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A HYBRID MEAN VALUE OF THE DEDEKIND SUMS

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Abstract. The main purpose of this paper is to use the M. Toyozumi's important work, the properties of the Dedekind sums and the estimates for character sums to study a hybrid mean value of the Dedekind sums, and give a sharper asymptotic formula for it.

Keywords: Dedekind sums, Dirichlet L -function, mean value

MSC 2010: 11L40, 11F20, 11M20

1. INTRODUCTION

For a positive integer q and an arbitrary integer h , the Dedekind sum $S(h, q)$ is defined as follows:

$$S(h, q) = \sum_{a=1}^q \left(\left(\frac{a}{q} \right) \right) \left(\left(\frac{ha}{q} \right) \right),$$

where

$$((x)) = \begin{cases} x - [x] - \frac{1}{2} & \text{if } x \text{ is not an integer;} \\ 0 & \text{if } x \text{ is an integer.} \end{cases}$$

The various properties of $S(h, q)$ were investigated by many authors. Maybe the most famous property of the Dedekind sums is the reciprocity formula (see references [1], [2], and [3]):

$$(1) \quad S(h, q) + S(q, h) = \frac{h^2 + q^2 + 1}{12hq} - \frac{1}{4}$$

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for all $(h, q) = 1$, $q > 0$, $h > 0$. J.B. Conrey etc. [4] studied the mean value distribution of $S(h, k)$, and first proved the important asymptotic formula

$$(2) \quad \sum_{h=1}^k {}' |S(h, k)|^{2m} = f_m(k) \left(\frac{k}{12}\right)^{2m} + O((k^{9/5} + k^{2m-1+(1/(m+1))}) \log^3 k),$$

where $\sum_h {}'$ denotes the summation over all h such that $(k, h) = 1$, and

$$\sum_{n=1}^{\infty} \frac{f_m(n)}{n^s} = 2 \frac{\zeta^2(2m)}{\zeta(4m)} \cdot \frac{\zeta(s+4m-1)}{\zeta^2(s+2m)} \zeta(s).$$

Especially for $m = 1$ and $k = p$, an odd prime, we may immediately get the asymptotic formula

$$(3) \quad \sum_{h=1}^{p-1} |S(h, p)|^2 = \frac{5}{144} p^2 + O(p^{9/5} \log^3 p).$$

In the spirit of [4] and [5], Wenpeng Zhang [6] obtained a sharper asymptotic formula for (2) with $m = 1$ and $k = p^n$. That is,

$$(4) \quad \sum_{h=1}^k {}' |S(h, p)|^2 = \frac{5}{144} k^2 \frac{(p^2 - 1)^2}{p(p^3 - 1)} + O\left(k \exp\left(\frac{3 \log k}{\log \log k}\right)\right),$$

and

$$(5) \quad \sum_{h=1}^k {}' \frac{S(h, p)}{h} = \frac{\pi^2}{72} k \left(1 - \frac{1}{p^2}\right) + O(k^{1/2}).$$

In this paper, we use M. Toyozumi's important work, the properties of the Dedekind sums and the estimates for character sums to study the mean value

$$\sum_{a=1}^{p-1} a^k S^2(a, p),$$

and give a sharper asymptotic formula for it. That is, we shall prove the following:

Theorem. For any odd prime p and positive integer k , we have the asymptotic formula

$$\sum_{a=1}^{p-1} a^k S^2(a, p) = Ap^{k+2} + O\left(p^{k+3/2} \exp\left(\frac{6 \log p}{\log \log p}\right)\right),$$

where the constant

$$A = \left(\frac{5}{144(k+1)} + \frac{1}{\pi^4} \sum_{1 \leq m \leq k/2} \frac{\binom{k}{2m-1} (2m-1)!}{(-1)^{m+1} (2\pi)^{2m}} \sum_{n=1}^{\infty} \sum_{v=1}^{\infty} \frac{\sum_{d_1|n} \sum_{d_2|n} \cos(2\pi d_1 v \bar{d}_2 / p)}{n^2 v^{2m}} \right).$$

For general positive integer q and $m \geq 1$, whether there exists an asymptotic formula for the mean value

$$\sum_{a=1}^q a^k S^{2m}(a, q)$$

is an open problem.

2. SOME LEMMAS

To prove our theorem, we need several lemmas.

