Junichi Aramaki

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ESTIMATE OF THE HAUSDORFF MEASURE OF THE SINGULAR SET OF A SOLUTION FOR A SEMI-LINEAR ELLIPTIC EQUATION ASSOCIATED WITH SUPERCONDUCTIVITY

Junichi Aramaki

Abstract. We study the boundedness of the Hausdorff measure of the singular set of any solution for a semi-linear elliptic equation in general dimensional Euclidean space $\mathbb{R}^n$. In our previous paper, we have clarified the structures of the nodal set and singular set of a solution for the semi-linear elliptic equation. In particular, we showed that the singular set is $(n-2)$-rectifiable. In this paper, we shall show that under some additive smoothness assumptions, the $(n-2)$-dimensional Hausdorff measure of singular set of any solution is locally finite.

1. Introduction

We consider a semi-linear elliptic equation

$$-\nabla_{\boldsymbol{A}}^2\psi = f(|\psi|^2)\psi \quad \text{in} \quad \Omega \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ and f is a real-valued, bounded function on $\overline{\mathbb{R}}_+ = [0,\infty)$. Here $\boldsymbol{A}$ is a real vector-valued function (called magnetic potential), ψ is a complex-valued function. $\nabla_{\boldsymbol{A}}$ and $\nabla_{\boldsymbol{A}}^2$ are defined by $\nabla_{\boldsymbol{A}} = \nabla - i\boldsymbol{A}$, ∇ is the gradient operator and

$$\nabla_{\boldsymbol{A}}^2\psi = \Delta\psi - i[2\boldsymbol{A}\cdot\nabla\psi + (\operatorname{div}\boldsymbol{A})\psi] - |\boldsymbol{A}|^2\psi\,.$$

This type of operator is considered in Aramaki [1, 2, 3, 5] and Pan and Kwek [24].

Associated with the magnetic potential $\boldsymbol{A} = (A_1, A_2, \dots, A_n)$, define an anti-symmetric $n\times n$ matrix $B = (B_{ij})$ called the magnetic vector field by

$$B_{ij} = \partial_{x_i}A_j - \partial_{x_j}A_i \quad \text{for} \quad i,j = 1,2,\dots,n\,.$$

Let us recall that superconductivity in two or three dimensional space can be described by a pair $(\psi,\boldsymbol{A})$, where ψ is a complex-valued function called the order parameter and $\boldsymbol{A}$ is a real vector-valued function called the magnetic potential,

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which is a minimizer of the so-called Ginzburg-Landau functional. The Euler equation becomes

$$(1.2)\qquad \begin{cases} -\nabla_{\boldsymbol{A}}^2\psi = \kappa^2(1-|\psi|^2)\psi & \text{in} \quad \Omega\,, \\ \boldsymbol{\nu}\cdot\nabla_{\boldsymbol{A}}\psi = 0 & \text{on} \quad \partial\Omega \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ with $n = 2$ or $n = 3$ is a bounded domain and $\boldsymbol{\nu}$ is the outer unit normal vector at $\partial\Omega$. It is well known that any solution of (1.2) satisfies $|\psi| \le 1$ in Ω. If we choose a bounded function f on $[0, \infty)$ so that $f(t) = \kappa^2(1-t)$ for $|t| \le 1$, the first equation of (1.2) is of the form (1.1).

In the superconductivity theory or the Landau-de Gennes model of liquid cristal, it is important to know the third critical field H_{c_3} or Q_{c_3}. It is associated with the lowest eigenvalue of the magnetic Schrödinger operator of type $-\nabla_{q\boldsymbol{A}}^2$, i.e.,

$$(1.3)\qquad \begin{cases} -\nabla_{q\boldsymbol{A}}^2\psi = \mu(q\boldsymbol{A})\psi & \text{in} \quad \Omega\,, \\ \boldsymbol{\nu}\cdot\nabla_{q\boldsymbol{A}}\psi = 0 & \text{on} \quad \partial\Omega\,. \end{cases}$$

If we put $f(t) = \mu(q\boldsymbol{A})$ which is a constant, the first equation of (1.3) is also of form (1.1). For the superconductivity theory, see Lu and Pan [17], [18] and Pan [22]. For the theory of liquid cristal with $\boldsymbol{A} = \boldsymbol{n}$ which is a unit vector field, see Pan [21]. Helffer and Mohamed [15] and Helffer and Morame [16] have extensively considered the eigenvalue problem for the magnetic Schrödinger operator of type $-\nabla_{\boldsymbol{A}}^2$ for $n \ge 2$.

In the equation (1.2), the nodal set $\{x \in \Omega\,;\, \psi(x) = 0\}$ means the normal state there. Pan [23] has studied the structure of the nodal set and the singular set $\{x \in \Omega\,;\, \psi(x) = 0, \nabla\psi(x) = 0\}$ of any non-trivial solution of (1.1) in the three dimensional domain.

In the previous paper Aramaki [4], we showed that the nodal set and the singular set of any non-trivial solution of (1.1) in the general n dimensional domain are $(n-1)$ and $(n-2)$-rectifiable, respectively.

For the second order linear elliptic equations with the real coefficients, there are many articles on the nodal set or the singular set. For example, see Garofalo and Lin [9], Han [11], [12] and Han et al. [13]. In particular, Hardt et al. [14] proved that for any non-trivial solution of a linear elliptic equation with real smooth coefficients, the $(n-2)$-dimensional Hausdorff measure of the singular set is locally finite.

However it seems that there are not many articles on the structure of the singular set of complex-valued solutions of equations of type (1.1) (cf. Elliot et al. [7]).

In this paper, we shall estimate the $(n-2)$-dimensional Hausdorff measure of singular set of any non-trivial complex-valued solution ψ of (1.1).

We assume that

(H) $\boldsymbol{A} \in L^\infty(\Omega;\mathbb{R}^n), \operatorname{div}\boldsymbol{A} \in L^q_{\rm loc}(\Omega)$ with $q > n/2$ if $n \ge 4$ and $q \ge 2$ if $n = 3$, and $B \in L^\infty(\Omega;\mathbb{R}^{n^2})$.

Our main result on the singular set is the following.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$ $(n \ge 3)$ be a bounded domain, assume that the hypothesis* (H) *holds, and let $\psi \in W^{1,2}_{\rm loc}(\Omega;\mathbb{C})$ be any non-trivial complex-valued weak solution of* (1.1) *with $\nabla_{\boldsymbol{A}}\psi \in W^{1,2}_{\rm loc}(\Omega;\mathbb{C}^n)$ and $f_\infty := \|f(|\psi|^2)\|_{L^\infty(\Omega)} < \infty$.*

Then there exists an integer $M > 0$ depending on ψ, f_∞ and $\|B\|_{L^\infty(\Omega)}$ such that if, in addition, $\boldsymbol{A} \in C^M(\Omega;\mathbb{R}^n)$, $\operatorname{div}\boldsymbol{A} \in C^M(\Omega)$ and $f \in C^M([0,\infty))$, then for any $\Omega' \Subset \Omega$, there exists a constant $C > 0$ depending on M, ψ, f_∞, $\|\boldsymbol{A}\|_{L^\infty(\Omega)}$, Ω', $C^M(\Omega'')$ norms of $\boldsymbol{A}$, $\operatorname{div}\boldsymbol{A}$, $f(|\psi|^2)$ for some $\Omega' \Subset \Omega'' \Subset \Omega$ such that

$$\mathcal{H}^{n-2}\big(\{x \in \Omega'; \psi(x) = 0, \nabla\psi(x) = 0\}\big) \le C \tag{1.4}$$

where $\mathcal{H}^{n-2}$ is the $(n-2)$-dimensional Hausdorff measure.

