\
&= -\frac{(x-1/\sqrt{n})}{n+1} L_n^{(1)}(x - 1/\sqrt{n}) \\
&\quad - \frac{2\sqrt{n}(x-1/\sqrt{n})^2}{(n+1)(n+2)} L_n^{(2)}(x - 1/\sqrt{n}) \\
&\quad - \frac{2n(x-1/\sqrt{n})^3}{(n+1)(n+2)(n+3)} L_n^{(3)}(x - 1/\sqrt{n}) \\
&\quad + \frac{2nx^3}{(n+1)(n+2)(n+3)} L_n^{(3)}(x).
\end{aligned} \tag{2}$$

Adding (1) and (2) together, we find that

$$\begin{aligned}\int_0^x L_n^{(0)}(x-s)Vf_n(s)\,\mathrm{d}s = &-\frac{\sqrt{n}(x-1/\sqrt{n})^2}{(n+1)(n+2)}L_n^{(2)}(x-1/\sqrt{n})\\ &-\frac{n(x-1/\sqrt{n})^3}{(n+1)(n+2)(n+3)}L_n^{(3)}(x-1/\sqrt{n})\\ &+\frac{nx^3}{(n+1)(n+2)(n+3)}L_n^{(3)}(x).\end{aligned}$$

The application of the reverse triangle inequality gives for every n large enough that

$$\begin{aligned}&n^{3/4}\|(I-V)^n V^2 f_n\|_2\\ &\quad\geqslant \left(\int_{\varepsilon+1/\sqrt{n}}^{1/4}\left(n^{3/4}\int_0^x L_n^{(0)}(x-s)Vf_n(s)\,\mathrm{d}s\right)^2\mathrm{d}x\right)^{1/2}\\ &\quad\geqslant \left(\int_{\varepsilon+1/\sqrt{n}}^{1/4}\left(\frac{n^{3/4}\sqrt{n}(x-1/\sqrt{n})^2}{(n+1)(n+2)}L_n^{(2)}(x-1/\sqrt{n})\right)^2\mathrm{d}x\right)^{1/2}\\ &\qquad-\left(\int_{\varepsilon+1/\sqrt{n}}^{1/4}\left(\frac{n^{3/4}n(x-1/\sqrt{n})^3}{(n+1)(n+2)(n+3)}L_n^{(3)}(x-1/\sqrt{n})\right)^2\mathrm{d}x\right)^{1/2}\\ &\qquad-\left(\int_{\varepsilon+1/\sqrt{n}}^{1/4}\left(\frac{n^{3/4}nx^3}{(n+1)(n+2)(n+3)}L_n^{(3)}(x)\right)^2\mathrm{d}x\right)^{1/2}.\end{aligned}$$

Let us apply Fejér's asymptotic formula (see Theorem 2.3) for $L_n^{(2)}(x)$. Then the first term above is asymptotically equal to

$$\frac{1}{\sqrt{\pi}}\left(\int_{\varepsilon}^{1/4-1/\sqrt{n}}\mathrm{e}^x x\sqrt{x}\cos^2(2\sqrt{nx}-5\pi/4)\,\mathrm{d}x\right)^{1/2}.$$

Performing the change of variable $y=\sqrt{x}$ and applying Lemma 2.4, we find that the integral tends to

$$\sqrt{\frac{2M}{\pi}}\left(\int_{\sqrt{\varepsilon}}^{1/2}y^4\mathrm{e}^{y^2}\,\mathrm{d}y\right)^{1/2},$$

when $n\to\infty$, where $M:=\pi^{-1}\int_0^{\pi}\sin^2 y\,\mathrm{d}y=(2\pi)^{-1}$.

