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Czechoslovak Mathematical Journal, Vol. 60 (2010), No. 4, 1079–1089

Persistent URL: `http://dml.cz/dmlcz/140807`

Czechoslovak Mathematical Journal, 60 (135) (2010), 1079–1089

ALGEBRAIC CONDITIONS FOR t-TOUGH GRAPHS

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(Received July 1, 2009)

Abstract. We give some algebraic conditions for t-tough graphs in terms of the Laplacian eigenvalues and adjacency eigenvalues of graphs.

Keywords: t-tough graph, Laplacian matrix, adjacent matrix, eigenvalues

MSC 2010: 05C50, 05C75, 15A18

1. Introduction

Let G be an undirected simple graph with vertices $v_1, v_2, \ldots, v_n$. The *adjacency matrix* $A = A(G) = (a_{ij})$ *of* G is the $n \times n$ symmetric matrix of 0's and 1's with $a_{ij} = 1$ if and only if v_i and v_j are joined by an edge of G. The eigenvalues of $A(G)$ are ordered as $\lambda_1(G) \geqslant \lambda_2(G) \geqslant \ldots \geqslant \lambda_n(G)$. Suppose the valence or degree of vertex v_i equals d_i for $i = 1, 2, \ldots, n$, and let $D = D(G)$ be the diagonal matrix whose (i,i)-entry is d_i. The matrix $L(G) = D(G) - A(G)$ is called the *Laplacian matrix of* G. The matrix $L(G)$ is positive semi-definite with row sum 0. Its eigenvalues are denoted by $0 = \mu_1 \leqslant \mu_2 \leqslant \ldots \leqslant \mu_n$. The eigenvalue μ_2 is often called the algebraic connectivity. $\mu_2 = 0$ if and only if the graph is disconnected.

A *vertex cut* of G is a subset V' of the vertex set $V(G)$ such that $G - V'$ is disconnected. G is a *t-tough* graph (where $t > 0$ is a real number) if, for every vertex cut S, the number of components of the graph $G - S$, denoted by $C(G - S)$, is at most $|S|/t$, that is, $C(G - S) \leqslant |S|/t$.

A *Hamiltonian circuit* in G is a circuit which contains every vertex of G. A graph which contains a Hamiltonian circuit is called a *Hamiltonian graph.* A *k-factor* of G

This work is supported by NNSF of China (No. 10771080) and SRFDP of China (No. 20070574006).

is a k-regular spanning subgraph of G. Therefore, G is Hamiltonian if and only if G has a connected 2-factor.

In this paper we concentrate on the case $0 < t \leqslant 2$.

The following simple but important theorem is due to V. Chvátal (see [3]).

Theorem 1.1 ([3]). *Let G be a Hamiltonian graph, and let S be any non-empty proper subset of the vertex-set $V(G)$. Then*

$$C(G - S) \leqslant |S|.$$

It follows from Theorem 1.1 that every Hamiltonian graph is 1-tough.

We can obtain the following lemma from Theorem 1.1.

Lemma 1.2. *Let G be a simple graph. If there exists a non-empty subset of the vertex-set $V(G)$ such that $C(G - S) > |S|$, then G is not a Hamiltonian graph.*

In [7], Jung proved the Chvátal conjecture (see [4]) as follows.

Theorem 1.3 ([7]). *If G is 1-tough, then either G is Hamiltonian, or its complement $\overline{G}$ contains the graph G_1 shown in Fig. 1 as a subgraph.*

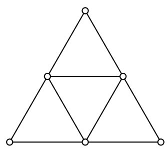

Fig. 1. G_1

Combining Theorems 1.1 and 1.3, we obtain

Lemma 1.4. *Let G be a simple graph whose complement $\overline{G}$ does not contain G_1 as a subgraph. Then G is a Hamiltonian graph if and only if G is 1-tough.*

It is known that 2-toughness is also related to some properties of graph theory. The following theorem is one of them.