2. Preliminaries

In this section, we shall list up some propositions which are needed later and held under the hypothesis (H). All the propositions and theorem are found in [4] (c.f. [23]).

At first, we have the regularity of the solution.

Proposition 2.1. *Assume that the hypothesis* (H) *holds and let $\psi \in W^{1,2}_{\mathrm{loc}}(\Omega;\mathbb{C})$ be any weak solution of* (1.1)*. Then $\psi \in W^{2,q}_{\mathrm{loc}}(\Omega;\mathbb{C}) \cap C^\alpha_{\mathrm{loc}}(\Omega;\mathbb{C})$ for some $\alpha \in (0,1)$, and for any $B_{2R}(x_0) \Subset \Omega$ and $1 < p \le q$, there exists a constant $C > 0$ depending on p, q and $\|\boldsymbol{A}\|_{L^\infty(\Omega)}$ such that*

$$\begin{aligned} &R^2\|D^2\psi\|_{L^p(B_R(x_0))} + R\|\nabla\psi\|_{L^p(B_R(x_0))} \\ &\qquad \le C\big\{\|\psi\|_{L^p(B_R(x_0))} + R^2\|f(|\psi|^2)\psi - i(\operatorname{div}\boldsymbol{A})\psi\|_{L^p(B_R(x_0))}\big\}\,. \end{aligned}$$

Next, we state the doubling property of solutions. Let $\psi \not\equiv 0$ be any weak solution of (1.1). For any $B_r(x_0) \Subset \Omega$, we define some quantities.

$$\begin{aligned} I(x_0,r) &= \int_{B_r(x_0)} \{|\nabla_{\boldsymbol{A}}\psi|^2 - f(|\psi|^2)|\psi|^2\}\,dx\,, \\ H(x_0,r) &= \int_{\partial B_r(x_0)} |\psi|^2\,dS_r\,, \quad D(x_0,r) = \int_{B_r(x_0)} |\nabla\psi|^2\,dx\,, \\ M(x_0,r) &= \frac{rI(x_0,r)}{H(x_0,r)}\,, \quad N(x_0,r) = \frac{rD(x_0,r)}{H(x_0,r)} \quad \text{if} \quad H(x_0,r) \ne 0 \end{aligned} \tag{2.1}$$

where dS_r denotes the surface area of $\partial B_r(x_0)$. Then we have

Proposition 2.2. *Assume that the conditions of Theorem 1.1 hold for any non-trivial weak solution $\psi \in W^{1,2}(\Omega;\mathbb{C})$. Then there exist $r_0, c_0, N > 0$ where r_0 depends only on f_∞, and c_0 and N depend only on Ω, ψ, f_∞ and $\|B\|_{L^\infty(\Omega)}$ such that for any $0 < r \le r_0/2$ with $B_{2r}(x_0) \Subset \Omega$, we have the following.*

(i) $M(x_0,r) \le c_0$,

(ii)

$$\int_{B_r(x_0)} |\psi|^2\,dx \le r\int_{\partial B_r(x_0)} |\psi|^2\,dS\,,$$

and the doubling property:

(iii)

$$\int_{B_{2r}(x_0)} |\psi|^2\,dx \le 4^N \int_{B_r(x_0)} |\psi|^2\,dx\,.$$

If $\psi \not\equiv 0$ is a weak solution of (1.1) and satisfies that $\int_{\partial B_r(x_0)} |\psi|^2 dS_r = 0$ for some $0 < r \le r_0/2$ with $B_{2r}(x_0) \Subset \Omega$, it follow from (ii) that $\int_{B_r(x_0)} |\psi|^2 dx = 0$. Therefore, from the unique continuation theorem (cf. Aronszajn [6]) or the doubling property (iii) (cf. [9]), $\psi \equiv 0$ in Ω. Thus we have

$$H(x_0, r) = \int_{\partial B_r(x_0)} |\psi|^2 \, dS_r \neq 0$$

for any $0 < r \le r_0/2$ with $B_{2r}(x_0) \Subset \Omega$, and so we see that $M(x_0, r)$ and $N(x_0, r)$ are well defined. From Proposition 2.2 (i) we see that $M(x_0, r) \le c_0$ for any $0 < r \le r_0/2$ with $B_{2r}(x_0) \Subset \Omega$.

We get an important fact.

Proposition 2.3 ([4] or [23]). *Assume that the conditions of Theorem 1.1 for any non-trivial weak solution $\psi \in W^{1,2}_{\mathrm{loc}}(\Omega; \mathbb{C})$ of* (1.1)*. Then we have*

$$\lim_{r\to 0} M(x_0, r) = \lim_{r \to 0} N(x_0, r) \quad \textit{for any } x_0 \in \Omega \tag{2.2}$$

and the limit is a non-negative integer.

From this proposition, we can define the vanishing order of ψ at $x_0 \in \Omega$ by

$$\mathcal{O}_\psi(x_0) = \lim_{r\to 0} M(x_0, r) = \lim_{r\to 0} N(x_0, r)\,.$$

Of course, if ψ is smooth enough, we see that

$$\begin{cases} D^\alpha \psi(x_0) = 0 & \text{for any } \alpha \ \text{ with } \ |\alpha| < \mathcal{O}_\psi(x_0)\,, \\ D^\beta \psi(x_0) \neq 0 & \text{for some } \beta \ \text{ with } |\beta| = \mathcal{O}_\psi(x_0) \end{cases}$$

where $D^\alpha = \partial^\alpha = \partial^{|\alpha|}/\partial x^\alpha$.

We note that the vanishing order of ψ is uniformly bounded in Ω, i.e.,

$$\mathcal{O}_\psi(x) \le c_0 \quad \text{for} \quad x \in \Omega \tag{2.3}$$

where c_0 is the constant as in Proposition 2.2 (i) and depends only on ψ, Ω, f_∞ and $\|B\|_{L^\infty(\Omega)}$.

Next, we state the decomposition of the solution of (1.1).

Proposition 2.4 (cf. [4], [23] and [11]). *Assume that the conditions of Theorem 1.1 hold for any non-trivial weak solution ψ of* (1.1)*. Then for any $0 < R \le r_0/2$ with $B_{2R}(x_0) \Subset \Omega$, there exists an integer $m \ge 0$ such that we can write*

$$\psi(x + x_0) = P_m(x) + \phi(x)\,, \quad x \in B_R(0) \tag{2.4}$$

where P_m is a non-zero, complex-valued homogeneous, harmonic polynomial of degree m, and ϕ satisfies

$$|\phi(x)| \le C|x|^{m+\alpha} \quad \textit{in} \quad B_R(0) \tag{2.5}$$

for some $\alpha \in (0,1)$, and a constant $C > 0$ which depends only on m, $\|\boldsymbol{A}\|_{L^\infty(\Omega)}$, $\|\mathrm{div}\, \boldsymbol{A}\|_{L^2(B_{2R}(x_0))}$ and f_∞.

