Czechoslovak Mathematical Journal

Osvaldo Gorosito; Gladis Pradolini; Oscar Salinas

Boundedness of fractional operators in weighted variable exponent spaces with non doubling measures

Czechoslovak Mathematical Journal, Vol. 60 (2010), No. 4, 1007–1023

Persistent URL: `http://dml.cz/dmlcz/140799`

Terms of use:

© Institute of Mathematics AS CR, 2010

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.

This document has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* `http://dml.cz`

Czechoslovak Mathematical Journal, 60 (135) (2010), 1007–1023

BOUNDEDNESS OF FRACTIONAL OPERATORS IN WEIGHTED VARIABLE EXPONENT SPACES WITH NON DOUBLING MEASURES

Osvaldo Gorosito, Gladis Pradolini, Oscar Salinas, Santa Fe

(Received May 12, 2009)

Abstract. In the context of variable exponent Lebesgue spaces equipped with a lower Ahlfors measure we obtain weighted norm inequalities over bounded domains for the centered fractional maximal function and the fractional integral operator.

Keywords: variable exponent, weighted spaces, non doubling measures

MSC 2010: 42B25

1. Introduction and statement of the main results

During the last two decades the variable exponent Lebesgue spaces $L^{p(x)}$ have been studied intensively. They seem to be the most adequate context for studying a great variety of problems related to certain classes of fluids that are characterized by their ability to undergo significant changes in their mechanical properties when an electric field is applied (see [13]).

Recently, a number of authors, interested in studying the continuity of certain classical operators from harmonic analysis, have succeeded in proving some boundedness results in such spaces. In fact, non-weighted strong inequalities for the Hardy-Littlewood and the fractional maximal operators were obtained in the euclidean setting, under certain property of regularity on the exponent. In particular, Lars Diening proved the continuity of the Hardy-Littlewood maximal operator in $\mathbb{R}^n$ by requiring an additional property of constancy on the exponent outside a fixed ball (see [4]). In proving his result, the technique developed by the author differs from the one applied in the classical theory, essentially based on interpolation. Cruz Uribe, Fiorenza and Neugebauer took advantage of these techniques and improved Diening's result by additionally assuming some type of logarithm decay on the exponent. Thus they obtained norm inequalities over open subsets of $\mathbb{R}^n$ (see [3]).

Under the hypothesis of continuity and logarithm decay on the exponent, the well known strong type inequality for the fractional maximal operator was proved in the variable context over open subsets of $\mathbb{R}^n$ (see [2]).

In addition, boundedness results for the Hardy-Littlewood maximal operator were obtained by Harjulehto, Hästö and Latvala in $\mathbb{R}^n$ over bounded domains with the novelty of using a non necessarily doubling measure (see [6]).

On the other hand, for the maximal operator, weighted norm inequalities involving power weights, were obtained by Kokilashvili and Samko ([8]) in $\mathbb{R}^n$ over open bounded domains.

In this article we prove weighted strong inequalities for fractional operators. For maximal operators we include the case of the classical Hardy-Littlewood maximal function in the setting of variable exponent spaces defined over bounded subsets of $\mathbb{R}^n$ which have been equipped with a non necessarily doubling measure. Such inequalities provide a weighted version of those contained in [6] for the Hardy-Littlewood maximal function as well as those proved in [2] for the fractional maximal operator in the standard context of Lebesgue measurable spaces.

Additionally the class of weights involved in our results is wider than that of power functions considered in [8] for the case of the Hardy-Littlewood maximal operator referred to the Lebesgue measure. It is worth mentioning that a weighted pointwise relationship between the fractional and the Hardy-Littlewood maximal functions is also proved and it is not only interesting in itself but essential for the proof of one of our main results as well.

In the same context as the one described at the beginning, i.e., over generalized Lebesgue spaces equipped with a non-doubling measure, we also obtain weighted strong inequalities for the integral fractional operator. A version of Welland's inequality in this variable setting is also given and it turns out to play a fundamental role in proving the boundedness of that operator.

In the context of Lebesgue standard measure spaces we give, in some sense, certain type of reverse Hölder inequality which proves to be appropriate to obtain a special class of weights for which the continuity of the fractional integral holds. We want to point out that this class is larger than that of power weights, generalizing in this way the results due to Samko in [14]. Finally we should say that the techniques developed by the author, essentially based on a Hedberg type inequality, differ from ours since Welland's inequality allows us to prove our results.

We first introduce the context in which we shall develop our results. Throughout this paper $Q = Q(x, l(Q))$ will denote a cube centered at x with side length $l(Q)$

and with sides parallel to the coordinate axes. Moreover, C will denote a positive constant not necessarily the same on each occurrence.

Let us now consider a non-negative Borel regular measure μ defined over subsets in $\mathbb{R}^n$. If Ω is a bounded μ-measurable set and $\beta\colon\ \Omega \to (0,\infty)$ is a bounded function, we shall say that μ is a lower Ahlfors $\beta(\cdot)$-regular measure in Ω if there exists a positive constant C such that the inequality

$$C \leqslant \frac{\mu(Q(x, l(Q)))}{l(Q)^{\beta(x)}} \tag{1.1}$$

holds for every $x \in \Omega$ and for every cube $Q \subset \mathbb{R}^n$ such that $0 < l(Q) < \operatorname{diam}(\Omega)$. In particular, whenever β is a constant function we shall simply say that μ is lower Ahlfors β-regular in Ω.

A measure μ is said to satisfy the doubling property if there exists a positive constant C such that the inequality $\mu(2Q) \leqslant C\mu(Q)$ holds for every cube Q. It is easy to see that a measure μ satisfying this property is a lower Ahlfors measure. However, there exist lower Ahlfors measures which fail to possess the doubling property (see an example at the end of Section 3).

The interest in studying these measures appears in connection with the notion of the dimension of a metric space. By dimension it is understood some quantity relating the measure of a cube to its side length. Examples of measures with variable dimensions are given in [6].

It is easy to see that lower Ahlfors $\beta(\cdot)$-regularity implies lower Ahlfors β^*-regularity, where $\beta^* = \sup\limits_{\Omega} \beta$. In this paper we shall consider lower Ahlfors β-regularity. It is not difficult to prove that all our results imply the corresponding results for the lower $\beta(\cdot)$-regularity case. In this article, β will denote a positive number related to the operators we shall be working with.

We now introduce the functional space we are going to deal with. For additional information see [9].

A non-negative μ-measurable function p from Ω to $[1,\infty)$ is called an exponent. For simplicity we write $p_* = \inf\limits_{\Omega} p$ and $p^* = \sup\limits_{\Omega} p$. Along this paper the exponent will be assumed to be bounded, that is $p^* < \infty$. We shall also suppose that $p_* > 1$. For any set $A \subset \Omega$ we denote $(p_A)_* = \inf\limits_{A} p$ and $(p_A)^* = \sup\limits_{A} p$.