We can apply a similar argument to $L_n^{(3)}(x)$ to show that the second and third terms above are asymptotically equal to

$$\frac{1}{\sqrt{\pi}}\left(\int_{\varepsilon}^{1/4-1/\sqrt{n}}\mathrm{e}^x x^2\sqrt{x}\cos^2(2\sqrt{nx}-7\pi/4)\,\mathrm{d}x\right)^{1/2}$$

and

$$\frac{1}{\sqrt{\pi}}\left(\int_{\varepsilon+1/\sqrt{n}}^{1/4} \mathrm{e}^{x} x^{2} \sqrt{x} \cos^{2}(2\sqrt{nx}-7\pi/4)\,\mathrm{d}x\right)^{1/2},$$

respectively. Making a change of variable ($y=\sqrt{x}$) and applying Lemma 2.4 once again, we see that these integrals converge to

$$\sqrt{\frac{2M}{\pi}}\left(\int_{\sqrt{\varepsilon}}^{1/2} y^{6}\mathrm{e}^{y^{2}}\,\mathrm{d}y\right)^{1/2}$$

when $n\to\infty$. Since $y^{4}\mathrm{e}^{y^{2}}-4y^{6}\mathrm{e}^{y^{2}}>0$ on the interval $(\sqrt{\varepsilon},1/2)$, the above sum appearing in the reverse triangle inequality is strictly positive; i.e., for n is large enough we have

$$n^{3/4}\|(I-V)^{n}V^{2}f_{n}\|\geqslant\sqrt{\frac{2M}{\pi}}\left[\left(\int_{\sqrt{\varepsilon}}^{1/2} y^{4}\mathrm{e}^{y^{2}}\,\mathrm{d}y\right)^{1/2}-\left(\int_{\sqrt{\varepsilon}}^{1/2} 4y^{6}\mathrm{e}^{y^{2}}\,\mathrm{d}y\right)^{1/2}\right]>0.$$

Thus there exists a positive constant C such that

$$\liminf_{n}\frac{n\|(I-V)^{n}V^{2}f_{n}\|_{2}}{\|f_{n}\|_{2}}>C,$$

which is what we intended to show. □

Proof of Theorem 2.2. Now the proof of the theorem follows immediately from Lemma 2.1 and Proposition 3.1. □

Remark 3.2. During the preparation of this paper Prof. Yuri Tomilov informed me that Stephen Montgomery-Smith had an unpublished note on the above problem. Using Fourier methods, he sketched a proof that $\liminf_{t>0} t\|(\mathrm{e}^{-V}-I)^{2}\mathrm{e}^{-tV}\|$ is positive. This means that the operator

$$\begin{pmatrix}\mathrm{e}^{-V} & \mathrm{e}^{-V}-I\\ O & \mathrm{e}^{-V}\end{pmatrix}$$

must be a good candidate for another counterexample to the question outlined in Introduction. I am grateful to Prof. Yuri Tomilov for bringing this information to my attention.

We have already seen that the operators $\mathcal{T}^{n+1}-\mathcal{T}^{n}$ do not converge to zero in the uniform operator norm. However, we can easily verify that the analogous strong statement is true.

We should mention here that the first sharper asymptotic estimates on certain Volterra operator pencils were given by D. Tsedenbayar [12]. These estimates led to improvements on an earlier result of T. Pytlik [9] (see also [8]). D. Tsedenbayar proved for the Volterra operator V in $L^2(0,1)$ that

$$\|(I-V)^{n+1}-(I-V)^n\| = O(n^{-1/2}).$$

Then one can easily verify that there exists a constant $C>0$ such that

$$\sup_{n\geqslant 0}\|\sqrt{n}(I-V)^n V\| \leqslant C$$

holds. Now, applying Tsedenbayar's result, we can exploit his theorem which proves the strong stability of the above sequence. Hence the following statement comes immediately.

Proposition 3.3. *For every $f\in L^2(0,1)$, we have*

$$\lim_{n\to\infty}\sqrt{n}(I-V)^n V f = 0.$$

Proof. Let us choose an $f\in L^2(0,1)$. Since the sequence $\sqrt{n}(I-V)^n V$ ($n=1,2,\dots$) is bounded in the operator norm and V is compact, we can select a subsequence $(n_i)_i$ and a $g\in L^2(0,1)$ such that

$$V\left(\sqrt{n_i}(I-V)^{n_i}Vf\right) = \sqrt{n_i}(I-V)^{n_i}V^2 f \to g \quad \text{if } \ i\to\infty.$$