We call $P_m = \psi_{x_0}$ the leading polynomial of ψ at x_0. We see that m is the vanishing order of ψ at x_0, so $m = \mathcal{O}_\psi(x_0) \le c_0$. Now we define the singular set of ψ by

$$\mathcal{S}(\psi) = \{x \in \Omega; \mathcal{O}_\psi(x) \ge 2\}\,.$$

In the previous paper [4], we showed

Theorem 2.5. *Assume the the conditions of Theorem 1.1 hold for any non-trivial weak solution $\psi \in W^{1,2}(\Omega;\mathbb{C})$. Then for any $\Omega' \Subset \Omega$, $\mathcal{S}(\psi) \cap \Omega'$ is countably $(n-2)$-rectifiable, more precisely, if we define*

$$\mathcal{S}_*(\psi) = \{x \in \mathcal{S}(\psi);\ \textit{the leading polynomial of } \psi \textit{ at } x \textit{ is a polynomial of two variables after some rotation of coordinates}\}\,,$$

then $\mathcal{S}(\psi) \cap \Omega' \setminus \mathcal{S}_*(\psi)$ is countably $(n-3)$-rectifiable. Thus

$$\mathcal{H}^{n-2}(\mathcal{S}(\psi) \cap \Omega' \setminus \mathcal{S}_*(\psi)) = 0\,. \tag{2.6}$$

3. Estimate of the singular set

In this section, we shall estimate the Hausdorff measure of the singular set of any non-trivial weak solution of (1.1). In addition to the hypothesis (H), we assume that for an integer $M \ge 1$,

$(\mathrm{K})_M$ $\quad \boldsymbol{A} \in C^M(\Omega;\mathbb{R}^n), \operatorname{div}\boldsymbol{A} \in C^M(\Omega)$ and $f \in C^M([0,\infty))$.

In the following, for any given $\Omega' \Subset \Omega$, we always choose

$$\Omega'' = \{x \in \Omega; \operatorname{dist}(x,\partial\Omega) > \min(r_0, \operatorname{dist}(\Omega',\partial\Omega))/3\}\,,$$

where r_0 is as in Proposition 2.2, and define

$$\Lambda(\Omega'') = |\boldsymbol{A}|_{C^M(\Omega'')} + |\boldsymbol{A}|^2_{C^M(\Omega'')} + |\operatorname{div}\boldsymbol{A}|_{C^M(\Omega'')} + |f(|\psi|^2)|_{C^M(\Omega'')}\,, \tag{3.1}$$

if $\psi \in C^M_{\mathrm{loc}}(\Omega;\mathbb{C})$. We also use the notations

$$|D^j\psi(x)| = \sum_{|\beta|=j} |D^\beta_x \psi(x)|,$$

$$|D^j\psi(x) - D^j\psi(y)| = |D^j(\psi(x)-\psi(y))|\,.$$

At first, we obtain the regularity of any solution of (1.1) under the hypotheses (H) and $(\mathrm{K})_M$.

Proposition 3.1 (Regularity)**.** *Addition to the hypothesis* (H)*, assume that* $(\mathrm{K})_M$ *holds for some integer $M \ge 1$. Let $\psi \in W^{1,2}_{\mathrm{loc}}(\Omega;\mathbb{C})$ be any weak solution of* (1.1)*. Then $\psi \in C^{M+1,\alpha}_{\mathrm{loc}}(\Omega;\mathbb{C})$ for some $\alpha \in (0,1)$. Moreover, we have the Schauder type estimate: for any $\Omega' \Subset \Omega$, there exists $R_0 > 0$ depending on n, M, α, Ω and $\Lambda(\Omega'')$ such that for all $0 < R \le R_0$ and $x_0 \in \Omega'$,*

$$\sum_{j=1}^{M+1} R^j \sup_{x\in B_R(x_0)} |D^j\psi(x)| + R^{M+1+\alpha} \sup_{\substack{x,y\in B_R(x_0)\\ x\ne y}} \frac{|D^{M+1}\psi(x) - D^{M+1}\psi(y)|}{|x-y|^\alpha} \le C \sup_{x\in B_{2R}(x_0)} |\psi(x)|$$

where the constant C depends on n, M, α, Ω, Ω' and $\Lambda(\Omega'')$.

Proof. By Proposition 2.1, we see that $\psi \in C^{\alpha}_{\rm loc}(\Omega;\mathbb{C}) \cap W^{2,q}_{\rm loc}(\Omega;\mathbb{C})$ for some $\alpha \in (0,1)$ and ψ satisfies the equation (1.1). We note that $\boldsymbol{A} \in C^M(\Omega;\mathbb{R}^n) \hookrightarrow C^{M-1,\alpha}_{\rm loc}(\Omega;\mathbb{R}^n)$. Similarly, $\operatorname{div}\boldsymbol{A}$, $|\boldsymbol{A}|^2$ belong to $C^{M-1,\alpha}_{\rm loc}(\Omega)$. Since $f \in C^M(\overline{\mathbb{R}}_+) \hookrightarrow C^{M-1,1}_{\rm loc}(\overline{\mathbb{R}}_+)$, we have $f(|\psi|^2)\psi \in C^{\alpha}_{\rm loc}(\Omega;\mathbb{C})$. Therefore, it follows from Gilbarg and Trudinger [10, Theorem 9.19] that $\psi \in C^{2,\alpha}_{\rm loc}(\Omega;\mathbb{C})$. By the boot-strap method, we see that $\psi \in C^{M+1,\alpha}_{\rm loc}(\Omega;\mathbb{C})$.

Next, we shall get the estimate. In order to do so, we write (1.1) into the form:

$$-\Delta\psi + 2i\boldsymbol{A}\cdot\nabla\psi + \big(i(\operatorname{div}\boldsymbol{A}) + |\boldsymbol{A}|^2\big)\psi = f(|\psi|^2)\psi \quad \text{in} \quad \Omega\,. \tag{3.2}$$

We simply write $3R_1 = \min\big(r_0, \operatorname{dist}(\Omega',\partial\Omega)\big)$ and we choose Ω'' as above. Then for any $0 < R < R_1$ and $x_0 \in \Omega'$, $B_{2R}(x_0) \subset \Omega''$. We shall apply the Schauder estimate [10, p.142] in $B_{2R}(x_0)$. We use the following notations as in [10]. For $g \in C^{k,\alpha}\big(B_r(x_0)\big)$,

$$|g|^{(\sigma)}_{k,\alpha,B_r(x_0)} = |g|^{(\sigma)}_{k,B_r(x_0)} + [g]^{(\sigma)}_{k,\alpha,B_r(x_0)}\,,$$

where

$$\begin{aligned}
|g|^{(\sigma)}_{k,B_r(x_0)} &= \sum_{j=0}^{k} \sup_{x\in B_r(x_0)} d_x^{j+\sigma}|D^j g(x)|\,,\\
[g]^{(\sigma)}_{k,\alpha,B_r(x_0)} &= \sup_{\substack{x,y\in B_r(x_0)\\ x\neq y}} d_{x,y}^{k+\alpha+\sigma}\frac{|D^k g(x) - D^k g(y)|}{|x-y|^{\alpha}}\,,\\
|g|^{*}_{k,\alpha,B_r(x_0)} &= |g|^{(0)}_{k,\alpha,B_r(x_0)}\,,\\
d_x = \operatorname{dist}\,&\big(x,\partial B_r(x_0)\big)\,,\quad d_{x,y} = \min(d_x,d_y)\,.
\end{aligned}$$

It follows from the hypothesis $(\mathrm{K})_M$ that

$$\begin{aligned}
|\boldsymbol{A}|^{(1)}_{M-1,\alpha,B_{2R}(x_0)} + |\operatorname{div}\boldsymbol{A}|^{(2)}_{M-1,\alpha,B_{2R}(x_0)} + ||\boldsymbol{A}|^2|^{(2)}_{M-1,\alpha,B_{2R}(x_0)}\\
\le C\Lambda(\Omega'') < \infty
\end{aligned}$$

for some constant C depending only on Ω. Therefore we can apply the result of [10] to (3.2) to get

$$|\psi|^{*}_{M+1,\alpha,B_{2R}(x_0)} \le C\big(|\psi|_{0,B_{2R}(x_0)} + |f(|\psi|^2)\psi|^{(2)}_{M-1,\alpha,B_{2R}(x_0)}\big) \tag{3.3}$$

where C depends on n, M, α, Ω and $\Lambda(\Omega'')$.