For any μ-measurable function f, the modular $\varrho_{p(\cdot),\Omega}$ is defined by

$$\varrho_{p(\cdot),\Omega}(f) = \int_{\Omega} |f(x)|^{p(x)}\, \mathrm{d}\mu(x)$$

and the formula

$$\|f\|_{p(\cdot),\Omega} = \inf\{\lambda > 0\colon\ \varrho_{p(\cdot),\Omega}(f/\lambda) \leqslant 1\}$$

is seen to define a norm.

The variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ consists of those μ-measurable functions f supported in Ω for which $\|f\|_{p(\cdot),\Omega} < \infty$. We should mention that these spaces are a special case of the Musielak-Orlicz spaces whose theory was developed a long time ago (see for example [10]).

If p is an exponent such that $p_* > 1$, let p' be the function defined by $1/p(x) + 1/p'(x) = 1$. Topics related to general properties of this space are treated in [9]. In particular, the generalized Hölder inequality (see [9])

$$\int_\Omega |fg| \, \mathrm{d}\mu \leqslant C\|f\|_{p(\cdot),\Omega}\|g\|_{p'(\cdot),\Omega} \tag{1.2}$$

holds and it shall be useful in our proofs.

We shall deal with a class of bounded exponents which satisfy certain property of regularity stronger than uniform continuity. More precisely, an exponent p is said to be log-Hölder continuous if it satisfies the following inequality

$$|p(x) - p(y)| \leqslant \frac{C}{\log(1/|x-y|)}, \quad x, y \in \Omega, \ \ |x-y| \leqslant 1/2.$$

It is worth mentioning that this condition guarantees regularity results on variable exponent spaces. In [4], the author proves that this condition along with the additional assumption that p is constant outside a fixed ball is sufficient for the maximal operator to be bounded in $L^{p(\cdot)}(\mathbb{R}^n)$. Moreover in [12] it is shown that the boundedness of this operator might fail for a general exponent p. In fact, the authors proved that the modulus of continuity is optimal.

We finally introduce the maximal functions we are interested in along with the class of weights involved with their properties of boundedness.

Let μ be a lower Ahlfors β-regular measure. For $0 \leqslant \alpha < \beta$ the centered fractional maximal operator of a locally integrable function f is defined by

$$M_\alpha f(x) = \sup_{r>0} \frac{1}{\mu(Q(x,r))^{1-\alpha/\beta}} \int_{Q(x,r)} |f(y)| \, \mathrm{d}\mu(y) \tag{1.3}$$

where $Q(x,r)$ denotes a cube centered at x with side-length equal to r. If $\alpha = 0$ in (1.3) we simply write $M_0 = M$ for the classical Hardy-Littlewood maximal function.

A version of the fractional integral operator associated to the maximal one defined above is given by

$$I_\alpha f(x) = \int_\Omega \frac{f(y)|x-y|^\alpha}{\mu(Q(x, 2|x-y|))} \, \mathrm{d}\mu(y). \tag{1.4}$$

The operator above is equivalent to the one defined in [1] in the context of a (quasi)-metric space equipped with a lower Ahlfors measure whenever the quasi-distance is a fixed multiple of the euclidean distance.

Let s be a real number such that $1 < s < \infty$. We say that a weight w belongs to the $A_s(\Omega)$ class if there exists a positive constant C such that the inequality

$$\left(\frac{1}{\mu(Q)}\int_{Q\cap\Omega} w\,\mathrm{d}\mu\right)\left(\frac{1}{\mu(Q)}\int_{Q\cap\Omega} w^{-1/(s-1)}\,\mathrm{d}\mu\right)^{s-1} \leqslant C$$

holds for each cube Q centered at a point in Ω.

If μ is the classical Lebesgue measure in $\mathbb{R}^n$, $0 < \alpha < n$, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$ and $s = 1 + q/p'$, it is well known that the $A_s(\mathbb{R}^n)$ class characterizes the boundedness of M_α from $L^p(w^{p/q})$ into $L^q(w)$ (see for example [11]). In particular, if $\alpha = 0$ and $1 < p < \infty$ this result gives the boundedness of M in $L^p(w)$.

Before stating our main results we introduce some additional notation.

Given a continuous function t defined in Ω we shall denote

$$\Omega_r^t = \{x \in \Omega\colon\ t(x) > r\}$$

for each $r \in \mathbb{R}$. Let us observe that this set is not empty whenever $r < t^*$. Given $\varepsilon > 0$, related to this set, we define

$$\Omega_{r,\varepsilon}^t = \Omega_r^t - \bigcup_{x\in\Omega-\Omega_r^t} B(x,\varepsilon). \tag{1.5}$$

It is easy to see that if μ is a lower Ahlfors β-regular measure in Ω then there exists $\varepsilon_0 > 0$ such that $\mu(\Omega_{r,\varepsilon}^t) > 0$ for every $\varepsilon \leqslant \varepsilon_0$.

Now we proceed to state our main results.

Theorem 1.1. *Let $0 \leqslant \alpha < \beta$ and let p be a log-Hölder continuous exponent such that $1 < p_* \leqslant p(x) \leqslant p^* < \beta/\alpha$. Let q and s be respectively defined by $1/q(x) = 1/p(x) - \alpha/\beta$ and $s(x) = 1 + q(x)/(p(x))'$. Let μ be a lower Ahlfors β-regular measure in Ω and $r \in (1, s^*)$. If w is a weight such that $w(\cdot)^{q(\cdot)} \in A_{r-\delta}(\Omega)$, for some $\delta \in (0, r-1]$ and $\int_{\Omega-\Omega_{r,\varepsilon_0}^s} w(x)^{-(p(x))'}\,\mathrm{d}\mu(x) < \infty$ for some $\varepsilon_0 > 0$, then there exists a positive constant $C = C(\varepsilon)$ such that*

$$\|wM_\alpha f\|_{q(\cdot),\Omega_{r,\varepsilon}^s} \leqslant C\|wf\|_{p(\cdot),\Omega}$$

for every function f such that $wf \in L^{p(\cdot)}(\Omega)$ and for every $\varepsilon \leqslant \varepsilon_0$.

Notice that if $\operatorname{supp} f \subset \Omega^s_{r,\varepsilon}$ then the hypothesis $\int_{\Omega-\Omega^s_{r,\varepsilon}} w(x)^{-(p(x))'}\,\mathrm{d}\mu(x) < \infty$ in the theorem above can be removed and we obtain

Corollary 1.1. *Let α, p, q, s and μ be as in Theorem 1.1 and let w be a weight such that $w(\cdot)^{q(\cdot)} \in A_{r-\delta}(\Omega)$, for some δ such that $0 < \delta \leqslant r-1$. Then there exists a positive constant C such that the inequality*

$$\|wM_\alpha f\|_{q(\cdot),\Omega^s_{r,\varepsilon}} \leqslant C\|wf\|_{p(\cdot),\Omega^s_{r,\varepsilon}}$$

holds for every function f such that $wf \in L^{p(\cdot)}(\Omega^s_{r,\varepsilon})$ and $\operatorname{supp} f \subset \Omega^s_{r,\varepsilon}$.