Theorem 1.5 ([5]). *All 2-tough graphs have a 2-factor.*

In this paper, we will consider the existence of t-tough graphs in terms of eigenvalues of the Laplacian matrix $L(G)$ and the adjacency matrix $A(G)$. The first theorem in this direction was given by Mohar (see Theorem 3.3 in [8]), but the condition in [8] only holds for regular graphs and also involves some rather complicated considerations. Later J. Vanden Heuvel ([9]) gave some results concerning a necessary condition for Hamiltonian graphs in terms of eigenvalues of $L(G)$ and $Q(G) = D(G) + A(G)$, while A. E. Brouwer derived lower bounds for toughness of a graph in terms of its eigenvalues (see Theorem 0.1 in [1]). Up to now there exist no more results that would show a relationship between t-tough graphs and eigenvalues of certain matrices associated with the graphs. In the sequel, we will give some conditions which are simpler than the conditions in [8] for a graph to be t-tough in terms of eigenvalues of $L(G)$ and $A(G)$.

2. Character of t-toughness in terms of eigenvalues of $L(G)$

To begin with, we want to obtain an algebraic condition for 2-tough graphs. In order to do that, now we establish some lemmas.

An inequality for disconnected vertex sets in a graph will be used, which is due to Haemers (see [6]). Two disjoint vertex sets A and B in a graph are disconnected if there are no edges between A and B.

Lemma 2.2 ([6]). *If A and B are disconnected vertex sets of a graph with n vertices and Laplacian eigenvalues $0 = \mu_1 \leqslant \mu_2 \leqslant \ldots \leqslant \mu_n$, then*

$$\frac{|A| \cdot |B|}{(n - |A|)(n - |B|)} \leqslant \Big(\frac{\mu_n - \mu_2}{\mu_n + \mu_2}\Big)^2.$$

Moreover, we have the following elementary lemmas.

Lemma 2.2 ([2]). *Let $x_1, x_2, \ldots, x_q$ be q positive integers such that $\sum\limits_{i=1}^{q} x_i = k \leqslant 2q - 1$. Then for every integer l satisfying $0 \leqslant l \leqslant k$ there exists a subset $I \subset \{1, 2, \ldots, q\}$ such that $\sum\limits_{i \in I} x_i = l$.*

Lemma 2.3. *The function $f(x) = (x - s - \frac{1}{2})/(x + s - \frac{1}{2})$ is an increasing function of x for $x > 0$, provided $s \geqslant 1$.*

Proof. Differentiating $f(x)$ we obtain $f'(x) = 2s/(x + s - 1)^2 > 0$. $\square$

Similarly, we have

Lemma 2.4. *$p(x) = x/(ns + x)$ is an increasing function of x for $x > 0$, provided $ns > 0$.*

Lemma 2.5. *$g(x) = 4sx/((6s-2)x - 3s^2 + 4s - 1)$ is an decreasing function of x for $x > 0$, provided $s \geqslant 1$.*

Now we shall prove our main results.

Theorem 2.6. *Let G be a simple graph with n vertices and Laplacian eigenvalues $0 = \mu_1 \leqslant \mu_2 \leqslant \ldots \leqslant \mu_n$. If $\mu_2 \geqslant \frac{2}{3}\mu_n$, then G is 2-tough.*

Proof. Assume that G is not 2-tough. By the definition of 2-toughness, there exists a vertex cut S of G such that

$$C(G-S) > \frac{|S|}{2}.$$

Denote $|S|$ by s. Then $G-S$ has $q \geqslant \lfloor \frac{1}{2}s \rfloor + 1$ components. Let x_i be the cardinality of the ith component, where $i = 1, 2, \ldots, q$.

We consider the following two cases.