We estimate the last term in the right hand side of (3.3). Since $\psi \in C^{M+1,\alpha}(B_{2R}(x_0)) \subset C^{M+1,\alpha}(\overline{\Omega''})$ $(M \geq 1)$, we have

$$\sum_{j=0}^{M-1} \sup_{x \in B_{2R}(x_0)} d_x^{j+2} |D^j[f(|\psi(x)|^2)\psi(x)]| \leq C \sum_{j=0}^{M-1} \sup_{x \in B_{2R}(x_0)} d_x^{j+2} |D^j \psi(x)|$$
$$\leq C_1 R^2 \sum_{j=0}^{M-1} \sup_{x \in B_{2R}(x_0)} d_x^{j} |D^j \psi(x)|$$

where the constant C_1 depends on $\Lambda(\Omega'')$. Similarly we can estimate

$$\sup_{\substack{x,y \in B_{2R}(x_0)\\ x \neq y}} d_{x,y}^{M+1+\alpha} \frac{|D^{M-1}[f(|\psi(x)|^2)\psi(x)] - D^{M-1}[f(|\psi(y)|^2)\psi(y)]|}{|x-y|^\alpha}$$
$$= \sup_{\substack{x,y \in B_{2R}(x_0)\\ x \neq y}} d_{x,y}^{M+1+\alpha} \frac{|D^{M-1}[f(|\psi(x)|^2)\psi(x)] - D^{M-1}[f(|\psi(y)|^2)\psi(y)]|}{|x-y|}$$
$$\times |x-y|^{1-\alpha}$$
$$\leq C_2 \sum_{j=0}^{M} \sup_{x,y \in B_{2R}(x_0)} \sup_{z \in B_{2R}(x_0)} d_{x,y}^{M+1+\alpha} |D^j \psi(z)| |x-y|^{1-\alpha}$$
$$\leq C_3 R^2 \sum_{j=0}^{M} \sup_{z \in B_{2R}(x_0)} d_z^{j} |D^j \psi(z)|$$

where the constant C_3 also depends on $\Lambda(\Omega'')$. Thus we see that

$$|f(|\psi|^2)\psi|^{(2)}_{M-1,\alpha,B_{2R}(x_0)} \leq C_4 R^2 |\psi|^*_{M+1,\alpha,B_{2R}(x_0)}$$

where C_4 depends on $\Lambda(\Omega'')$. If we choose $R_0 > 0$ so that $CC_4R_0^2 < 1/2$ where C is as in (3.3) and $R_0 < R_1$, then it follows from (3.3) that for all $0 < R \leq R_0$

$$|\psi|^*_{M+1,\alpha,B_{2R}(x_0)} \leq C|\psi|_{0,B_{2R}(x_0)}$$

where the constant C depends on n, M, α, Ω and $\Lambda(\Omega'')$. Since $d_x = \mathrm{dist}\,(x, \partial B_{2R}(x_0)) \geq R$ for $x \in B_R(x_0)$, we obtain the conclusion. □

We choose an integer $M \geq 1$ in Theorem 1.1 so that

$$M \geq 2c_0^2 \tag{3.4}$$

where c_0 is the constant as in Proposition 2.2 (i). We note that it follows from (2.3) that the vanishing order of ψ is uniformly bounded in Ω : $\mathcal{O}_\psi(x) \leq c_0$ for all $x \in \Omega$.

Let ψ be any non-trivial weak solution ψ of (1.1) and $\Omega' \Subset \Omega$. Then for all $x_0 \in \Omega'$ and $0 < R < R_0$ where R_0 is as in Proposition 3.1, ψ has a decomposition in $B_{2R}(x_0) \Subset \Omega''$:

$$\psi(x + x_0) = P_m(x) + \phi(x), \quad x \in B_R(0) \tag{3.5}$$

where P_m is a non-zero complex-valued homogeneous, harmonic polynomial of degree m and ϕ satisfies (2.5).

We estimate the remainder term ϕ.

Lemma 3.2. *Assume that the hypotheses* (H) *and* $(\mathrm{K})_M$ *hold. Then in the decomposition of* ψ *in* (3.5), ϕ *satisfies*

$$|D^j\phi(x)| \le \begin{cases} CR^{m-j+\alpha}, & j = 0, 1, \dots, m, \\ C, & j = m+1, \dots, M+1 \end{cases}$$

in $B_R(0)$ *where the constant* C *depends on* n, M, ψ, Ω *and* $\Lambda(\Omega'')$.

Proof. Since P_m is harmonic in $\mathbb{R}^n$, we have

$$\Delta\phi(x) = -\Delta\psi(x + x_0) \quad \text{in} \quad B_R(0).$$

Therefore we can apply the Schauder estimate as in the proof of Proposition 3.1, so we can get

$$\begin{aligned} &\sum_{j=0}^{M+1} R^j \sup_{x\in B_R(0)} |D^j\phi(x)| \\ &\qquad \le C\Big\{ \sup_{x\in B_R(0)} |\phi(x)| + \sum_{j=0}^{M-1} R^{j+2} \sup_{x\in B_R(x_0)} |D^j\Delta\psi(x)| \\ &\qquad\qquad + R^{M+1+\alpha} \sup_{\substack{x,y\in B_R(x_0)\\ x\ne y}} \frac{|D^j\Delta\psi(x) - D^j\Delta\psi(y)|}{|x-y|^\alpha}\Big\} \end{aligned} \tag{3.6}$$

where C depends on n, M and Ω. We write the equation (1.1) into the form

$$\Delta\psi = 2i\boldsymbol{A}\cdot\nabla\psi + i(\operatorname{div}\boldsymbol{A})\psi + |\boldsymbol{A}|^2\psi - f(|\psi|^2)\psi.$$

Then applying Proposition 3.1, we shall estimate the last two terms in (3.6). In the following we denote constants depending only on Ω and $\Lambda(\Omega'')$ by C which may vary from line to line. For $0 \le j \le M-1$, we have

$$\begin{aligned} &R^{j+2} \sup_{x\in B_R(x_0)} |D^j\Delta\psi(x)| \\ &\quad = R^{j+2} \sup_{x\in B_R(x_0)} \big|D^j[2i\boldsymbol{A}\cdot\nabla\psi + i(\operatorname{div}\boldsymbol{A})\psi + |\boldsymbol{A}|^2\psi - f(|\psi|^2)\psi]\big| \\ &\quad \le CR^{j+2} \sum_{k=0}^{j+1} \sup_{x\in B_R(x_0)} |D^k\psi(x)| \\ &\quad \le CR \sum_{k=0}^{j+1} R^k \sup_{x\in B_R(x_0)} |D^k\psi(x)| \\ &\quad \le CR \sup_{x\in B_{2R}(x_0)} |\psi(x)| \le CR^{m+1}. \end{aligned}$$

We can similarly estimate the last term in (3.6). Thus we get

$$\sum_{j=0}^{M+1} R^j \sup_{x\in B_R(0)} |D^j\phi(x)| \le C\{ \sup_{x\in B_R(0)} |\phi(x)| + R^{m+1}\} \le CR^{m+\alpha}.$$

Therefore, for $j = 0, 1, \ldots, m$, we have

$$|D^j \phi(x)| \le C R^{m-j+\alpha} \quad \text{in} \quad B_R(0).$$

For $j = m+1, \ldots, M+1$, since $D^j P_m \equiv 0$, we see that $D^j \phi(x) = D^j \psi(x + x_0)$ in $B_R(0)$. Thus we have

$$|D^j \phi(x)| \le \sup_{x \in B_R(x_0)} |D^j \psi(x)| \le \sup_{x \in \Omega''} |D^j \psi(x)| \le C_j \quad \text{in} \quad B_R(0)\,.$$

This completes the proof. □

Now we show a property of a complex-valued harmonic polynomial.

