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A NOTE ON A PROBLEM ARISING FROM RISK THEORY

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Abstract. In this note we give an answer to a problem of Gheorghiţă Zbăganu that arose from the study of the properties of the moments of the iterates of the integrated tail operator.

Keywords: improper integrals, moments, integrated tail operator

MSC 2010: 44A60, 91B30

1. Introduction

The goal of this note is to give an answer to the following problem of Professor Gheorghita Zbaganu.

Problem 1.1 (Zbaganu). *Find the subclass of all monotone decreasing functions* $f\colon [0,\infty) \to (0,\infty)$ *with*

$$\int_0^\infty x^n f(x)\,\mathrm{d}x < \infty, \qquad \forall\, n = 0, 1, \ldots$$

such that the following limit exists:

$$\lim_{n\to\infty} \frac{n\int_0^\infty x^n f(x)\,\mathrm{d}x}{\int_0^\infty x^{n+1} f(x)\,\mathrm{d}x}.$$

The problem that arose from insurance/risk theory problems, is related to the study of the properties of the moments of the iterates of the so called integrated tail operator, and during the last three years it has also been studied by the mathematics community. For a history of how this operator was introduced in the mathematical literature the interested reader is referred to [1] and [2]. Straightforward properties of the integrated tail operator as well as the moment calculations of the iterates of it are given in [3]. Before we give an answer to Problem 1.1 we prove the following theorem.

Theorem 1.2. *If* $h, g\colon [0,\infty) \to (0,\infty)$ *are integrable functions such that for all nonnegative integers* n

$$\int_0^\infty h(x)\, x^n \,\mathrm{d}x < \infty \quad \textit{and} \quad \int_0^\infty g(x)\, x^n \,\mathrm{d}x < \infty,$$

then

$$\lim_{n\to\infty} \frac{\int_0^\infty h(x)\, x^n \,\mathrm{d}x}{\int_0^\infty g(x) x^n \,\mathrm{d}x} = \lim_{x\to\infty} \frac{h(x)}{g(x)}$$

provided that the right-hand side limit exists.

Proof. Let $L = \lim\limits_{x\to\infty} h(x)/g(x)$. We will discuss, in detail, only the case when L is finite, since the case when $L = \infty$ is treated similarly. Let $\varepsilon > 0$ be a fixed positive real number. There exists $\delta > 0$ such that

$$L - \varepsilon < \frac{h(x)}{g(x)} < L + \varepsilon, \quad x > \delta. \tag{1.1}$$

We have

$$\int_0^\delta h(x)\; x^n \,\mathrm{d}x \leqslant \delta^n \int_0^\delta h(x) \,\mathrm{d}x$$

and

$$\int_\delta^\infty h(x)\, x^n \,\mathrm{d}x \geqslant \int_{\delta+1}^{\delta+2} h(x)\, x^n \,\mathrm{d}x \geqslant (\delta+1)^n \int_{\delta+1}^{\delta+2} h(x) \,\mathrm{d}x.$$

It follows that

$$0 < \frac{\int_0^\delta h(x) x^n \,\mathrm{d}x}{\int_\delta^\infty h(x) x^n \,\mathrm{d}x} \leqslant \Big(\frac{\delta}{\delta+1}\Big)^n \frac{\int_0^\delta h(x) \,\mathrm{d}x}{\int_{\delta+1}^{\delta+2} h(x) \,\mathrm{d}x},$$
$$0 < \frac{\int_0^\delta g(x) x^n \,\mathrm{d}x}{\int_\delta^\infty g(x) x^n \,\mathrm{d}x} \leqslant \Big(\frac{\delta}{\delta+1}\Big)^n \frac{\int_0^\delta g(x) \,\mathrm{d}x}{\int_{\delta+1}^{\delta+2} g(x) \,\mathrm{d}x},$$

and hence,

$$\lim_{n\to\infty} \frac{\int_0^\delta h(x)\, x^n \,\mathrm{d}x}{\int_\delta^\infty h(x)\, x^n \,\mathrm{d}x} = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{\int_0^\delta g(x)\, x^n \,\mathrm{d}x}{\int_\delta^\infty g(x)\, x^n \,\mathrm{d}x} = 0. \tag{1.2}$$