Case 1: $n \leqslant 2\lfloor \frac{1}{2}s \rfloor + s + 1$. Then $\sum\limits_{i=1}^{q} x_i = n - s \leqslant 2\lfloor \frac{1}{2}s \rfloor + 1 \leqslant 2(q-1) + 1 = 2q - 1$. By Lemma 2.2, G has a pair of disconnected vertex sets A and B with $|A| = \lfloor \frac{1}{2}(n-s) \rfloor$ and $|B| = \lceil \frac{1}{2}(n-s) \rceil$. From Lemma 2.1 we have

$$\left(\frac{\mu_n - \mu_2}{\mu_n + \mu_2}\right)^2 \geqslant \frac{|A| \cdot |B|}{ns + |A| \cdot |B|}. \tag{2.1}$$

Thus

$$\left(\frac{\mu_n - \mu_2}{\mu_n + \mu_2}\right)^2 \geqslant \frac{(n-s)^2}{(n+s)^2} \quad \text{if } n-s \text{ is even};$$
$$\left(\frac{\mu_n - \mu_2}{\mu_n + \mu_2}\right)^2 \geqslant \frac{(n-s)^2 - 1}{(n+s)^2 - 1} \quad \text{if } n-s \text{ is odd}.$$

That is to say,

$$\left(\frac{\mu_n - \mu_2}{\mu_n + \mu_2}\right)^2 \geqslant \frac{(n-s)^2 - 1}{(n+s)^2 - 1}.$$

Observing that $\lfloor \frac{1}{2}s \rfloor + s + 1 \leqslant n \leqslant 2\lfloor \frac{1}{2}s \rfloor + s + 1$ and S is a vertex cut of G, we have $s \geqslant 2$. Next, we prove the following inequality:

$$\frac{(n-s)^2 - 1}{(n+s)^2 - 1} > \left(\frac{n - s - \frac{1}{2}}{n + s - \frac{1}{2}}\right)^2.$$

Since

$$\begin{aligned}
&\frac{(n-s)^2-1}{(n+s)^2-1}-\Big(\frac{n-s-\frac{1}{2}}{n+s-\frac{1}{2}}\Big)^2\\
&\qquad=\frac{[(n-s)^2-1](n+s-\frac{1}{2})^2-(n-s-\frac{1}{2})^2[(n+s)^2-1]}{[(n+s)^2-1](n+s-\frac{1}{2})^2}\\
&\qquad=\frac{s(2n^2-5n+2-2s^2)}{[(n+s)^2-1](n+s-\frac{1}{2})^2}\\
&\qquad\geqslant\frac{s[2(\lfloor\frac{1}{2}s\rfloor+s+1)^2-5(\lfloor\frac{1}{2}s\rfloor+s+1)+2-2s^2]}{[(n+s)^2-1](n+s-\frac{1}{2})^2}\\
&\qquad\geqslant\frac{s[2(\frac{1}{2}(s-1)+s+1)^2-5(\frac{1}{2}(s-1)+s+1)+2-2s^2]}{[(n+s)^2-1](n+s-\frac{1}{2})^2}\\
&\qquad\geqslant\frac{s(5s^2-9s)}{[(n+s)^2-1](n+s-\frac{1}{2})^2}>0,
\end{aligned}$$

we have

$$\Big(\frac{\mu_n-\mu_2}{\mu_n+\mu_2}\Big)^2\geqslant\frac{(n-s)^2-1}{(n+s)^2-1}>\Big(\frac{n-s-\frac{1}{2}}{n+s-\frac{1}{2}}\Big)^2.$$

Thus

$$\frac{\mu_n-\mu_2}{\mu_n+\mu_2}>\frac{n-s-\frac{1}{2}}{n+s-\frac{1}{2}}.$$

Since S is non-empty and $q\geqslant s+1$, $n\geqslant\lfloor\frac{1}{2}s\rfloor+s+1$, by Lemma 2.3 we have

$$\frac{\mu_n-\mu_2}{\mu_n+\mu_2}>\frac{\lfloor\frac{1}{2}s\rfloor+s+1-s-\frac{1}{2}}{\lfloor\frac{1}{2}s\rfloor+s+1+s-\frac{1}{2}}=\frac{\lfloor\frac{1}{2}s\rfloor+\frac{1}{2}}{\lfloor\frac{1}{2}s\rfloor+2s+\frac{1}{2}}\geqslant\frac{\frac{1}{2}(s-1)+\frac{1}{2}}{\frac{1}{2}(s-1)+2s+\frac{1}{2}}=\frac{s}{5s}=\frac{1}{5}.$$

Hence $\mu_2<\frac{2}{3}\mu_n$. This is contrary to the given condition.