Lemma 3.3 (cf. [13]). *Let P be a complex-valued non-zero homogeneous, harmonic polynomial of degree $m \ge 2$, and of two variables in $\mathbb{R}^n$. Then there exist $\delta_*, r_* > 0$ depending on P such that if $\varphi \in C^{2m^2}(B_1(0); \mathbb{C})$ satisfies $|\varphi - P|_{C^{2m^2}(B_1(0);\mathbb{C})} < \delta_*$, then*

$$\mathcal{H}^{n-2}(|\nabla \varphi|^{-1}\{0\} \cap B_r(0)) \le c(n)(m-1)^2 r^{n-2}$$

for all $0 < r \le r_*$.

Proof.

Step 1. It suffices to prove the case where P and φ are real-valued.

In fact, assume that Lemma 3.3 holds for the case where P and φ are real-valued. Let P and φ be complex-valued functions satisfying the hypotheses in the lemma. Since either of $\Re P$ or $\Im P$ is non-zero, let $\Re P \not\equiv 0$. We choose δ_* and r_* corresponding to $\Re P$. If $|\varphi - P|_{C^{2m^2}(B_1(0);\mathbb{C})} < \delta_*$, then $|\Re\varphi - \Re P|_{C^{2m^2}(B_1(0);\mathbb{C})} < \delta_*$. Since $|\nabla \varphi|^{-1}\{0\} \subset |\nabla \Re\varphi|^{-1}\{0\}$, we get the conclusion.

Step 2. We shall show the lemma for the real case. Though the proof is identical as [13, Lemma 3.2], we introduce an outline of the proof. We choose a coordinates $\widetilde{x} = (\widetilde{x}_1, \widetilde{x}_2, \ldots, \widetilde{x}_n) \in \mathbb{R}^n$, and the polar coordinates $\widetilde{x}_1 = r\cos\theta, \widetilde{x}_2 = r\sin\theta$ in $\mathbb{R}^2$. By the hypothesis on P, we may assume that

$$P(\widetilde{x}) = r^m \cos m\theta\,.$$

Then we have

$$D_{\widetilde{x}_1} P(\widetilde{x}) = m r^{m-1} \cos(m-1)\theta\,,$$

$$D_{\widetilde{x}_2} P(\widetilde{x}) = m r^{m-1} \sin(m-1)\theta\,.$$

By the formulae:

$$\cos(m-1)\theta = 2^{m-2} \prod_{r=1}^{m-1} \sin\left(\theta + \frac{(2r-1)\pi}{2(m-1)}\right),$$

$$\sin(m-1)\theta = 2^{m-2} \prod_{r=1}^{m-1} \sin\left(\theta + \frac{(r-1)\pi}{m-1}\right),$$

there exists $2(m-1)$ non-zero vectors $\nu_k^i \in \mathbb{R}^2$ $(i=1,2,k=1,2,\ldots,m-1)$ such that

$$D_{\widetilde{x}_i} P(\widetilde{x}) = \prod_{k=1}^{m-1} \left(\nu_k^i \cdot (\widetilde{x}_1, \widetilde{x}_2)\right) \quad \text{for} \quad i=1,2\,.$$

We note that

$$\det(\nu_k^i, \nu_l^j) \neq 0 \quad \text{for} \quad (i,k) \neq (j,l) \tag{3.7}$$

and $D_{\widetilde{x}_i} P(\widetilde{x}) = 0$ for $i = 3,4,\ldots,n$. Here for $\nu_k^i = (\nu_{k,1}^i, \nu_{k,2}^i) \in \mathbb{R}^2$, $\det(\nu_k^i, \nu_l^j) = \det \begin{pmatrix} \nu_{k,1}^i & \nu_{l,1}^j \\ \nu_{k,2}^i & \nu_{l,2}^j \end{pmatrix}$. We take a change of coordinates $\widetilde{x} = Ox$ with an orthogonal matrix $O = (o_{ij})$ to be chosen. Let $\eta_i = (o_{1i}, o_{2i}) \in \mathbb{R}^2$, $i=1,2,\ldots,n$. Then we get

$$\begin{aligned} D_{x_i} P(x) &= o_{1i} \prod_{k=1}^{m-1} \left((\eta_1 \cdot \nu_k^1) x_1 + \cdots + (\eta_n \cdot \nu_k^1) x_n\right) \\ &\quad + o_{2i} \prod_{k=1}^{m-1} \left((\eta_1 \cdot \nu_k^2) x_1 + \cdots + (\eta_n \cdot \nu_k^2) x_n\right) \end{aligned} \tag{3.8}$$

for $i=1,2,\ldots,n$. We note that if $D_{x_i} P(x)$ does not vanish, it is a homogeneous polynomial of degree $m-1$, and that (3.8) contains only first two rows of O. For any $1 \le i < j \le n$ and $p \in \mathbb{R}^n$, we define two dimensional planes

$$\mathbb{P}_{ij}(p) = \left\{(p_1, \ldots, p_{i-1}, x_i, p_{i+1}, \ldots, p_{j-1}, x_j, p_{j+1}, \ldots, p_n)\right\}$$

and $\mathbb{P}_{ij} = \mathbb{P}_{ij}(0)$. Then it follows from (3.8) that

$$\begin{pmatrix} D_{x_i} P|_{\mathbb{P}_{ij}} \\ D_{x_j} P|_{\mathbb{P}_{ij}} \end{pmatrix} = \begin{pmatrix} o_{1i} & o_{2i} \\ o_{1j} & o_{2j} \end{pmatrix} \begin{pmatrix} \prod_{k=1}^{m-1} \left((\eta_i \cdot \nu_k^1) x_i + (\eta_j \cdot \nu_k^1) x_j\right) \\ \prod_{k=1}^{m-1} \left((\eta_i \cdot \nu_k^2) x_i + (\eta_j \cdot \nu_k^2) x_j\right) \end{pmatrix}. \tag{3.9}$$

If we require that

$$\det(\eta_i, \eta_j) = \det \begin{pmatrix} o_{1i} & o_{1j} \\ o_{2i} & o_{2j} \end{pmatrix} \neq 0\,, \tag{3.10}$$

then that $D_{x_i} P = D_{x_j} P = 0$ on $\mathbb{P}_{ij}$ is equivalent to

$$\prod_{k=1}^{m-1} \left((\eta_i \cdot \nu_k^1) x_i + (\eta_j \cdot \nu_k^1) x_j\right) = \prod_{k=1}^{m-1} \left((\eta_i \cdot \nu_k^2) x_i + (\eta_j \cdot \nu_k^2) x_j\right) = 0\,.$$

By (3.7) and (3.10), for any $1 \le k, l \le m-1$,

$$\det \begin{pmatrix} \eta_i \cdot \nu_k^1 & \eta_j \cdot \nu_k^1 \\ \eta_i \cdot \nu_l^2 & \eta_j \cdot \nu_l^2 \end{pmatrix} = \det(\eta_i, \eta_j) \det(\nu_k^1, \nu_l^2) \neq 0\,.$$