Lemma 1. Let $q \geq 3$ be an integer and $(h, q) = 1$. Then we have the identity

$$S(h, q) = \frac{1}{\pi^2 q} \sum_{d|q} \frac{d^2}{\varphi(d)} \sum_{\substack{\chi \pmod{d} \\ \chi(-1) = -1}} \chi(h) |L(1, \chi)|^2,$$

where $\varphi(d)$ is the Euler function, χ denotes a Dirichlet character modulo d with $\chi(-1) = -1$, and $L(s, \chi)$ denotes the Dirichlet L -function corresponding to χ .

Proof. See Lemma 2 of [6]. □

Lemma 2. Let $q \geq 3$ be an integer, let χ denote any primitive character mod q . Then we have the identity

$$\sum_{a=1}^q a^n \chi(a) = \begin{cases} 2q^n \tau(\chi) \sum_{1 \leq m \leq n/2} \frac{\binom{n}{2m-1} (2m-1)! L(2m, \bar{\chi})}{(-1)^{m+1} (2\pi)^{2m}} & \text{if } \chi(-1) = 1, \\ 2q^n \tau(\chi) \sum_{0 \leq m \leq (n-1)/2} \frac{\binom{n}{2m} (2m)! L(2m+1, \bar{\chi})}{(-1)^{m+1} (2\pi)^{2m+1} i} & \text{if } \chi(-1) = -1, \end{cases}$$

where $\tau(\chi) = \sum_{a=1}^q \chi(a)e(a/q)$ is the Gaussian sum, $e(y) = e^{2\pi iy}$, $L(s, \chi)$ is the Dirichlet L -function corresponding to χ , and $\binom{n}{m} = n!/(m!(n-m)!)$.

Proof. See [7]. □

Lemma 3. Let $q \geq 3$ be an integer. Then we have the asymptotic formula

$$\sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} |L(1, \chi)|^4 = \frac{5\pi^4}{144} \varphi(q) \prod_{p|q} \frac{(p^2-1)^3}{p^4(p^2+1)} + O\left(\frac{\varphi(q)}{q} \exp\left(\frac{3 \ln q}{\ln \ln q}\right)\right),$$

where $\prod_{p|q}$ denotes the product over all distinct prime divisors of q .

Proof. See Lemma 3 of [6]. □

Lemma 4. Let $p \geq 3$ be a prime, let χ^0 denote the principal character mod p , and let $\chi_1 \neq \chi^0$ be any even character mod p . Then we have the asymptotic formula

$$\sum_{\substack{\chi_1 \bmod p \\ \chi_1(-1)=-1}} |L(1, \chi_1)|^2 |L(1, \chi_1 \bar{\chi}_1)|^2 = \frac{\varphi(p)}{2} \sum_{n=1}^{\infty} \frac{|\sum_{d|n} \chi_1(d)|^2}{n^2} + O\left(\exp\left(\frac{6 \log p}{\log \log p}\right)\right),$$

where $\sum_{n=1}^{\infty}$ denotes the summation over all n such that $(n, p) = 1$.

Proof. Let $r(n) = \sum_{d|n} \bar{\chi}_1(d)$, let $\chi_1 \neq \chi^0$ be any even character mod p . Then for a parameter $N \geq p$, applying Abel's identity we have

$$(6) \quad L(1, \chi_1)L(1, \bar{\chi}_1) = \sum_{n=1}^{\infty} \frac{\chi_1(n)r(n)}{n} = \sum_{1 \leq n \leq N} \frac{\chi_1(n)r(n)}{n} + \int_N^{\infty} \frac{A(y, \chi_1)}{y^2} dy$$

where $A(y, \chi_1) = \sum_{N < n \leq y} \chi_1(n)r(n)$.

Note the partition identities

$$\begin{aligned} A(y, \chi_1) &= \sum_{n \leq \sqrt{y}} \chi_1(n) \sum_{N/n \leq m \leq y/n} \chi_1(m) \bar{\chi}_1(m) + \sum_{m \leq \sqrt{y}} \chi_1(m) \bar{\chi}_1(m) \sum_{N/n \leq n \leq y/n} \chi_1(n) \\ &\quad - \left(\sum_{\sqrt{N} \leq n \leq \sqrt{y}} \chi_1(n) \right) \left(\sum_{\sqrt{N} \leq n \leq \sqrt{y}} \chi_1(n) \bar{\chi}_1(n) \right). \end{aligned}$$