From the fact that $\Omega^s_{s_*-\delta,\varepsilon} = \Omega$ we immediately obtain the following result.

Corollary 1.2. *Let α, p, q, s and μ be as in Theorem 1.1. If w is a weight such that $w(\cdot)^{q(\cdot)} \in A_{s_*-\delta}(\Omega)$, for some δ such that $0 < \delta < s_*-1$ then there exists a positive constant C such that the inequality*

$$\|wM_\alpha f\|_{q(\cdot),\Omega} \leqslant C\|wf\|_{p(\cdot),\Omega}$$

holds for every function f such that $wf \in L^{p(\cdot)}(\Omega)$.

Corollary 1.3. *Let $\beta = n$, and α, p, q and s be as in Theorem 1.1, and μ be the Lebesgue measure in $\mathbb{R}^n$. Let $x_0 \in \Omega$ and $w(x) = |x-x_0|^\eta$, $\eta \in \mathbb{R}$. If $-n/q(x_0) < \eta < n/(p(x_0))'$ then there exists a positive constant C such that the inequality*

$$\|wM_\alpha f\|_{q(\cdot),\Omega} \leqslant C\|wf\|_{p(\cdot),\Omega}$$

holds for every function f such that $wf \in L^{p(\cdot)}(\Omega)$.

Remark 1.1. The case $\alpha = 0$ in the corollary above was proved in [8]. The authors also show that the range of η is sharp by proving that the reciprocal result is true whenever $x_0 \in \Omega$. Following similar arguments, an analogous result can be obtained for the case $\alpha > 0$.

An application of Corollaries 1.2 and 1.3 allows us to obtain two results concerning one-weighted-type inequalities for the fractional integral operator defined in (1.4).

Let $0 < \alpha < \beta$. Let p be an exponent such that $1 < p_* < p(x) < p^* < \infty$ for every $x \in \Omega$ and let q be the function defined by $1/q(x) = 1/p(x) - \alpha/\beta$.

If $0 < \varepsilon < \min\{\alpha, \beta-\alpha, \beta/q^*, \beta(1/p^*-1/q_*)\}$, let q^+_ε, q^-_ε, s^+_ε and s^-_ε be the functions defined by

$$\frac{1}{q^+_\varepsilon(x)} = \frac{1}{p(x)} - \frac{\alpha+\varepsilon}{\beta}, \qquad \frac{1}{q^-_\varepsilon(x)} = \frac{1}{p(x)} - \frac{\alpha-\varepsilon}{\beta},$$

and

$$s_\varepsilon^+(x) = 1 + \frac{q_\varepsilon^+(x)}{(p(x))'}, \qquad s_\varepsilon^-(x) = 1 + \frac{q_\varepsilon^-(x)}{(p(x))'}.$$

Theorem 1.2. *Let $0 < \alpha < \beta$ and let μ be a lower Ahlfors β-regular measure in Ω. Let p be a log-Hölder continuous exponent such that $1 < p_* \leqslant p(x) \leqslant p^* < \beta/\alpha$ and q be defined by $1/q(x) = 1/p(x) - \alpha/\beta$. Let w be a weight such that $w^{q_\varepsilon^+} \in A_{(s_\varepsilon^+)_*}$ and $w^{q_\varepsilon^-} \in A_{(s_\varepsilon^-)_*}$. Then there exists a positive constant C such that*

$$\|wI_\alpha f\|_{q(\cdot),\Omega} \leqslant C\|wf\|_{p(\cdot),\Omega}.$$

In the non-weighted case, the theorem above was proved by Almeida and Samko in [1] in the context of (quasi)-metric measure spaces. The authors give boundedness results with measures either satisfying the doubling condition or the lower Ahlfors condition. Moreover, for doubling measures they generalized their result for a variable index α.

In the classical Lebesgue context and for a particular class of weights in A_1, we prove that certain variable powers of such weights also remain in that class. This property allows us to obtain weights for which the boundedness of the fractional integral holds. In order to make the results more precise we introduce those special weights.

Let Q_0 be a cube in $\mathbb{R}^n$ and let μ be the standard Lebesgue measure. We shall be interested in those weights w belonging to $A_1(Q_0)$ for which the following properties hold:

i) For almost every $x \in Q_0$, $w(x) \geqslant 1$.
ii) The weight w has m singularities $x_1, x_2, \dots, x_m$ in Q_0.
iii) There exist two positive numbers θ and r such that $w(x) \leqslant |x - x_i|^{-\theta}$, for almost every $x \in Q(x_i, r) \cap \bar{Q}_0$ and for each $i = 1, 2, \dots, m$.

Theorem 1.3. *Let (Q_0, μ) be the measure space consisting of the cube Q_0 in $\mathbb{R}^n$ and of the classical Lebesgue measure μ. Let w be a weight in the $A_1(Q_0)$ class that satisfies the properties stated above.*
Then there exists a positive number δ such that w^α also belongs to the $A_1(Q_0)$ class for every function α satisfying both a log-Hölder condition and the inequality $1 \leqslant \alpha(x) \leqslant 1 + \delta$ for almost every point x in Q_0.

Corollary 1.4. *Let α, p and q be as in Corollary 1.3, and μ be the Lebesgue measure in a cube Q_0. For a pair of weights w_1 and w_2 in the $A_1(Q_0)$ class that satisfy the hypotheses of Theorem 1.3 and for $\varepsilon > 0$ small enough, let w be the weight* *defined by* $w = w_1^{1/q_\varepsilon^-} w_2^{(1/q_\varepsilon^-)(1-(s_\varepsilon^-)_*)}$. *Then there exists a positive constant* C *such that the inequality*

$$\|wI_\alpha f\|_{q(\cdot),Q_0} \leqslant C\|wf\|_{p(\cdot),Q_0}$$

holds for every function f *such that* $wf \in L^{p(\cdot)}(Q_0)$.

Remark 1.2. When w is a product of a finite number of power weights, the corollary above can be proved using similar techniques, for Q_0 replaced by a more general bounded set Ω. In particular, when w is a power weight, this result was proved by Samko ([14]) but for a variable index α.

The structure of this paper goes on as follows. Section 2 contains certain types of inequalities frequently used in the variable context. A pointwise estimate relating both operators M and M_α is also shown. Section 3 is devoted to proving our main results. Finally, an example of weights in a bounded subset equipped with a non doubling measure is also given.

2. Preliminary results

The following result is a version for cubes of Lemma 3.6 in [7] relating lower Ahlfors regularity and log-Hölder continuity of the exponent and it is essential to prove Theorem 2.1. We omit its proof since it is similar to the one for balls.