Case 2: $n>2\lfloor\frac{1}{2}s\rfloor+s+1$. If $s\geqslant 2$, G has a pair of disconnected vertex sets A and B such that

$$|A|+|B|=\sum_{i=1}^{q}x_i=n-s,\ \min(|A|,|B|)\geqslant\Big\lfloor\frac{s}{2}\Big\rfloor.$$

Thus

$$|A|\cdot|B|\geqslant\Big\lfloor\frac{s}{2}\Big\rfloor\Big(n-s-\Big\lfloor\frac{s}{2}\Big\rfloor\Big).$$

By Inequality (2.1), Lemmas 2.4 and 2.5,

$$\begin{aligned}
\left(\frac{\mu_n-\mu_2}{\mu_n+\mu_2}\right)^2 &\geqslant \frac{|A|\cdot|B|}{ns+|A|\cdot|B|} \geqslant \frac{\lfloor\frac{1}{2}s\rfloor(n-s-\lfloor\frac{1}{2}s\rfloor)}{ns+\lfloor\frac{1}{2}s\rfloor(n-s-\lfloor\frac{1}{2}s\rfloor)}\\
&\geqslant \frac{\frac{1}{2}(s-1)(n-s-\frac{1}{2}(s-1))}{ns+\frac{1}{2}(s-1)(n-s-\frac{1}{2}(s-1))} = \frac{2ns-3s^2+4s-2n-1}{6ns-3s^2+4s-2n-1}\\
&= 1-\frac{4ns}{6ns-3s^2+4s-2n-1}\\
&> 1-\frac{4s\cdot 2s}{6s\cdot 2s-3s^2+4s-2\cdot 2s-1}\\
&= 1-\frac{8s^2}{9s^2-1} > 1-\frac{8s^2}{9s^2-\frac{1}{2}s^2} = \frac{1}{17} > \frac{1}{25}.
\end{aligned}$$

Thus

$$\frac{\mu_n-\mu_2}{\mu_n+\mu_2} > \frac{1}{5}.$$

If $s=1$, then G also has a pair of disconnected vertex sets A' and B' such that

$$|A'|+|B'| = n-1, \quad \min(|A'|,|B'|) \geqslant 1.$$

Then

$$|A'|\cdot|B'| \geqslant 1\cdot(n-1-1) = n-2.$$

By Inequality (2.1) and Lemma 2.4,

$$\left(\frac{\mu_n-\mu_2}{\mu_n+\mu_2}\right)^2 \geqslant \frac{|A'|\cdot|B'|}{n+|A'|\cdot|B'|} \geqslant \frac{n-2}{n+n-2} = \frac{1}{2}-\frac{1}{2(n-1)} \geqslant \frac{1}{4} > \frac{1}{25}.$$

So we have

$$\frac{\mu_n-\mu_2}{\mu_n+\mu_2} > \frac{1}{5}.$$

Hence $\mu_2 < \frac{2}{3}\mu_n$, which is also contrary to the given condition.

Consequently, we have proved that G is 2-tough. □

Example 1. Consider the graph G_2 of order 6 in Fig. 2. G_2 has Laplacian eigenvalues $\mu_2 = 4$ and $\mu_6 = 6$. Note that $\mu_2 \geqslant \frac{2}{3}\mu_6$. By Theorem 2.6, G_2 is 2-tough.

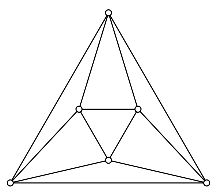

Fig. 2. G_2

Similarly, we can prove the following results.

Theorem 2.7. *Let G be a simple graph with n vertices and Laplacian eigenvalues $0 = \mu_1 \leqslant \mu_2 \leqslant \ldots \leqslant \mu_n$. For $1/k > 0$, $k \in \mathbb{Z}^+$, if $\mu_2 \geqslant k/(k+1)\mu_n$, then G is $1/k$-tough.*

Corollary 2.8. *Let G be a simple graph with n vertices and Laplacian eigenvalues $0 = \mu_1 \leqslant \mu_2 \leqslant \ldots \leqslant \mu_n$. If $\mu_2 \geqslant \frac{1}{2}\mu_n$, then G is 1-tough.*

Example 2. The complete bipartite graph $K_{n,n}$ has Laplacian eigenvalues $\mu_2 = n$ and $\mu_n = 2n$. This implies that $2\mu_2 \geqslant \mu_n$. By Corollary 2.8, $K_{n,n}$ is 1-tough.