Let us assume that $g\neq 0$. Pick an $0<\varepsilon<\frac{1}{4}\|g\|_2$. By the Esterle-Katznelson-Tzafriri theorem, we have the uniform convergence $\lim\limits_{n\to\infty}V(I-V)^n=0$. Then we can choose an index m such that for every $i\geqslant m$ both

$$\|V(I-V)^i f\|_2 < \frac{\varepsilon}{C} \quad \text{and} \quad \|\sqrt{n_i}(I-V)^{n_i}V^2 f - g\|_2 < \varepsilon$$

hold. Pick an arbitrary $i\geqslant m$. Applying the reverse triangle inequality again, we may infer that for any sufficiently large $k\geqslant i$

$$\begin{aligned}
&\|\sqrt{n_i}(I-V)^{n_i}V^2 f - \sqrt{n_i+k}(I-V)^{n_i+k}V^2 f\|_2 \\
&\qquad \geqslant \big|\|\sqrt{n_i}(I-V)^{n_i}V^2 f - \sqrt{n_i}(I-V)^{n_i+k}V^2 f\|_2 \\
&\qquad\quad - \|\sqrt{n_i}(I-V)^{n_i+k}V^2 f - \sqrt{n_i+k}(I-V)^{n_i+k}V^2 f\|_2\big| \\
&\qquad \geqslant \|\sqrt{n_i}(I-V)^{n_i}V^2 f\|_2 - \|\sqrt{n_i}(I-V)^{n_i}V^2(I-V)^k f\|_2 \\
&\qquad\quad - \frac{k}{\sqrt{n_i}+\sqrt{n_i+k}}\|(I-V)^k V^2 (I-V)^{n_i} f\|_2 \\
&\qquad \geqslant \|g\|_2 - \varepsilon - C\frac{\varepsilon}{C} - C\frac{\varepsilon}{C} \\
&\qquad \geqslant \frac{1}{4}\|g\|_2.
\end{aligned}$$

But this inequality contradicts the convergence of the sequence $\{\sqrt{n_i}(I-V)^{n_i}V^2f\}_i$. This means that $g=0$ must hold; that is,

$$\lim_{n\to\infty} \sqrt{n}(I-V)^nV^2f = 0$$

follows. Since the range of V is dense in $L^2(0,1)$ and the operators $\sqrt{n}(I-V)^nV$ are uniformly bounded, the statement readily follows. $\square$

An immediate corollary of this stability proposition is that the operator sequence $(\sqrt{n}(I-V)^nV)^2$ $(n=1,2,\dots)$ is also strongly stable but not uniformly (see Proposition 3.1).

Corollary 3.4. *For every $f \in L^2(0,1)$, we have*

$$\lim_{n\to\infty} n(I-V)^nV^2f = 0.$$

Combining this corollary with the above propositions, we obtain the following result.

Corollary 3.5. *For the operator $\mathcal{T}$ on $L^2(0,1)\oplus L^2(0,1)$, we have that $\mathcal{T}^{n+1}-\mathcal{T}^n$ tends to zero strongly and $\lim\limits_{n\to\infty} \|\mathcal{T}^n(\mathcal{T}-I)^2\| = 0$.*

P r o o f. The Esterle-Katznelson-Tzafriri theorem implies that $\lim\limits_{n\to\infty}(I-V)^nV = 0$. From Lemma 2.1 and the previous corollary we readily obtain the strong convergence result.

To prove the second statement, we recall that V is compact and hence from Corollary 3.4 we get that $\lim\limits_{n\to\infty} n\|(I-V)^nV^3\| = 0$. Now an application of Lemma 2.1 gives the statement. $\square$

Question 1. Let T be a Cesàro bounded operator on a Hilbert space $\mathcal{H}$ such that $\sigma(T)=\{1\}$. Does it follow that $(T^{n+1}-T^n)h \to 0$ (if $n\to\infty$) holds for every $h \in \mathcal{H}$?

Example 4.1 in [11] shows that the spectral assumption is essential here. We note that open questions concerning local versions of the Esterle-Katznelson-Tzafriri theorem for Cesàro bounded operators can be also found in [13, p. 378].

Finally we mention another question which is related to the Esterle-Katznelson-Tzafriri theorem.

Question 2. Let T be a positive Cesàro bounded operator on a Hilbert space $\mathcal{H}$ such that $\sigma(T)=\{1\}$. Does it follow that $\|T^{n+1}-T^n\| \to 0$ if $n\to\infty$?

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