Lemma 2.1. *Let* μ *be a lower Ahlfors* β*-regular measure and let* p *be a log-Hölder continuous exponent. Then, there exists a positive constant* C *such that*

$$\mu(Q)^{(p_Q)_*-(p_Q)^*} \leqslant C \tag{2.1}$$

for every cube Q *centered at* Ω.

Theorem 2.1. *Let* μ *be a lower Ahlfors* β*-regular measure in a bounded* μ*-measurable set* Ω. *Let* $0 < t < t^*$ *be a log-Hölder continuous function in* Ω *and* $\varepsilon > 0$. *Then there exists a positive constant* $C = C(\varepsilon)$ *such that the inequality*

$$Mf(x)^{t(x)} \leqslant C(1 + M(|f|(\cdot)^{t(\cdot)})(x)) \tag{2.2}$$

holds for almost every $x \in \Omega^t_{1,\varepsilon}$ *and for every function* f *such that* $\|f\|_{t(\cdot),\Omega^t_{1,\varepsilon}} \leqslant 1$ *and* $\int_{\Omega-\Omega^t_{1,\varepsilon}} |f|\,\mathrm{d}\mu \leqslant 1$, *where* $\Omega^t_{1,\varepsilon}$ *is defined as in* (1.5).

Remark 2.1. The fact that t is allowed to take positive values is essential in the proof of Theorem 1.1. If $t_* > 1$ and μ is the standard Lebesgue measure the inequality above is proved in [8]. Inequalities of this type were originally proved in [4].

P r o o f. Let x be a fixed point in $\Omega^t_{1,\varepsilon}$. If Q is any cube centered at x, it is enough to prove that the inequality

$$\Big(\frac{1}{\mu(Q)}\int_Q |f(y)|\,\mathrm{d}\mu(y)\Big)^{t(x)} \leqslant C\Big(1+\frac{1}{\mu(Q)}\int_Q |f(y)|^{t(y)}\,\mathrm{d}\mu(y)\Big) \tag{2.3}$$

holds for some positive constant C.

Let us first assume that $\mu(Q)\geqslant 1/2$. Note that

$$\int_Q |f(y)|\,\mathrm{d}\mu(y) = \int_{Q\cap\Omega^t_{1,\varepsilon}} |f(y)|\,\mathrm{d}\mu(y) + \int_{Q\cap(\Omega-\Omega^t_{1,\varepsilon})} |f(y)|\,\mathrm{d}\mu(y).$$

The second term in the inequality above is bounded by 1 by hypothesis. For the first we take into account that in $Q\cap\Omega^t_{1,\varepsilon}$ we have that $t(x)>1$. Then, from the fact that

$$\int_{Q\cap\Omega^t_{1,\varepsilon}} \Big(\frac{1}{\mu(Q)+1}\Big)^{(t(x))'}\,\mathrm{d}\mu(x) \leqslant \int_Q \frac{1}{\mu(Q)+1}\,\mathrm{d}\mu(x) < 1,$$

we have that $\|\chi_Q\|_{t'(\cdot),Q\cap\Omega^t_{1,\varepsilon}} \leqslant \mu(Q)+1$. Then, by the generalized the Hölder inequality (1.2), the remark above and the hypotheses we obtain

$$\begin{aligned}\Big(\frac{1}{\mu(Q)}\int_{Q\cap\Omega^t_{1,\varepsilon}} |f(x)|\,\mathrm{d}\mu(x)\Big)^{t(x)} &\leqslant \frac{1}{\mu(Q)^{t(x)}}\|f\|^{t(x)}_{t(\cdot),Q\cap\Omega^t_{1,\varepsilon}}\|\chi_Q\|^{t(x)}_{t'(\cdot),Q\cap\Omega^t_{1,\varepsilon}}\\ &\leqslant \Big(1+\frac{1}{\mu(Q)}\Big)^{t(x)} \leqslant C.\end{aligned}$$

Now we assume that $\mu(Q)<1/2$ and $l(Q)>C\varepsilon$ where C is a constant depending on the dimension. Then, from the definition of μ a constant C depending on ε and the dimension can be found so that $\mu(Q)\geqslant C$. Then we proceed as in the case above to obtain the result.

If $\mu(Q)<1/2$ and $l(Q)<C\varepsilon$ then it is easy to check that $(\Omega-\Omega_1)\cap Q=\emptyset$. Then $t(x)>1$ in Q.

If $t_Q=\min\limits_{y\in Q} t(y)$, by applying the Hölder inequality we obtain

$$\Big(\frac{1}{\mu(Q)}\int_Q |f(y)|\,\mathrm{d}\mu(y)\Big)^{t(x)} \leqslant \frac{1}{\mu(Q)^{t(x)/t_Q}}\Big(\int_Q |f(y)|^{t_Q}\,\mathrm{d}\mu(y)\Big)^{t(x)/t_Q}. \tag{2.4}$$

Since

$$\begin{aligned}\int_Q |f(y)|^{t_Q}\,\mathrm{d}\mu(y) &= \int_{Q\cap\{|f|\leqslant 1\}} |f(y)|^{t_Q}\,\mathrm{d}\mu(y) + \int_{Q\cap\{|f|>1\}} |f(y)|^{t_Q}\,\mathrm{d}\mu(y)\\ &\leqslant 2\Big(\mu(Q)+\frac{1}{2}\int_Q |f(y)|^{t(y)}\,\mathrm{d}\mu(y)\Big),\end{aligned}$$

and as the expression in brackets is less than 1, from (2.4) we get

$$\left(\frac{1}{\mu(Q)}\int_Q |f(y)|\,\mathrm{d}\mu(y)\right)^{t(x)} \leqslant C\mu(Q)^{1-t(x)/t_Q}\left(1+\frac{1}{\mu(Q)}\int_Q |f(y)|^{t(y)}\,\mathrm{d}\mu(y)\right).$$

But this is (2.3) because of (2.1). □

The following lemma gives a pointwise estimation relating both operators M and M_α and it proves to be essential to obtain our main results.

Lemma 2.2. *Let μ be a lower Ahlfors β-regular measure in Ω. Let $0<\alpha<\beta$ and p be an exponent such that $1<p_*\leqslant p(x)\leqslant p^*<\beta/\alpha$. Let q be defined by $1/q(x)=1/p(x)-\alpha/\beta$. If $s(x)=1+q(x)/p'(x)$, then the following inequality*

$$M_\alpha(f/w)(x)\leqslant (M(|f|^{p(\cdot)/s(\cdot)}w^{-q(\cdot)/s(\cdot)})(x))^{s(x)/q(x)}\left(\int_Q |f|(y)^{p(y)}\,\mathrm{d}\mu(y)\right)^{\alpha/\beta}$$

holds for every function f and for every weight w.