Applying the Cauchy inequality and the estimates for character sums

$$\begin{aligned} \sum_{\chi_1 \neq \chi^0} \left| \sum_{N \leq n \leq M} \chi_1(n) \right|^2 &= \sum_{\chi_1 \neq \chi^0} \left| \sum_{N \leq n \leq M \leq N+p} \chi_1(n) \right|^2 \\ &= \varphi(p) \sum_{N \leq n \leq M \leq N+p} \chi^0(n) - \left| \sum_{N \leq n \leq M \leq N+p} \chi^0(n) \right|^2 \leq \frac{\varphi^2(p)}{4} \end{aligned}$$

we have

$$\begin{aligned} (7) \quad \sum_{\substack{\chi_1 \bmod p \\ \chi_1(-1)=-1}} |A(y, \chi_1)|^2 &\ll \sqrt{y} \sum_{n \leq \sqrt{y}} \sum_{\substack{\chi_1 \bmod p \\ \chi_1(-1)=-1}} \left| \sum_{N/n \leq m \leq y/n} \chi_1(m) \bar{\chi}(m) \right|^2 \\ &\quad + \sqrt{y} \sum_{m \leq \sqrt{y}} \sum_{\substack{\chi_1 \bmod p \\ \chi_1(-1)=-1}} \left| \sum_{N/n \leq n \leq y/n} \chi_1(n) \right|^2 \\ &\quad + \sum_{\substack{\chi_1 \bmod p \\ \chi_1(-1)=-1}} \left| \sum_{\sqrt{N} \leq n \leq \sqrt{y}} \chi_1(n) \right|^2 \left| \sum_{\sqrt{N} \leq n \leq \sqrt{y}} \chi_1(n) \bar{\chi}(n) \right|^2 \\ &\ll y \varphi^2(p). \end{aligned}$$

Thus from (7) and the Cauchy inequality we obtain

$$\begin{aligned} (8) \quad \sum_{\substack{\chi_1 \bmod p \\ \chi_1(-1)=-1}} \left| \int_N^\infty \frac{A(y, \chi_1)}{y^2} dy \right|^2 &\leq \int_N^\infty \int_N^\infty \frac{1}{y^2 z^2} \left(\sum_{\substack{\chi_1 \bmod p \\ \chi_1(-1)=-1}} |A(y, \chi_1)| |A(z, \chi_1)| \right) dy dz \\ &\ll \left(\int_N^\infty \frac{1}{y^2} \sum_{\substack{\chi_1 \bmod p \\ \chi_1(-1)=-1}} |A(y, \chi_1)|^2 dy \right)^{1/2} \\ &\ll \frac{\varphi^2(p)}{N}. \end{aligned}$$

Note that for $(ab, p) = 1$, the orthogonality relation for character sums modulo p yields

$$\sum_{\substack{\chi_1 \bmod p \\ \chi_1(-1)=-1}} \chi_1(a) \chi_1(b) = \begin{cases} \frac{\varphi(p)}{2} & \text{if } a \equiv b \pmod{p}; \\ -\frac{\varphi(p)}{2} & \text{if } a \equiv -b \pmod{p}; \\ 0, & \text{otherwise,} \end{cases}$$

so that

$$\begin{aligned}
 (9) \quad & \sum_{\substack{\chi_1 \bmod p \\ \chi_1(-1)=-1}} \left| \sum_{1 \leq n \leq N} \frac{\chi_1(n)r(n)}{n} \right|^2 \\
 &= \frac{\varphi(p)}{2} \sum'_{\substack{1 \leq a, b \leq N \\ a \equiv b \pmod{p}}} \frac{r(a)\overline{r(b)}}{ab} - \frac{\varphi(p)}{2} \sum'_{\substack{1 \leq a, b \leq N \\ a \equiv -b \pmod{p}}} \frac{r(a)\overline{r(b)}}{ab} \\
 &= \frac{\varphi(p)}{2} \sum'_{1 \leq a \leq N} \frac{|r(a)|^2}{a^2} + O\left(\varphi(p) \sum_{b=1}^N \sum_{l=1}^{[N/p]} \frac{d(b)d(lp+b)}{(lp+b)b}\right) \\
 &\quad + O\left(\varphi(p) \sum_{a=1}^{p-1} \frac{d(a)d(p-a)}{a(p-a)}\right) \\
 &\quad + O\left(\varphi(p) \sum_{a=1}^N \sum_{(1+a/p) \leq l \leq N/p}^{[N/p]} \frac{d(a)d(lp-a)}{a(lp-a)}\right) \\
 &= \frac{\varphi(p)}{2} \sum'_{n=1}^{\infty} \frac{|r(n)|^2}{n^2} + O\left(\exp\left(\frac{2 \log N}{\log \log N}\right)\right),
 \end{aligned}$$

where $d(n)$ is the divisor function and $|r(n)| \leq d(n) \ll \exp((1 + \varepsilon) \frac{\log 2 \log n}{\log \log n})$.