On the other hand,

$$\frac{\int_0^\infty h(x)\,x^n\,\mathrm{d}x}{\int_0^\infty g(x)\,x^n\,\mathrm{d}x} = \frac{\int_0^\delta h(x)\,x^n\,\mathrm{d}x + \int_\delta^\infty h(x)x^n\,\mathrm{d}x}{\int_0^\delta g(x)x^n\,\mathrm{d}x + \int_\delta^\infty g(x)x^n\,\mathrm{d}x}$$
$$= \frac{\int_\delta^\infty h(x)\,x^n\,\mathrm{d}x}{\int_\delta^\infty g(x)x^n\,\mathrm{d}x} \cdot \alpha_n,$$

where

$$\alpha_n = \frac{1 + \int_0^\delta h(x)\,x^n\,\mathrm{d}x / \int_\delta^\infty h(x)x^n\,\mathrm{d}x}{1 + \int_0^\delta g(x)\,x^n\,\mathrm{d}x / \int_\delta^\infty g(x)\,x^n\,\mathrm{d}x}.$$

We have, by virtue of (1.2), that $\lim\limits_{n\to\infty} \alpha_n = 1$. On the other hand, it follows from (1.1) that

$$L - \varepsilon \leqslant \frac{\int_\delta^\infty h(x)x^n\,\mathrm{d}x}{\int_\delta^\infty g(x)x^n\,\mathrm{d}x} \leqslant L + \varepsilon,$$

and hence,

$$(L-\varepsilon)\alpha_n \leqslant \frac{\int_0^\infty h(x)x^n\,\mathrm{d}x}{\int_0^\infty g(x)x^n\,\mathrm{d}x} \leqslant (L+\varepsilon)\alpha_n.$$

Letting n tend to ∞ in the preceding inequality we obtain that

$$L - \varepsilon \leqslant \lim_{n\to\infty} \frac{\int_0^\infty h(x)x^n\,\mathrm{d}x}{\int_0^\infty g(x)x^n\,\mathrm{d}x} \leqslant L + \varepsilon.$$

Since ε is arbitrary we get that the desired limit relation holds and the theorem is proved. □

2. An answer to Zbaganu's problem

In this section we prove Theorem 2.1 below which gives an answer to Problem 1.1.

Theorem 2.1. *Let $f\colon [0,\infty) \to (0,\infty)$ be a positive and monotone decreasing function such that for all nonnegative integers n*

$$\int_0^\infty x^n f(x)\,\mathrm{d}x < \infty, \tag{2.1}$$

and let $F(x) = \int_x^\infty f(t)\,\mathrm{d}t$. Then

$$\lim_{n\to\infty} \frac{n\int_0^\infty x^n f(x)\,\mathrm{d}x}{\int_0^\infty x^{n+1} f(x)\,\mathrm{d}x} = \lim_{x\to\infty} \frac{f(x)}{F(x)} \tag{2.2}$$

provided that the right-hand side limit exists.