P r o o f. Let f be a non negative function and let g be the function defined by $g^s=f^pw^{-q}$. Since $f/w=g^{s/p}w^{q/p-1}=g^{1-\alpha/\beta}g^{s/p+\alpha/\beta-1}w^{\alpha q/\beta}$ then, by the Hölder inequality we have that

$$\begin{aligned}\frac{1}{\mu(Q)^{1-\alpha/\beta}}\int_Q \frac{f}{w}\,\mathrm{d}\mu &\leqslant \frac{1}{\mu(Q)^{1-\alpha/\beta}}\int_Q g^{s/p}w^{q/p-1}\,\mathrm{d}\mu\\ &\leqslant \left(\frac{1}{\mu(Q)}\int_Q g\,\mathrm{d}\mu\right)^{1-\alpha/\beta}\left(\int_Q g^{(s/p+\alpha/\beta-1)(\beta/\alpha)}w^q\,\mathrm{d}\mu\right)^{\alpha/\beta}.\end{aligned}$$

Since $s/q=1-\alpha/\beta$ and $(s/p+\alpha/\beta-1)\beta/\alpha=s$ the last expression is bounded by

$$(Mg(x))^{s(x)/q(x)}\left(\int_Q g^sw^q\,\mathrm{d}\mu\right)^{\alpha/\beta}\leqslant (Mg(x))^{s(x)/q(x)}\left(\int_Q f^p\,\mathrm{d}\mu\right)^{\alpha/\beta}.$$

□

The following result gives a Welland type inequality in the context of lower Ahlfors measures. The proof in the euclidean setting is given in [15]. For more general measures see, for instance, [5].

Lemma 2.3. *Let $0<\alpha<\beta$ and $0<\varepsilon<\min\{\alpha,\beta-\alpha\}$. Then the inequality*

$$|I_\alpha f(x)|\leqslant C(M_{\alpha+\varepsilon}f(x)M_{\alpha-\varepsilon}f(x))^{1/2} \tag{2.5}$$

holds.

P r o o f. Let s be a positive number. We split I_α as follows

$$I_\alpha f(x) = \int_{|x-y|<s} \frac{|f(y)||x-y|^\alpha}{\mu(Q(x,2|x-y|))}\,\mathrm{d}\mu(y) + \int_{|x-y|\geqslant s} \frac{|f(y)||x-y|^\alpha}{\mu(Q(x,2|x-y|))}\,\mathrm{d}\mu(y) = I + II. \tag{2.6}$$

By using the property of the measure μ, for the first term we have

$$\begin{aligned} I &= \sum_{k=0}^{\infty} \int_{2^{-k-1}s\leqslant|x-y|<2^{-k}s} \frac{|f(y)||x-y|^\alpha}{\mu(Q(x,2|x-y|))}\,\mathrm{d}\mu(y) \\ &\leqslant s^\alpha \sum_{k=0}^{\infty} \frac{2^{-k\alpha}}{\mu(Q(x,2^{-k}s))} \int_{|x-y|<2^{-k}s} |f(y)|\,\mathrm{d}\mu(y) \\ &\leqslant s^\alpha \sum_{k=0}^{\infty} \frac{2^{-k\alpha}}{\mu(Q(x,2^{-k}s))^{(\alpha-\varepsilon)/\beta}} \frac{1}{\mu(Q(x,2^{-k}s))^{1-(\alpha-\varepsilon)/\beta}} \int_{|x-y|<2^{-k}s} |f(y)|\,\mathrm{d}\mu(y) \\ &\leqslant Cs^\varepsilon M_{\alpha-\varepsilon} f(x) \sum_{k=0}^{\infty} 2^{-k\varepsilon} \leqslant Cs^\varepsilon M_{\alpha-\varepsilon} f(x). \end{aligned}$$

For the second term we have

$$\begin{aligned} II &= \sum_{k=0}^{\infty} \int_{2^{k}s\leqslant|x-y|<2^{k+1}s} \frac{|f(y)||x-y|^\alpha}{\mu(Q(x,2|x-y|))}\,\mathrm{d}\mu(y) \\ &\leqslant s^\alpha \sum_{k=0}^{\infty} \frac{2^{(k+1)\alpha}}{\mu(Q(x,2^{k+1}s))} \int_{|x-y|<2^{k+1}s} |f(y)|\,\mathrm{d}\mu(y) \\ &\leqslant s^\alpha \sum_{k=0}^{\infty} \frac{2^{(k+1)\alpha}}{\mu(Q(x,2^{k+1}s))^{(\alpha+\varepsilon)/\beta}} \frac{1}{\mu(Q(x,2^{k+1}s))^{1-(\alpha+\varepsilon)/\beta}} \\ &\quad\times \int_{|x-y|<2^{k+1}s} |f(y)|\,\mathrm{d}\mu(y) \\ &\leqslant Cs^{-\varepsilon} M_{\alpha+\varepsilon} f(x) \sum_{k=0}^{\infty} 2^{-k\varepsilon} \leqslant Cs^{-\varepsilon} M_{\alpha+\varepsilon} f(x). \end{aligned}$$

Combining both estimates from (2.6) we obtain

$$I_\alpha f(x) \leqslant C(s^\varepsilon M_{\alpha-\varepsilon} f(x) + s^{-\varepsilon} M_{\alpha+\varepsilon} f(x))$$

and thus, by minimizing the expression in brackets in the inequality above as a function of s we obtain the desired result. $\square$

3. Proofs of the main results

We now proceed to prove our main results.

Proof of Theorem 1.1. It is enough to prove that the inequality

$$\|wM_\alpha(f/w)\|_{q(\cdot),\Omega^s_{r,\varepsilon}} \leqslant C\|f\|_{p(\cdot),\Omega}$$

holds for every function f such that $\|f\|_{p(\cdot),\Omega} \leqslant C$. But, by Lemma 2.2, it is enough to prove that

$$\|w(M(|f|^{p/s}w^{-q/s}))^{s/q}\|_{q(\cdot),\Omega^s_{r,\varepsilon}} \leqslant C\|f\|_{p(\cdot),\Omega}$$

which is equivalent to seing that the inequality $\varrho_q(\chi_{\Omega^s_{r,\varepsilon}} w(M(|f|^{p/s}w^{-q/s}))^{s/q}) \leqslant C$ holds whenever $\|f\|_{p(\cdot),\Omega} \leqslant C$.