Now

$$\begin{aligned}
 (10) \quad & \sum_{\substack{\chi_1 \bmod p \\ \chi_1(-1)=-1}} \left(\sum_{1 \leq n \leq N} \frac{\chi_1(n)r(n)}{n} \right) \left(\int_N^{\infty} \frac{\overline{A(y, \chi_1)}}{y^2} dy \right) \\
 &\ll \ln^2 N \int_N^{\infty} \frac{1}{y^2} \left(\sum_{\substack{\chi_1 \bmod p \\ \chi_1(-1)=-1}} |A(y, \overline{\chi_1})| \right) dy \\
 &\ll \varphi^{3/2}(p) N^{-1/2} \ln^2 N.
 \end{aligned}$$

Taking the parameter $N = p^3$, from (6), (7), (8), (9), and (10) we immediately obtain

$$\sum_{\substack{\chi_1 \bmod p \\ \chi_1(-1)=-1}} |L(1, \chi_1)|^2 |L(1, \overline{\chi_1})|^2 = \frac{\varphi(p)}{2} \sum'_{n=1}^{\infty} \frac{|\sum_{d|n} \chi(d)|^2}{n^2} + O\left(\exp\left(\frac{6 \log p}{\log \log p}\right)\right).$$

This completes the proof of Lemma 4. □

Lemma 5. Let $p \geq 3$ be a prime and m a positive integer, let χ^0 denote the principal character mod p , and let χ_1 and χ_2 be any two odd characters mod p . Then we have the asymptotic formula

$$\begin{aligned}
 E &:= \sum_{\substack{\chi_1 \bmod p \\ \chi_1(-1)=-1}} \sum_{\substack{\chi_2 \bmod p \\ \chi_2(-1)=-1 \\ \chi_1 \chi_2 \neq \chi^0}} \tau(\chi_1 \chi_2) L(2m, \overline{\chi_1 \chi_2}) |L(1, \chi_1)|^2 |L(1, \chi_2)|^2 \\
 &= \frac{\varphi^2(p)}{2} \sum_{n=1}^{\infty} \sum_{v=1}^{\infty} \frac{\sum_{d_1|n} \sum_{d_2|n} \cos(2\pi d_1 v \overline{d_2}/p)}{n^2 v^{2m}} + O\left(p^{3/2} \exp\left(\frac{6 \log p}{\log \log p}\right)\right).
 \end{aligned}$$

Proof. For any odd characters χ_1 and $\chi_2 \bmod p$, let $\chi = \chi_1 \chi_2$. Note that for $(ab, p) = 1$, the orthogonality relation for character sums modulo p implies

$$\sum_{\substack{\chi \bmod p \\ \chi(-1)=1}} \chi(a) \chi(b) = \begin{cases} \frac{\varphi(p)}{2} & \text{if } a \equiv \pm b \pmod{p}, \\ 0 & \text{otherwise,} \end{cases}$$

and from Lemma 4 we have

$$\begin{aligned}
 E &:= \sum_{\substack{\chi_1 \bmod p \\ \chi_1(-1)=-1}} \sum_{\substack{\chi \bmod p \\ \chi(-1)=1 \\ \chi \neq \chi^0}} \tau(\chi) L(2m, \overline{\chi}) |L(1, \chi_1)|^2 |L(1, \chi \overline{\chi_1})|^2 \\
 &= \sum_{\substack{\chi \bmod p \\ \chi(-1)=1 \\ \chi \neq \chi^0}} \tau(\chi) L(2m, \overline{\chi}) \sum_{\substack{\chi_1 \bmod p \\ \chi_1(-1)=-1}} |L(1, \chi_1)|^2 |L(1, \chi \overline{\chi_1})|^2 \\
 &= \frac{\varphi(p)}{2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{\substack{\chi \bmod p \\ \chi(-1)=1 \\ \chi \neq \chi^0}} \tau(\chi) L(2m, \overline{\chi}) \left| \sum_{d|n} \overline{\chi(d)} \right|^2 + O\left(p^{3/2} \exp\left(\frac{6 \log p}{\log \log p}\right)\right) \\
 &= \frac{\varphi(p)}{2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{\substack{\chi \bmod p \\ \chi(-1)=1}} \tau(\chi) L(2m, \overline{\chi}) \left| \sum_{d|n} \overline{\chi(d)} \right|^2 \\
 &\quad - \frac{\varphi(p)}{2} \sum_{n=1}^{\infty} \frac{1}{n^2} \tau(\chi^0) L(2m, \chi^0) \left| \sum_{d|n} \chi^0(d) \right|^2 + O\left(p^{3/2} \exp\left(\frac{6 \log p}{\log \log p}\right)\right) \\
 &= \frac{\varphi(p)}{2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{d_1|n} \sum_{d_2|n} \sum_{\substack{\chi \bmod p \\ \chi(-1)=1}} \tau(\chi) L(2m, \overline{\chi}) \overline{\chi}(d_1) \chi(d_2) \\
 &\quad + O\left(p^{3/2} \exp\left(\frac{6 \log p}{\log \log p}\right)\right)
 \end{aligned}$$