Thus if we require that in the orthogonal matrix O,

$$\det \begin{pmatrix} o_{1i} & o_{2i} \\ o_{1j} & o_{2j} \end{pmatrix} \neq 0 \quad \text{for all} \quad 1 \le i < j \le n\,,$$

we obtain that for fixed $1 \le i < j \le n$, $f_{ij} = (D_{x_i} P, D_{x_j} P)|_{\mathbb{P}_{ij}} : \mathbb{R}^2 \to \mathbb{R}^2$ has only one zero at $x_i = x_j = 0$. If we replace (x_i, x_j) in (3.9) with $(z_i, z_j) \in \mathbb{C}^2$, we see that $f_{ij}\colon \mathbb{C}^2 \to \mathbb{C}^2$ has also only zero at $z_i = z_j = 0$. We apply [13, Theorem 4.1] with $n = 2$ due to Hilbert's Nullstellensatz. For fixed $1 \le i < j \le n$, there exist $\delta_{ij} > 0$ and $r_{ij} > 0$ depending on f_{ij} such that for any $v \in C^M(B^2_{1/2}(0); \mathbb{R}^2)$ with $|v - f_{ij}|_{C^M(B_{1/2}(0);\mathbb{R}^2)} < \delta_{ij}$, we have

$$\text{card}\,\big(v^{-1}\{0\} \cap B^2_{r_{ij}}(0)\big) \le (m-1)^2 \tag{3.11}$$

where $B^2_r(0)$ denotes the ball centered at the origin with radius r in $\mathbb{R}^2$ and $M = 2(m-1)^2$ which is independent of i, j. Put $\delta_* = \frac{1}{2}\min_{1\le i<j\le n}\delta_{ij}$ and $r_* = \min_{1\le i<j\le n} r_{ij}$. Moreover, we assume that $\varphi \in C^{2m^2}(B_1(0);\mathbb{R})$ satisfies $|\varphi - P|_{C^{2m^2}(B_1(0);\mathbb{R})} < \delta_*$. If we take r_* smaller if necessary, for any $0 < r < r_*$ and for any $p \in B_r(0)$, $v_{ij,p} := (D_{x_i}\varphi, D_{x_j}\varphi)|_{\mathbb{P}_{ij}(p)}$ satisfies $|v_{ij,p} - f_{ij}|_{C^M(B^2_{1/2}(0);\mathbb{R})} < 2\delta \le \delta_{ij}$. Hence from (3.11)

$$\text{card}\,(v^{-1}_{ij,p}\{0\} \cap B^2_r(0)) \le (m-1)^2\,.$$

Since $|\nabla\varphi|^{-1}\{0\} \cap \mathbb{P}_{ij}(p) \subset v^{-1}_{ij,p}\{0\}$, if we set the projection

$$\pi_{ij}(x_1,\dots,x_n) = (x_1,\dots,x_{i-1},x_{i+1},\dots,x_{j-1},x_{j+1},\dots,x_n)\,,$$

then for any $q \in B^{n-2}_r(0) \subset \mathbb{R}^{n-2}$ and any $1 \le i < j \le n$,

$$\text{card}\,\big(|\nabla\varphi|^{-1}\{0\} \cap \pi^{-1}_{ij}(q) \cap B_r(0)\big) \le (m-1)^2\,.$$

By the general area-coarea formula (cf. Federer [8, 3.3.22] or Morgan [20, 3.13]), we have

$$\begin{aligned} &\mathcal{H}^{n-2}\big(|\nabla\varphi|^{-1}\{0\} \cap B_r(0)\big) \\ &\quad\le \sum_{1\le i<j\le n}\int_{B^{n-2}_r(0)} \text{card}\,\big(|\nabla\varphi|^{-1}\{0\} \cap \pi^{-1}_{ij}(q) \cap B_r(0)\big)\,d\mathcal{H}^{n-2}q \\ &\quad\le c(n)(m-1)^2 r^{n-2}\,. \end{aligned}\tag{3.12}$$

Since $M + 1 = 2(m-1)^2 + 1 \le 2m^2$, this completes the proof. □

Now we can get the following

Proposition 3.4. *Assume that the conditions of Theorem 1.1 hold. Then for any $\Omega' \Subset \Omega$ and any $\varepsilon > 0$, there exist $C(\varepsilon) = C\varepsilon^{n-2}$ and $\gamma(\varepsilon) = \gamma\varepsilon^{n-2}$ where C and γ depend on ψ, Ω' and $\Lambda(\Omega'')$, and a collection of finitely many balls $\{B_{r_i}(x_i)\}_i$ with $r_i \le \varepsilon$, $x_i \in \mathcal{S}(\psi) \cap \Omega'$ such that*

$$\begin{gathered} \mathcal{H}^{n-2}\Big(\mathcal{S}(\psi) \cap \Omega' \setminus \bigcup_i B_{r_i}(x_i)\Big) \le C(\varepsilon)\,, \\ \sum_i r_i^{n-2} \le \gamma(\varepsilon)\,. \end{gathered}\tag{3.13}$$

Proof. By (2.6), we have $\mathcal{H}^{n-2}\big(\mathcal{S}(\psi) \cap \Omega' \setminus \mathcal{S}_*(\psi)\big) = 0$. Therefore $(n-2)$-spherical measure of $\mathcal{S}(\psi) \cap \Omega' \setminus \mathcal{S}_*(\psi)$ is equal to zero (cf. Mattila [19, p. 75]). Thus for any $\varepsilon > 0$, there exist at most countably many balls $\{B_{r_i}(x_i)\}$ with $r_i \le \varepsilon$ and $x_i \in \mathcal{S}(\psi) \cap \Omega' \setminus \mathcal{S}_*(\psi)$ such that

$$\sum_i r_i^{n-2} \le \gamma(\varepsilon, \psi, \Omega')$$

where $\gamma(\varepsilon, \psi, \Omega') \to 0$ as $\varepsilon \to 0$.

Then we shall show the following claim.

Claim: For any $y \in S_*(\psi) \cap \Omega'$, there exist $R = R(y, \psi, \Omega')$ and $c = c(y, \psi, \Omega')$ with $R \le R_0$ where R_0 is as in Proposition 3.1 such that for any $0 < r < R$,

$$\mathcal{H}^{n-2}(\mathcal{S}(\psi) \cap \Omega' \cap B_r(y)) \le cr^{n-2}\,.$$

Here R and c depend only on y, ψ, Ω' but also on $\Lambda(\Omega'')$.

We prove the claim. Let $y \in \mathcal{S}_*(\psi) \cap \Omega'$. By the construction of Ω'', $B_{2R_0}(y) \Subset \Omega''$. If $R < R_0$, it follows from Proposition 2.4 that we can write

$$\psi(x+y) = P_m(x) + \phi(x) \quad \text{in} \quad B_R(0)$$

where P_m is a non-zero homogeneous, harmonic polynomial of degree $m \ge 2$ of two variables after some rotation of coordinates, and from Lemma 3.2, we have

$$|D^j\phi(x)| \le \begin{cases} CR^{m-j+\alpha} & j = 0, 1, \ldots, m\,, \\ C & j = m+1, \ldots M+1 \end{cases}$$

in $B_R(0)$. If we choose $R = R(y, \psi, \Omega')$ small enough with $R \le R_0$, we have

$$\Big|\frac{1}{R^m}\phi\Big|^*_{C^{M+1}(B_R(0));\mathbb{C}} < \delta_* \tag{3.14}$$

where δ_* is as in Lemma 3.3. In fact,

$$\begin{aligned} \Big|\frac{1}{R^m}\phi\Big|^*_{C^{M+1}(B_R(0);\mathbb{C})} &= \sum_{j=0}^{M+1} \frac{R^j}{R^m} \sup_{x \in B_R(0)} |D^j\phi(x)| \\ &= \sum_{j=0}^{m} + \sum_{j=m+1}^{M+1} \\ &\le C\Big[\sum_{j=0}^{m} \frac{R^j}{R^m} R^{m-j+\alpha} + \sum_{j=m+1}^{M+1} \frac{R^j}{R^m}\Big] \\ &\le CR^\alpha \end{aligned}$$

where C depends on y, ψ, Ω and $\Lambda(\Omega'')$. Thus if we choose $R > 0$ small enough, we get (3.14). That is to say, we have

$$\Big|\frac{1}{R^m}(\psi(\cdot + y) - P_m)\Big|^*_{C^{M+1}(B_R(0);\mathbb{C})} < \delta_*\,.$$