Let $\tilde{s}(x) = s(x)/r$. Since $w^q \in A_r(\Omega)$, Hölder's inequality implies that

$$\begin{aligned}\varrho_{\tilde{s}}(\chi_{\Omega^s_{r,\varepsilon}}|f|^{p/s}w^{-q/s}) &= \int_{\Omega^s_{r,\varepsilon}} |f|^{p/r}w^{-q/r}\,\mathrm{d}\mu\\ &\leqslant C\Big(\int_\Omega |f|(x)^{p(x)}\,\mathrm{d}\mu(x)\Big)^{1/r}\Big(\int_\Omega w(x)^{-q(x)/(r-1)}\,\mathrm{d}\mu(x)\Big)^{1/r'}\\ &\leqslant C.\end{aligned}$$

Thus $\||f|^{p/s}w^{-q/s}\|_{\tilde{s}(\cdot),\Omega^s_{r,\varepsilon}} \leqslant C$. On the other hand, the hypothesis on the weight gives

$$\int_{\Omega-\Omega^s_{r,\varepsilon}} |f|^{p/s}w^{-q/s}\,\mathrm{d}\mu(x) \leqslant C\||f|^{p/s}\|_{s(\cdot),\Omega-\Omega^s_{r,\varepsilon}}\|w^{-q/s}\|_{(s(\cdot))',\Omega-\Omega^s_{r,\varepsilon}} \leqslant C.$$

Thus Theorem 2.1 can be applied by choosing $t(x) = \tilde{s}(x)$ and taking into account that $\Omega^s_{r,\varepsilon} = \Omega^{\tilde{s}}_{1,\varepsilon}$. Then we get

$$\begin{aligned}\varrho_q(\chi_{\Omega^s_{r,\varepsilon}} w(M(|f|^{p/s}w^{-q/s}))^{sq}) &= \int_{\Omega^s_{r,\varepsilon}} (M(|f|^{p/s}w^{-q/s})(x))^{s(x)}w(x)^{q(x)}\,\mathrm{d}\mu(x)\\ &= \int_{\Omega^s_{r,\varepsilon}} ((M(|f|^{p/s}w^{-q/s})(x))^{\tilde{s}(x)})^r w(x)^{q(x)}\,\mathrm{d}\mu(x)\\ &\leqslant C\int_{\Omega^s_{r,\varepsilon}} (1+M((|f|^{p/s}w^{-q/s})^{\tilde{s}(\cdot)}))^r w^q\,\mathrm{d}\mu\\ &\leqslant C + C\int_\Omega (M((|f|^{p/s}w^{-q/s})^{\tilde{s}(\cdot)}))^r w^q\,\mathrm{d}\mu.\end{aligned}$$

From the fact that $w^q \in A_{r-\delta}$, Marcinkiewicz interpolation theorem can be applied in order to obtain the boundedness of the maximal operator M in $L^r(w^q)$. Then, from the estimation above we get that

$$\varrho_q(\chi_{\Omega^s_{r,\varepsilon}} w(M(|f|^{p/s} w^{-q/s}))^{s/q}) \leqslant C + C\int_\Omega |f|^{p(x)}\,\mathrm{d}\mu(x) \leqslant C.$$

$\square$

Proof of Corollary 1.3. Without loss of generality we may assume that $\|f\|_{p(\cdot),\Omega} = 1$. Thus, we have to prove that $\|wM_\alpha(f/w)\|_{q(\cdot),\Omega} \leqslant C$.

Since p is a log-Hölder continuous exponent it is easy to check that so is q and $w(x)^{q(x)} \sim w(x)^{q(x_0)}$. In fact, this can be obtained from the property of continuity of q if $|x-x_0| \leqslant 1/2$. Otherwise the statement is immediate because the boundedness of Ω.

Since $n/q(x_0) < \beta < n/(p(x_0))'$ then $w^q \in A_{s(x_0)}(\mathbb{R}^n)$ and there exists a positive number η such that $w^q \in A_{s(x_0)-\eta}(\mathbb{R}^n)$; in particular, $w^q \in A_{s(x_0)-\eta}(\Omega)$. By virtue of the continuity of p and s two positive numbers δ and ε can be chosen satisfying $Q(x_0,\delta) \subset \Omega^s_{s(x_0)-\eta/2,\varepsilon}$ and $\beta(p(x))' < n$ for $x \in Q(x_0,\delta)$. Thus, we have

$$\begin{aligned}\|wM_\alpha(f/w)\|_{q(\cdot),\Omega} &\leqslant \|wM_\alpha(f/w)\|_{q(\cdot),Q(x_0,\delta)} + \|\chi_{\Omega\setminus Q(x_0,\delta)} wM_\alpha(f/w)\|_{q(\cdot),\Omega}\\ &\leqslant \|wM_\alpha(f/w)\|_{q(\cdot),\Omega^s_{s(x_0)-\eta/2,\varepsilon}} + \|\chi_{\Omega\setminus Q(x_0,\delta)} wM_\alpha(f/w)\|_{q(\cdot),\Omega}.\end{aligned}$$

It is easy to see that the hypotheses on the weight are satisfied as Ω is bounded and $\Omega \setminus \Omega^s_{s(x_0)-\eta/2,\varepsilon} \subset \{x \in \Omega\colon |x-x_0| > C\delta\}$. Thus Theorem 1.1 can be applied to estimate the first term.

If $\beta \leqslant 0$, w is bounded below for $x \in \Omega \setminus Q(x_0,\delta)$ and its reciprocal is bounded above in Ω and the boundedness of the second term follows by virtue of the non-weighted norm inequality for M_α. On the other hand, if $\beta > 0$, let Q be a cube such that $l(Q) \leqslant \delta/2$. Since $|y-x_0| \geqslant \delta/2$ when $y \in Q$ then

$$\frac{|x-x_0|^\beta}{|Q|^{1-\alpha/n}}\int_Q \frac{f(y)}{|y-x_0|^\beta}\,\mathrm{d}y \leqslant C(\operatorname{diam}\Omega)^\beta M_\alpha f(x).$$

On the other hand, if $l(Q) \geqslant \delta/2$ we have

$$\frac{|x-x_0|^\beta}{|Q|^{1-\alpha/n}}\int_Q \frac{f(y)}{|y-x_0|^\beta}\,\mathrm{d}y \leqslant (\operatorname{diam}\Omega)^\beta(I + II)$$

where

$$I = \frac{1}{|Q|^{1-\alpha/n}}\int_{Q\cap\{|y-x_0|\leqslant\delta/2\}} \frac{f(y)}{|y-x_0|^\beta}\,\mathrm{d}y$$

and

$$II = \frac{1}{|Q|^{1-\alpha/n}}\int_{Q\cap\{|y-x_0|\geqslant\delta/2\}} \frac{f(y)}{|y-x_0|^\beta}\,\mathrm{d}y.$$