$$\begin{aligned}
&= \frac{\varphi^2(p)}{4} \sum'_{n=1}^{\infty} \frac{1}{n^2} \sum_{d_1|n} \sum_{d_2|n} \sum_{u=1}^{p-1} e\left(\frac{u}{p}\right) \sum'_{v=1}^{\infty} \frac{1}{v^{2m}} + O\left(p^{3/2} \exp\left(\frac{6 \log p}{\log \log p}\right)\right) \\
&= \frac{\varphi^2(p)}{2} \sum'_{n=1}^{\infty} \sum'_{v=1}^{\infty} \frac{\sum_{d_1|n} \sum_{d_2|n} \cos(2\pi d_1 v \bar{d}_2/p)}{n^2 v^{2m}} + O\left(p^{3/2} \exp\left(\frac{6 \log p}{\log \log p}\right)\right).
\end{aligned}$$

This completes the proof of Lemma 5. □

3. PROOF OF THEOREM

In this section, we complete the proof of our theorem. From Lemma 1, Lemma 2, Lemma 3, Lemma 4, and Lemma 5 we have

$$\begin{aligned}
\sum_{a=1}^{p-1} a^k S^2(a, p) &= \sum_{a=1}^{p-1} a^k \left(\frac{p}{\pi^2 \varphi(p)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(a) |L(1, \chi)|^2 \right)^2 \\
&= \frac{p^2}{\pi^4 \varphi^2(p)} \sum_{\substack{\chi_1 \bmod p \\ \chi_1(-1)=-1}} |L(1, \chi_1)|^4 \sum_{a=1}^{p-1} a^k \chi^0(a) \\
&\quad + \frac{p^2}{\pi^4 \varphi^2(p)} \sum_{\substack{\chi_1 \bmod p \\ \chi_1(-1)=-1}} \sum_{\substack{\chi_2 \bmod p \\ \chi_2(-1)=-1 \\ \chi_1 \chi_2 \neq \chi^0}} |L(1, \chi_1)|^2 |L(1, \chi_2)|^2 \sum_{a=1}^{p-1} a^k \chi_1 \chi_2(a) \\
&= \frac{p^2}{\pi^4 \varphi^2(p)} \left(\frac{5\pi^4 p}{144} + O\left(\exp\left(\frac{3 \ln p}{\ln \ln p}\right)\right) \right) \left(\frac{(p-1)^{k+1}}{k+1} + O((p-1)^k) \right) \\
&\quad + \frac{2p^{k+2}}{\pi^4 \varphi^2(p)} \sum_{1 \leq m \leq k/2} \frac{\binom{k}{2m-1} (2m-1)!}{(-1)^{m+1} (2\pi)^{2m}} \\
&\quad \times \sum_{\substack{\chi_1 \bmod p \\ \chi_1(-1)=-1}} \sum_{\substack{\chi_2 \bmod p \\ \chi_2(-1)=-1 \\ \chi_1 \chi_2 \neq \chi^0}} \tau(\chi_1 \chi_2) L(2m, \overline{\chi_1 \chi_2}) |L(1, \chi_1)|^2 |L(1, \chi_2)|^2 \\
&= Ap^{k+2} + O\left(p^{k+3/2} \exp\left(\frac{6 \log p}{\log \log p}\right)\right),
\end{aligned}$$

where

$$\begin{aligned}
A &= \left(\frac{5}{144(k+1)} \right. \\
&\quad \left. + \frac{1}{\pi^4} \sum_{1 \leq m \leq k/2} \frac{\binom{k}{2m-1} (2m-1)!}{(-1)^{m+1} (2\pi)^{2m}} \sum'_{n=1}^{\infty} \sum'_{v=1}^{\infty} \frac{\sum_{d_1|n} \sum_{d_2|n} \cos(2\pi d_1 v \bar{d}_2/p)}{n^2 v^{2m}} \right).
\end{aligned}$$

This completes the proof. □

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