By scaling: $x \mapsto Rx$ and using the homogeneity of P_m, we have

$$\Big|\frac{1}{R^m}\psi(y + Rx) - P_m(x)\Big|_{C^{M+1}(B_1(0);\mathbb{C})} < \delta_*\,.$$

Since from (3.4) and (2.3), $M \geq 2c_0^2 \geq 2m^2$, we can apply Lemma 3.3 to $\frac{1}{R^m}\psi(y+Rx)$ and get

$$\begin{aligned}
c(n)(m-1)^2 r^{n-2} &\geq \mathcal{H}^{n-2}\big(\{x; \nabla_x \psi(y+Rx)=0\} \cap B_r(0)\big) \\
&= \mathcal{H}^{n-2}\Big(\Big\{\frac{z-y}{R}; \nabla_z \psi(z)=0\Big\} \cap \Big\{\frac{z-y}{R}; |\frac{z-y}{R}| < r\Big\}\Big) \\
&= \frac{1}{R^{n-2}} \mathcal{H}^{n-2}\big(|\nabla\psi|^{-1}\{0\} \cap B_r(y)\big)
\end{aligned}$$

for all $0 < r < r_*$. Here we may assume that $R < 1$. Therefore, we get

$$\mathcal{H}^{n-2}\big(|\nabla\psi|^{-1}\{0\} \cap B_r(y)\big) \leq c(n)(m-1)^2 r^{n-2} \tag{3.15}$$

for all $0 < r < Rr_*$. Since we can replace R with a smaller one, we can take $Rr_* \leq \varepsilon$. Thus the claim holds.

Therefore, since

$$\mathcal{S}_*(\psi) \subset \bigcup_{y \in \mathcal{S}_*(\psi)} B_{r(y)}(y)\,,$$

we have

$$\begin{aligned}
\mathcal{S}(\psi) \cap \Omega' &= (\mathcal{S}(\psi) \cap \Omega' \setminus \mathcal{S}_*(\psi)) \cup \mathcal{S}_*(\psi) \\
&\subset \bigcup_i B_{r_i}(x_i) \cup \bigcup_{y \in \mathcal{S}_*(\psi)} B_{r(y)}(y)\,.
\end{aligned}$$

Since $\mathcal{S}(\psi) \cap \Omega'$ is relatively compact, there exist finitely many $x_i \in \mathcal{S}(\psi) \cap \Omega' \setminus \mathcal{S}_*(\psi)$ $(i = 1, 2, \dots, k = k(\varepsilon, \psi))$ and $y_j \in \mathcal{S}_*(\psi)$ $(j = 1, 2, \dots, l = l(\varepsilon, \psi))$ such that

$$\mathcal{S}(\psi) \cap \Omega' \subset \bigcup_{i=1}^k B_{r_i}(x_i) \cup \bigcup_{j=1}^l B_{s_j}(y_j)$$

and

$$\sum_{i=1}^k r_i^{n-2} \leq \sum_{i=1}^k \varepsilon^{n-2} := \gamma(\varepsilon, \psi) = k\varepsilon^{n-2}\,.$$

Thus it follows from the claim that

$$\begin{aligned}
&\mathcal{H}^{n-2}\Big(\mathcal{S}(\psi) \cap \Omega' \setminus \bigcup_{i=1}^k B_{r_i}(x_i)\Big) \\
&\qquad \leq \sum_{j=1}^l \mathcal{H}^{n-2}\big(\mathcal{S}(\psi) \cap \Omega' \cap B_{s_j}(y_j)\big) \\
&\qquad \leq C \sum_{j=1}^l s_j^{n-2} \leq C \sum_{j=1}^l \varepsilon^{n-2} = Cl\varepsilon^{n-2}\,.
\end{aligned}$$

This completes the proof. □

Finally, we have

Theorem 3.5. *Assume that the conditions in Theorem 1.1 hold. For any $\Omega' \Subset \Omega$, there exists a constant $C > 0$ depending on ψ, Ω' and $\Lambda(\Omega'')$ such that*

$$\mathcal{H}^{n-2}(\mathcal{S}(\psi) \cap \Omega') \le C\,.$$

Proof. Let $0 < R < R_0$ where R_0 is as in Proposition 3.1. Since $\{B_R(x)\}_{x\in\Omega'}$ is an open covering of a compact set $\overline{\Omega'}$, there exists finitely many points $x_1, \dots, x_{k_0} \in \Omega'$ such that $\overline{\Omega'} \subset \bigcup_{i=1}^{k_0} B_R(x_i)$. We put a collection of the balls $\phi_0 = \{B_R(x_i)\}_{i=1}^{k_0}$. Fix any $\varepsilon > 0$. Then we have the following

Claim: There exist collections of balls $\phi_1, \phi_2, \dots$ such that for any $l \ge 1$,

(i) $\mathrm{rad}\,(B) \le (2\varepsilon)^l R_0$ for all $B \in \phi_l$ where $\mathrm{rad}\,(B)$ denotes the radius of the ball B.

(ii) The center of B is contained in Ω' for all $B \in \phi_l$.

(iii) $\sum_{B\in\phi_l} (\mathrm{rad}\,(B))^{n-2} \le \gamma(\varepsilon)^l$.

(iv) $\mathcal{H}^{n-2}\big(\mathcal{S}(\psi) \cap \Omega' \cap \big(\bigcup_{B\in\phi_{l-1}} B \sim \bigcup_{B\in\phi_l} B\big)\big) \le C(\varepsilon)\gamma(\varepsilon)^{l-1}$ where $\gamma(\varepsilon)$ and $C(\varepsilon)$ are as in Proposition 3.4.

First, we show that the claim implies Theorem 3.5. In order to do so, we show that

$$\begin{aligned}\mathcal{S}(\psi) \cap \Omega' \subset \bigcup_{l=1}^{\infty} \Big(\mathcal{S}(\psi) \cap \Omega' \cap \Big(\bigcup_{B\in\phi_{l-1}} B \sim \bigcup_{B\in\phi_l} B\Big)\Big)\\ \cup \bigcap_{l=0}^{\infty} \Big(\mathcal{S}(\psi) \cap \Omega' \cap \Big(\bigcup_{j=l}^{\infty} \bigcup_{B\in\phi_j} B\Big)\Big).\end{aligned} \tag{3.16}$$

In fact, let $p \in \mathcal{S}(\psi) \cap \Omega'$ and assume that

$$p \notin \bigcup_{B\in\phi_{l-1}} B \sim \bigcup_{B\in\phi_l} B \tag{3.17}$$

for all $l \ge 1$. Since $p \in \bigcup_{B\in\phi_0} B$, clearly $p \in \mathcal{S}(\psi) \cap \Omega' \cap \big(\bigcup_{j=0}^{\infty} \bigcup_{B\in\phi_j} B\big)$. It suffices to show that for any $k \ge 0$,

$$p \in \bigcap_{l=0}^{k} \Big(\mathcal{S}(\psi) \cap \Omega' \cap \Big(\bigcup_{j=l}^{\infty} \bigcup_{B\in\phi_j} B\Big)\Big). \tag{3.18}$$