It is easy to see that $II \leqslant C_\delta M_\alpha f(x)$. Thus we proceed to estimate I. Let $\bar{p} = (p_{Q(x_0,\delta/2)})_*$, by applying the Hölder inequality and taking into account that $\beta\bar{p}' < n$, we obtain that

$$
\begin{aligned}
I &\leqslant \frac{1}{|Q|^{1-\alpha/n}}\Big(\int_{Q\cap\{|y-x_0|\leqslant\delta/2\}}|f|^{\bar p}\Big)^{1/\bar p}\Big(\int_{\{|y-x_0|\leqslant\delta/2\}}|y-x_0|^{-\beta\bar p'}\Big)^{1/\bar p'}\\
&\leqslant C\Big(\frac{1}{|Q|}\int_{Q\cap\{|y-x_0|\leqslant\delta/2\}}|f|^{\bar p}\Big)^{1/\bar p}\\
&\leqslant C\Big(\frac{1}{|Q|}\int_{Q\cap\{|f|\leqslant 1\}}|f|^{\bar p}+C_\delta\int_{\{|f|\geqslant 1\}\cap\{\{|y-x_0|\leqslant\delta/2\}\}}|f|^{\bar p}\Big)^{1/\bar p}\\
&\leqslant \Big(C+C_\delta\int_{\{|f|\geqslant 1\}\cap\{\{|y-x_0|\leqslant\delta/2\}\}}|f|^{p(y)}\,\mathrm{d}y\Big)^{1/\bar p}\leqslant C
\end{aligned}
$$

where in the last inequality we have used that $\|f\|_{p,\Omega}=1$. Thus we have the pointwise inequality

$$w(x)M_\alpha(f/w)(x)\leqslant C+CM_\alpha f(x),$$

which allows us to obtain the desired result by using the non-weighted classical boundedness of M_α and the fact that Ω is bounded. □

P r o o f of Theorem 1.2. It is enough to prove that the inequality

$$\|wI_\alpha(f/w)\|_{q(\cdot),\Omega}\leqslant C\|f\|_{p(\cdot),\Omega}$$

holds for every function f such that $\|f\|_{p(\cdot),\Omega}\leqslant C$.

If we define $q^+(x)=2q_\varepsilon^+/q(x)$ and $q^-(x)=2q_\varepsilon^-/q(x)$ then $1/q^+(x)+1/q^-(x)=1$. By applying Welland's inequality and Young's inequality we obtain

$$
\begin{aligned}
(3.1)\quad &\int_\Omega\Big|I_\alpha\Big(\frac{f}{w}\Big)\Big|^q w^q\,\mathrm{d}\mu\\
&\leqslant C\Big(\int_\Omega\frac{1}{q^+}M_{\alpha+\varepsilon}\Big(\frac{f}{w}\Big)^{qq^+/2}w^{qq^+/2}\,\mathrm{d}\mu+\int_\Omega\frac{1}{q^-}M_{\alpha-\varepsilon}\Big(\frac{f}{w}\Big)^{qq^-/2}w^{qq^-/2}\,\mathrm{d}\mu\Big)\\
&\leqslant C\Big(\int_\Omega M_{\alpha+\varepsilon}\Big(\frac{f}{w}\Big)^{q_\varepsilon^+}w^{q_\varepsilon^+}\,\mathrm{d}\mu+\int_\Omega M_{\alpha-\varepsilon}\Big(\frac{f}{w}\Big)^{q_\varepsilon^-}w^{q_\varepsilon^-}\,\mathrm{d}\mu\Big).
\end{aligned}
$$

Now the desired inequality follows immediately because of the hypothesis on the weights and by virtue of Corollary 1.2. □

P r o o f of Theorem 1.3. Let us see that $w^\alpha\in A_1(Q_0)$. In fact, a positive constant τ can be chosen in such a way that, if $\mu(Q)\leqslant\tau$ then either Q does contain only one singularity or it does not contain any at all.

If Q contains no singularity it is easy to see that $w \cong C$ and then, from the fact that $w \geqslant 1$ we obtain the result. Now let x_i be the only singularity contained in Q. Since $1 < w(x) \leqslant |x - x_i|^{-\theta}$ and α satisfies a log-Hölder condition, by taking logarithms, we obtain

$$0 \leqslant |\alpha(x) - \alpha(x_i)| \log w \leqslant C$$

which allows us to immediately obtain that $w^{\alpha(x)} \cong w^{\alpha(x_i)}$ for almost every $x \in Q$. Thus the result follows easily whenever $\mu(Q)$ is small enough.

Let us now consider those cubes Q for which $\mu(Q) > \tau$. By a well-known property of Muckenhoupt classes there exists a positive number δ such that $w^{1+\delta} \in A_1(Q_0)$. Let us see that this number does work. Let α be a function as in the hypothesis. Since $1 < \alpha(x) < 1 + \delta$, for almost every x in Q, we have

$$\frac{w^{\alpha}(Q)}{\mu(Q)} \leqslant \frac{w^{1+\delta}(Q)}{\mu(Q)} \leqslant C\Big(\frac{w(Q)}{\mu(Q)}\Big)^{1+\delta} \leqslant Cw(\Omega)^{1+\delta} \leqslant Cw(x)^{\alpha(x)}.$$

□

P r o o f of Corollary 1.4. The thesis follows by observing that w satisfies the hypothesis in Theorem 1.2 in the context of the measure space (Q_0, μ), where Q_0 is a cube and μ is the Lebesgue measure. As $w_1,\ w_2 \in A_1$, there exist two positive numbers δ_1 and δ_2 such that both weights $w_1^{1+\delta_1}$ and $w_2^{1+\delta_2}$ also belong to that class. We choose $\delta = \min\{\delta_1, \delta_2\}$. Now let ε be a positive number as defined in Theorem 1.2 and let α_ε be the function defined by $\alpha_\varepsilon(x) = q_\varepsilon^+(x)/q_\varepsilon^-(x)$. If $\varepsilon < \delta\beta/((2+\delta)q^*)$, we then have that $1 < \alpha_\varepsilon < 1 + \delta$. Moreover, since q has the log-Hölder property, it is easy to see that so does α_ε. Thus $w^{q_\varepsilon^-} \in A_{(s_\varepsilon^-)_*}$ since $w_1 w_2^{1-(s_\varepsilon^-)_*} \in A_{(s_\varepsilon^-)_*}$. Moreover Theorem 1.3 can now be applied to conclude that $w^{q_\varepsilon^+} = w_1^{\alpha_\varepsilon} w_2^{\alpha_\varepsilon(1-(s_\varepsilon^-)_*)} \in A_{(s_\varepsilon^-)_*}$. By virtue of the monotonic character of the Muckenhoupt classes it is also true that $w^{q_\varepsilon^+} \in A_{(s_\varepsilon^+)_*}$ and we are done. □

We finally obtain a family of weights in the $A_{p(\cdot)}(\Omega)$ class where Ω has been equipped with a measure μ that fails to have the standard doubling property.