3. Character of 1-toughness in terms of eigenvalues of $A(G)$

In this section, we continue to investigate the condition of 1-toughness. For regular graphs, the conditions obtained in the previous section are improved. First of all, we establish the following lemmas.

Lemma 3.1 ([7]). *The largest adjacency eigenvalue of a graph is bounded from below by the average degree with equality if and only if the graph is regular.*

Theorem 3.2. *A connected k-regular graph on n vertices with adjacency eigenvalues $k = \lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n$ which satisfies*

$$\lambda_2 \leqslant \begin{cases} k - 1 + \dfrac{3}{k+1}, & k \text{ even;} \\[2ex] k - 1 + \dfrac{2}{k+1}, & k \text{ odd;} \end{cases}$$

is 1-tough.

Proof. Let $G = (V, E)$ be a connected k-regular graph with $|V| = n$ and not 1-tough. By the definition of 1-toughness, there exists a non-empty proper subset S of $V(G)$ such that

$$C(G - S) > |S|.$$

Denote $|S|$ by s. Then $G - S$ has $q \geqslant s + 1$ components $G_1, G_2, \ldots, G_q$. Let x_i be the cardinality of G_i, $i = 1, 2, \ldots, q$, and let t_i denote the number of edges in G between S and G_i. Since G is connected, $t_i \geqslant 1$. Then clearly

$$\sum_{i=1}^{q} t_i \leqslant ks, \quad s \geqslant 1. \tag{3.1}$$

Hence $t_i < k$ for at least two values of i, say $i = 1, 2$. If not, say $t_i \geqslant k$, $i = 1, 2, \ldots, q-1$, then

$$\sum_{i=1}^{q} t_i \geqslant (q-1)k \geqslant sk \quad (\text{since } q \geqslant s+1).$$

It implies $t_q \leqslant 0$ from Inequality (3.1). This is contrary to $t_i \geqslant 1$, $i = 1, 2, \ldots, q$. Moreover, since G is k-regular, we have $x_i > 1$, where $i = 1, 2$.

Let l_i denote the largest adjacency eigenvalue of G_i and assume $l_1 \geqslant l_2$. The eigenvalue interlacing (see for example in [6]) applied to the subgraph induced by $G_1 \cup G_2$ gives

$$l_i \leqslant \lambda_i \quad \text{for } i = 1, 2. \tag{3.2}$$

Consider G_2 with x_2 vertices and e_2 edges. Then $2e_2 = kx_2 - t_2 \leqslant x_2(x_2-1)$. Since $t_2 < k$ and $x_2 > 1$,

$$kx_2 - k < kx_2 - t_2 \leqslant x_2(x_2-1).$$

Hence

$$k < x_2. \tag{3.3}$$

Moreover, let the average degree of G_2 be $\bar{d_2}$. Then

$$\bar{d_2} = \frac{2e_2}{x_2} = \frac{kx_2 - t_2}{x_2} = k - \frac{t_2}{x_2}. \tag{3.4}$$

If k is even, then by $2e_2 = kx_2 - t_2$, t_2 must be even and hence $t_2 \leqslant k-2$. By (3.3) and (3.4), $\bar{d_2} \geqslant k - (k-2)/(k+1) = k - 1 + 3/(k+1)$.

If k is odd, then $\bar{d_2} \geqslant k - (k-1)/(k+1) = k - 1 + 2/(k+1)$.

Note that $t_2 < k < x_2$, hence G_2 cannot be regular. By Lemma 3.1 and (3.2), we have

$$\lambda_2 \geqslant l_2 > \bar{d_2}.$$

This completes the proof. □

From the above it is clear that $\lambda_2 \leqslant k-1$ implies 1-toughness of a k-regular graph. Noting that $\mu_2 = k - \lambda_2$, we can obtain the following corollary in terms of the Laplacian matrix.