We show $(3.18)_k$ by induction on k. When $k = 1$, by (3.17), $p \notin \big(\bigcup_{B\in\phi_0} B\big) \setminus \big(\bigcup_{B\in\phi_1} B\big)$ and $p \notin \big(\bigcup_{B\in\phi_1} B\big) \setminus \big(\bigcup_{B\in\phi_0} B\big)$. Since $p \in \bigcup_{B\in\phi_0} B$, we see that

$$p \in \mathcal{S}(\psi) \cap \Omega' \cap \Big(\bigcup_{B\in\phi_1} B\Big) \subset \mathcal{S}(\psi) \cap \Omega' \cap \Big(\bigcup_{j=1}^{\infty} \bigcup_{B\in\phi_j} B\Big).$$

Thus we have

$$p \in \bigcap_{l=0}^{1} \Big(\mathcal{S}(\psi) \cap \Omega' \cap \Big(\bigcup_{j=l}^{\infty} \bigcup_{B\in\phi_j} B\Big)\Big).$$

Therefore $(3.18)_1$ holds. Assume that $(3.18)_j$ holds for $j \le k$ $(k \ge 1)$. Then $p \in \mathcal{S}(\psi) \cap \Omega'$ and $p \in \bigcup_{j=k}^{\infty} \bigcup_{B\in\phi_j} B$. That is to say, for some $j \ge k$, $p \in \bigcup_{B\in\phi_j} B$.

If $j \geq k+1$, then $p \in \bigcup_{j=k+1}^{\infty} \bigcup_{B \in \phi_j} B$. Hence $(3.18)_{k+1}$ holds. If $j = k$, then $p \in \bigcup_{B \in \phi_k} B$. From (3.17), $p \in \bigcup_{B \in \phi_{k+1}} B \subset \bigcup_{j=k+1}^{\infty} \bigcup_{B \in \phi_j} B$. Thus $(3.18)_{k+1}$ holds. Therefore, $(3.18)_k$ holds for any $k \geq 0$.

From (3.16), we see that

$$\mathcal{H}^{n-2}\big(\mathcal{S}(\psi) \cap \Omega'\big) \leq \sum_{l=1}^{\infty} \mathcal{H}^{n-2}\Big(\mathcal{S}(\psi) \cap \Omega' \cap \Big(\bigcup_{B \in \phi_{l-1}} B \sim \bigcup_{B \in \phi_l} B\Big)\Big) + \inf_{l \geq 0} \sum_{j=l}^{\infty} \sum_{B \in \phi_j} \mathcal{H}^{n-2}\big(\mathcal{S}(\psi) \cap \Omega' \cap B\big) .$$

Here it follows from (3.15) that

$$\begin{aligned} \mathcal{H}^{n-2}\big(\mathcal{S}(\psi) \cap \Omega' \cap B\big) &\leq \mathcal{H}^{n-2}\big(|\nabla \psi|^{-1}\{0\} \cap B\big) \\ &\leq c(n)(m-1)^2 \big(\operatorname{rad}(B)\big)^{n-2} \leq c(n) M \big(\operatorname{rad}(B)\big)^{n-2} . \end{aligned}$$

Therefore, from the claim, we have

$$\begin{aligned} &\mathcal{H}^{n-2}\big(\mathcal{S}(\psi) \cap \Omega'\big) \\ &\leq C(\varepsilon) \sum_{l=1}^{\infty} \gamma(\varepsilon)^{l-1} + c(n) M \inf_{l \geq 0} \sum_{j=l}^{\infty} \sum_{B \in \phi_j} \big(\operatorname{rad}(B)\big)^{n-2} \\ &\leq C(\varepsilon) \sum_{l=1}^{\infty} \gamma(\varepsilon)^{l-1} + c(n) M \inf_{l \geq 0} \sum_{j=l}^{\infty} \gamma(\varepsilon)^{j} . \end{aligned}$$

If we choose $\varepsilon > 0$ small enough so that $\gamma(\varepsilon) \leq 1/2$, we have

$$\mathcal{H}^{n-2}\big(\mathcal{S}(\psi) \cap \Omega'\big) \leq 2C(\varepsilon) .$$

Thus the theorem holds.

Finally we prove the claim. Note that $\phi_0 = \{B_R(x_i)\}_{i=1}^{k_0}$ is independent of ε. We shall construct $\{\phi_l\}_{l \geq 1}$ by induction on l. Assume that we have constructed $\phi_0, \phi_1, \ldots, \phi_{l-1}$ $(l \geq 1)$. Let $B = B_r(y) \in \phi_{l-1}$, $(y \in \Omega')$. If we choose $\varepsilon > 0$ small enough, if necessary, by induction hypothesis, $r \leq (2\varepsilon)^{l-1} R_0 \leq R_0$. Then $B_{2r}(y) \Subset \Omega$ and ψ satisfies the equation

$$-\nabla_{\boldsymbol{A}}^2 \psi = f(|\psi|^2)\psi \quad \text{in} \quad B_{2r}(y) .$$

By Proposition 3.4 replaced ε with εr, there exist a finitely many balls $\{B_{r_i}(x_i)\}_i$ with $r_i \leq \varepsilon r$ and $x_i \in \mathcal{S}(\psi) \cap \Omega' \cap B_r(y)$ such that

$$\mathcal{H}^{n-2}\Big(\mathcal{S}(\psi) \cap \Omega' \cap B_r(y) \setminus \Big(\bigcup_i B_{r_i}(x_i)\Big)\Big) \leq C(\varepsilon) r^{n-2} \tag{3.19}$$

and

$$\sum_i r_i^{n-2} \leq \gamma(r\varepsilon)^{n-2} = \gamma(\varepsilon) r^{n-2} .$$

Let $\phi_l^B = \{B_{r_i}(x_i)\}_i$ and $\phi_l = \{\phi_l^B\}_{B \in \phi_{l-1}}$. Then

$$r_i \leq \varepsilon r \leq \varepsilon (2\varepsilon)^{l-1} R_0 \leq (2\varepsilon)^l R_0 .$$

Thus (i) in the claim holds. (ii) is clear. Since

$$\sum_{B\in\phi_l}\big(\mathrm{rad}\,(B)\big)^{n-2}=\sum_{B\in\phi_{l-1}}\sum_i r_i^{n-2}\le\gamma(\varepsilon)\gamma(\varepsilon)^{l-1}=\gamma(\varepsilon)^l\,,$$

(iii) holds. It follows from (3.19) and the induction hypothesis that

$$\begin{aligned}\mathcal{H}^{n-2}&\Big(\mathcal{S}(\psi)\cap\Omega'\cap\Big(\bigcup_{B\in\phi_{l-1}}B\sim\bigcup_{B\in\phi_l}B\Big)\Big)\\&\le\sum_{B_r(y)\in\phi_{l-1}}\mathcal{H}^{n-2}\Big(\mathcal{S}(\psi)\cap\Omega'\cap B_r(y)\setminus\Big(\bigcup_i B_{r_i}(x_i)\Big)\Big)\\&\le C(\varepsilon)\sum_{B_r(y)\in\phi_{l-1}}r^{n-2}\le C(\varepsilon)\gamma(\varepsilon)^{l-1}\,.\end{aligned}$$

This proves (iv). Thus the claim holds. This completes the proof of Theorem 3.5. □

It is clear that this Theorem 3.5 implies Theorem 1.1.

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Department of Mathematical Sciences, Faculty of Science and Engineering,
Tokyo Denki University, Hatoyama-machi, Saitama 350-0394, Japan
E-mail: aramaki@mail.dendai.ac.jp