Let $X_1 = \{(x, x),\ x \in (0, 1)\}$, $X_2 = (-1, 0)^2$ and $\Omega = X_1 \cup X_2$. If Ω is any cube containing Ω and μ_i is the i-dimensional Lebesgue measure for $i = 1, 2$, let μ be the measure supported in Ω and defined by $\mu = \mu_i$ in X_i for $i = 1, 2$. It is easy to prove that μ is lower Ahlfors 2-regular. If $1 < p < \infty$, let w be the weight defined in Ω by

$$w(x, y) = \begin{cases} x^{\alpha} & \text{if } (x, y) \in X_1, \\ |xy|^{\alpha} & \text{if } (x, y) \in X_2, \end{cases}$$

with $-1 < \alpha < p - 1$. Then w belongs to the class $A_p(\Omega)$. In fact, if Q is a cube contained in Ω, it might happen that Q is a proper subset of X_1 or of X_2, otherwise

Q intersects both of them. In the first two cases the statement follows immediately from the A_p conditions for the 1-dimensional and 2-dimensional Lebesgue measure respectively.

Let us prove the remaining case. Given $b \leqslant a < 0$, and $l > 0$ such that $0 < b+l < a+l$ let $Q = (a, a+l) \times (b, b+l)$. Let $I = (a, 0)$, $J = (b, 0)$ and $K = (0, b+l)$ thus $Q \cap \Omega = (I \times J) \cup \{(x, x),\, x \in K\}$ and $\mu(Q) = \mu(Q \cap \Omega) = \mu_2(I \times J) + \sqrt{2}\mu_1(K)$. By the definition of w and Tonelli's theorem we obtain

$$\begin{aligned}
&\Big(\frac{1}{\mu(Q)}\int_Q w \,\mathrm{d}\mu\Big)\Big(\frac{1}{\mu(Q)}\int_Q w^{-1/(p-1)} \,\mathrm{d}\mu\Big)^{p-1} \\
&\quad \leqslant \frac{1}{\mu_1(I)^p}\Big(\int_I |x|^{\alpha} \,\mathrm{d}x\Big)\Big(\int_I |x|^{\alpha/(1-p)} \,\mathrm{d}x\Big)^{p-1} \\
&\quad\quad \times \frac{1}{\mu_1(J)^p}\Big(\int_J |x|^{\alpha} \,\mathrm{d}x\Big)\Big(\int_J |x|^{\alpha/(1-p)} \,\mathrm{d}x\Big)^{p-1} \\
&\quad\quad + \frac{1}{\mu_1(K)^p}\Big(\int_K |x|^{\alpha} \,\mathrm{d}x\Big)\Big(\int_K |x|^{\alpha/(1-p)} \,\mathrm{d}x\Big)^{p-1} \\
&\quad\quad + C\Big(\frac{\mu_1(I)\mu_1(J)}{\mu_1(I)\mu_1(J) + \sqrt{2}\mu_1(K)}\Big)^{\alpha+1}\Big(\frac{\sqrt{2}\mu_1(K)}{\mu_1(I)\mu_1(J) + \sqrt{2}\mu_1(K)}\Big)^{p-\alpha-1} \\
&\quad\quad + C\Big(\frac{\mu_1(I)\mu_1(J)}{\mu_1(I)\mu_1(J) + \sqrt{2}\mu_1(K)}\Big)^{p-\alpha-1}\Big(\frac{\sqrt{2}\mu_1(K)}{\mu_1(I)\mu_1(J) + \sqrt{2}\mu_1(K)}\Big)^{\alpha+1} \\
&\quad \leqslant C
\end{aligned}$$

where we have used the one dimensional A_p inequality for both the first and the second terms and the range of α for the boundedness of the last two terms.

References

[1] *A. Almeida, S. Samko*: Fractional and hypersingular operators in variable exponent spaces on metric measure spaces. Mediterr. J. Math. *6* (2009), 215–232.

[2] *C. Capone, D. Cruz-Uribe, A. Fiorenza*: The fractional maximal operator and fractional integrals on variable L^p spaces. Rev. Mat. Iberoam. *23* (2007), 743–770.

[3] *D. Cruz-Uribe, A. Fiorenza, C. J. Neugebauer*: The maximal function on variable L^p spaces. Ann. Acad. Sci. Fenn., Math. *28* (2003), 223–238.

[4] *L. Diening*: Maximal function on generalized Lebesgue spaces $L^{p(\cdot)}$. Math. Inequal. Appl. *7* (2004), 245–254.

[5] *J. García Cuerva, J. M. Martell*: Two-weight norm inequalities for maximal operators and fractional integrals on non-homogeneous spaces. Indiana Univ. Math. J. *50* (2001), 1241–1280.

[6] *P. Harjulehto, P. Hästö, V. Latvala*: Sobolev embeddings in metric measure spaces with variable dimension. Math. Z. *254* (2006), 591–609.

[7] *P. Harjulehto, P. Hästö, M. Pere*: Variable exponent Lebesgue spaces on metric spaces: The Hardy-Littlewood maximal operator. Real Anal. Exch. *30* (2004/2005), 87–104.

[8] *V. Kokilashvili, S. Samko*: Maximal and fractional operators in weighted $L^{p(x)}$ spaces. Rev. Mat. Iberoam. 20 (2004), 493–515.
[9] *O. Kováčik, J. Rákosník*: On spaces $L^{p(x)}$ and $W^{k,p(x)}$. Czechoslovak Math. J. *41(116)* (1991), 592–618.
[10] *J. Musielak*: Orlicz spaces and Modular spaces. Lecture Notes in Math. Vol. 1034. Springer, Berlin, 1983.
[11] *B. Muckenhoupt, R. L. Wheeden*: Weighted norm inequalities for fractional integrals. Trans. Am. Math. Soc. *192* (1974), 261–274.
[12] *L. Pick, M. Růžička*: An example of a space $L^{p(x)}$ on which the Hardy-Littlewood maximal operator is not bounded. Expo. Math. *4* (2001), 369–372.
[13] *M. Růžička*: Electrorheological Fluids: Modeling and Mathematical Theory. Lecture Notes in Math. Vol. 1748. Springer, Berlin, 2000.
[14] *S. Samko*: Hardy-Littlewood-Stein-Weiss inequality in the Lebesgue spaces with variable exponent. Fract. Calc. Appl. Anal. *6* (2003), 421–440.
[15] *G. V. Welland*: Weighted norm inequalities for fractional integrals. Proc. Am. Math. Soc. *51* (1975), 143–148.

Authors' addresses: Osvaldo Gorosito, Facultad de Ingeniería Química de la Universidad Nacional del Litoral, Santiago del Estero 2829, Santa Fe (3000), Argentina, e-mail: ogoro@fiq.unl.edu.ar; Gladis Pradolini, Consejo Nacional de Investigaciones Científicas y Técnicas, Facultad de Ingeniería Química de la Universidad Nacional del Litoral, Güemes 3450, Santa Fe (3000), Argentina, e-mail: gladis.pradolini@gmail.com; Oscar Salinas, Consejo Nacional de Investigaciones Científicas y Técnicas, Facultad de Ingeniería Química de la Universidad Nacional del Litoral, Güemes 3450, Santa Fe (3000), Argentina, e-mail: salinas@santafe-conicet.gov.ar.