P r o o f. Since f decreases, condition (2.1) implies that $\lim\limits_{x\to\infty} x^n f(x) = 0$. An application of the l'Hospital Rule shows that

$$\lim_{x\to\infty} x^{n+1}F(x) = \lim_{x\to\infty} \frac{F(x)}{x^{-n-1}} = \frac{1}{n+1} \lim_{x\to\infty} x^{n+2} f(x) = 0.$$

Integrating by parts, we obtain that

$$\int_0^\infty x^{n+1} f(x)\,\mathrm{d}x = (n+1)\int_0^\infty x^n F(x)\,\mathrm{d}x,$$

and the proof follows as an application of Theorem 1.2. □

Remark 2.2. It is worth mentioning that there are positive functions such that even though the right-hand side limit of (2.2) does not exist, the left-hand side limit does. To see this, let $f(x) = (2+\sin x)\mathrm{e}^{-x}$. We have

$$\frac{n\int_0^\infty x^n f(x)\,\mathrm{d}x}{\int_0^\infty x^{n+1} f(x)\,\mathrm{d}x} = \frac{n}{n+1}\Big(1 + \frac{\sin\frac{1}{4}n\pi}{4\cdot 2^{n/2} + \cos\frac{1}{4}n\pi}\Big) \to 1.$$

On the other hand, the function $f(x)/F(x) = (4+2\sin x)/(4+\cos x+\sin x)$, has no limit as $x\to\infty$.

The next corollary is a consequence of Theorem 2.1.

Corollary 2.3. *If $f \in C^1[0,\infty)$ is a positive and monotone decreasing function such that for all nonnegative integers n*

$$\int_0^\infty x^n f(x)\,\mathrm{d}x < \infty,$$

then

$$\lim_{n\to\infty} \frac{n\int_0^\infty x^n f(x)\,\mathrm{d}x}{\int_0^\infty x^{n+1} f(x)\,\mathrm{d}x} = -\lim_{x\to\infty}\frac{f'(x)}{f(x)}$$

provided that the right-hand side limit exists.

Remark 2.4. We mention that if $f \in C^1[0,\infty)$ is logarithmically convex/concave, i.e., $\log f$ is convex/concave, then the limit $\lim\limits_{x\to\infty} f'(x)/f(x)$ exists.

Remark 2.5. Under the conditions of Theorem 2.1 we have

$$\lim_{n\to\infty} \frac{n\int_0^\infty x^n f(x)\,\mathrm{d}x}{\int_0^\infty x^{n+1} f(x)\,\mathrm{d}x} = \lambda \in (0,\infty)$$

if and only if

$$f(x) = \mathrm{e}^{-(\lambda+o(1))x} \quad \text{as } x\to\infty.$$

Example 2.6. Let

$$L_n(f) = \frac{n \int_0^\infty x^n f(x)\,\mathrm{d}x}{\int_0^\infty x^{n+1} f(x)\,\mathrm{d}x}.$$

A calculation shows that

$$\begin{aligned} L_n(\mathrm{e}^{-x^2}) &= \frac{2\Gamma\left(\frac{1}{2}(1+n)\right)}{\Gamma\left(\frac{1}{2}n\right)} \to \infty, \\ L_n(\mathrm{e}^{-x}) &= \frac{n}{1+n} \to 1, \\ L_n(\mathrm{e}^{-\sqrt{x}}) &= \frac{n}{6+10n+4n^2} \to 0. \end{aligned}$$

We conclude this note with the following generalization of Theorem 1.2.

Theorem 2.7. *If* $f, g, \varrho_n \colon [0,\infty) \to (0,\infty)$, $n = 0, 1, \ldots$, *are integrable functions such that for all* n

$$\int_0^\infty f(x)\varrho_n(x)\,\mathrm{d}x < \infty, \quad \int_0^\infty g(x)\varrho_n(x)\,\mathrm{d}x < \infty,$$

ϱ_n *being strictly increasing functions and* $\lim_{n\to\infty} \varrho_n(a)/\varrho_n(b) = 0$ *with* $a < b$, *then*

$$\lim_{n\to\infty} \frac{\int_0^\infty f(x)\varrho_n(x)\,\mathrm{d}x}{\int_0^\infty g(x)\varrho_n(x)\,\mathrm{d}x} = \lim_{x\to\infty} \frac{f(x)}{g(x)}$$

provided that the right-hand side limit exists.

The proof is identical to the proof of Theorem 1.2.

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