Corollary 3.3. *A regular graph with algebraic connectivity at least* 1 *is* 1*-tough.*

In the proof of Theorem 3.2, we saw that $t_i < x_i$ for $i = 1, 2$. Hence there exist vertices u and v in G_1 and G_2 respectively which are not adjacent to a vertex of S. Therefore the distance between u and v is at least 4. Hence we have

Corollary 3.4. *A regular graph with diameter at most* 3 *is* 1*-tough.*

Remark 1. For regular graphs, the condition of Theorem 3.2 is better than that of Corollary 2.8. That is to say, for a connected k-regular graph G, if $\mu_2 \geqslant \frac{1}{2}\mu_n$, then $\lambda_2 \leqslant k - 1 + 2/(k+1)$.

P r o o f. Since G is a connected k-regular graph, we have

$$\mu_2 = k - \lambda_2, \quad \mu_n = k - \lambda_n.$$

Then

$$2(k - \lambda_2) \geqslant k - \lambda_n.$$

That is,

$$\lambda_2 \leqslant \frac{k}{2} + \frac{\lambda_n}{2}.$$

Noting that $\sum\limits_{i=1}^{n} \lambda_i = 0$, $\lambda_1 = k$, we can get $\lambda_n < 0$ immediately. Therefore,

$$\lambda_2 \leqslant \frac{k}{2} + \frac{\lambda_n}{2} < \frac{k}{2}.$$

On the other hand,

$$k - 1 + \frac{2}{k+1} = \frac{k}{2} + \frac{(k-2)(k+1)+4}{2(k+1)} = \frac{k}{2} + \frac{(k-\frac{1}{2})^2 + \frac{7}{4}}{2(k+1)} > \frac{k}{2} > \lambda_2.$$

Then $\lambda_2 < k - 1 + 2/(k+1)$. $\square$

Example 3. There exist 1-tough regular graphs that satisfy the condition of Theorem 3.2 but $\mu_2 < \frac{1}{2}\mu_n$. G_3 of order 6 in Fig. 3 is an example, whose $\lambda_2 = 1 \leqslant 2\frac{1}{2}$, but $2\mu_2 = 2 \times 2 < \mu_6 = 5$.

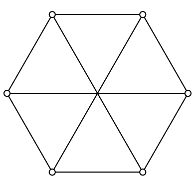

Fig. 3. G_3

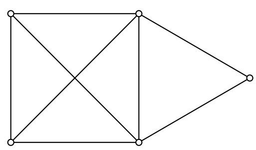

Fig. 4. G_4

Remark 2. There exist 1-tough graphs that do not satisfy the condition of Corollary 2.8. That is to say, the condition of Corollary 2.8 is not necessary. G_4 of order 5 in Fig. 4 is an example, whose $\mu_2 = 2$, $\mu_5 = 5$.

Remark 3. There also exist 1-tough connected regular graphs that do not satisfy the condition of Theorem 3.2. That is to say, the condition of Theorem 3.2 is not necessary. G_5 of order 10 in Fig. 5 is an example, whose $\lambda_2 = 2.56 \leqslant 2.5$.

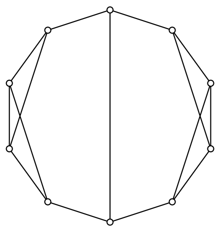

Fig. 5. G_5

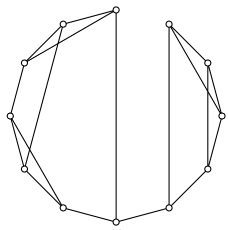

Fig. 6. G_6

Remark 4. There exist connected regular graphs with diameter 5 that are not 1-tough. G_6 of order 10 in Fig. 6 is an example. In fact, if $S = \{v\}$, then $C(G-S) = 2 > |S| = 1$.

Finaly, we pose the following question.

Question. What is the smallest diameter which regular but not 1-tough graphs are connected with?

Acknowledgements. The authors are grateful to the referees for their comments and suggestions which improved the presentation of the paper.

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