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ON COMPUTATION OF C-STATIONARY POINTS FOR EQUILIBRIUM PROBLEMS WITH LINEAR COMPLEMENTARITY CONSTRAINTS VIA HOMOTOPY METHOD

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In the paper we consider EPCCs with convex quadratic objective functions and one set of complementarity constraints. For this class of problems we propose a possible generalization of the homotopy method for finding stationary points of MPCCs. We analyze the difficulties which arise from this generalization. Numerical results illustrate the performance for randomly generated test problems.

Keywords: equilibrium problems with complementarity constraints, homotopy, C-stationarity

Classification: 90C31, 90C33, 90C20

1. INTRODUCTION

Complementarity problems are in the focus of mathematicians since the formulation of Karush–Kuhn–Tucker (KKT) conditions in linear and quadratic programming. These problems arise among constraints of optimization problems, known as mathematical problems with complementarity constraints (MPCCs). MPCCs form an important subclass of mathematical programs with equilibrium constraints (MPECs), optimization problems where, among the constraints, there is a special one in the form of a variational inequality, generalized equation or a complementarity problem. Many engineering problems, e. g. shape optimization problems with unilateral contact conditions, can be formulated as MPCCs. Such problems occur often also in economics (e. g. Stackelberg problem, facility location and production problem), in biology or in chemistry.

In this paper we consider equilibrium problems with complementarity constraints (EPCCs), a class of problems to find an equilibrium point that simultaneously solves several MPCCs, each of which are parameterized by decision variables of other MPCCs. EPCCs, a subclass of equilibrium problems with equilibrium constraints (EPECs), were introduced in the past decade and they receive an interest of mathematicians particularly for their use in the analysis of the market power of participants in deregulated electricity markets.

In this paper, we are interested in EPCCs associated with n parametric mathematical programs with convex-quadratic objective function and a linear complementarity constraint. The value of each objective depends on the vector $x := (x^1, \ldots, x^n)$ with $x^i \in \mathbb{R}^l, i = 1, \ldots, n$, and on $y \in \mathbb{R}$. The ith MPCC is the mathematical program in variables x^i and y

$$\begin{aligned} &\underset{x^i, y}{\text{minimize}}\ \frac{1}{2}\begin{pmatrix} x \\ y \end{pmatrix}^\top Q^i \begin{pmatrix} x \\ y \end{pmatrix} + (c^i)^\top \begin{pmatrix} x \\ y \end{pmatrix} \\ &\text{subject to } 0 \leq Ax + by + a \perp y \geq 0, \end{aligned} \tag{1}$$

with the symmetric matrix

$$Q^i = \begin{pmatrix} Q^i_{xx} & Q^i_{xy} \\ Q^i_{yx} & Q^i_{yy} \end{pmatrix} \in \mathbb{R}^{(nl+1)\times(nl+1)},$$

$Q^i_{xx} \in \mathbb{R}^{nl\times nl}, Q^i_{xy} = (Q^i_{yx})^\top \in \mathbb{R}^{nl\times 1}, Q^i_{yy} \in \mathbb{R}$, a row vector $A \in \mathbb{R}^{1\times nl}$, vectors $c^i \in \mathbb{R}^{nl+1}$ and real constants $a, b \in \mathbb{R}$. The symbol $\perp$ denotes orthogonality. Since in (1) all but variables x^i and y are fixed, to ensure convexity of the objective assume that the square submatrix which results from Q^i by deletion of rows and columns with indices corresponding to components of

$$x^{-i} := (x^1, \ldots, x^{i-1}, x^{i+1}, \ldots, x^n)$$

is positive definite.

There are only few proposed numerical methods, e. g. nonlinear Jacobi and nonlinear Gauss-Seidel diagonalization methods, sequential nonlinear complementarity method, cf. [8], which, under certain restrictive conditions, solve EPCCs. As an alternative to these methods, we aim to modify a piecewise affine homotopy method (the homotopy method I, [5]) which searches for so-called C-stationary points of MPCC [6], into a homotopy method for finding stationary points of EPCC, with main interest lying on so-called C-stationary points of EPCC. We also introduce an EPCC version of the C-index of nondegenerate C-stationary points. Analogously to the MPCC case, a vanishing C-index corresponds to a strongly stationary point which for the EPCC associated with convex MPCCs (1) coincides with a local solution (equilibrium point).

There are more attractive stationarity concepts for MPCCs to study, e. g., the mentioned strong stationarity. Also the development of effective numerical methods which converge to strongly stationary points [2] makes further study of C-stationarity for MPCCs less important. On the other hand, still very little is known about the structure of stationary points of EPCCs. From this perspective, the results known about C-stationarity for MPCCs become a useful tool for the study of EPCCs and for development of numerical methods for this class of problems.

We admit that studying only EPCCs with only one-dimensional lower-level decision variable seem too restrictive. Also, in the view of restrictions we impose upon the data, cf. the next section, nontrivial description of the algorithm and the fact that the proposed numerical method may not find any C-stationary point even if there is one, the practical use of our modified homotopy method is questionable. On the other hand, during the process we gained a detailed, previously unknown information about the structure of the sets of stationary points and solutions to the considered class of problems.

The rest of this paper is organized as follows. In section 2 we study genericity of the key assumptions imposed on the data needed for proper definition of the homotopy method. Section 3 studies the structure of the set of C-stationary points of one-parametric EPCCs. We show that although we cannot avoid singularities in a generic case, with the use of a generic "regularity" assumption these singular points can be well treated.

In section 4 we design the homotopy method which tracks the boundary of the set of C-stationary points of one-parametric EPCCs. We also briefly recall the homotopy method I for MPCCs which we apply to auxiliary MPCC to find an initial feasible point of an EPCC. Similarly to the MPCC case, this so-called Phase I approach may fail and as an alternative, one can use a method analyzing the polyhedral patches of the feasible set of the original problem.

Section 5 is devoted to our numerical results for randomly generated problems. Interestingly, very often the algorithm finds a stationary point satisfying stronger conditions of the so-called M-stationarity.

Throughout the paper, the upper index $i = 1, \ldots, n$ is used as a reference to data and variables of the ith MPCC. We denote the rows of a matrix $M \in \mathbb{R}^{p \times q}$ by $M_i, i = 1, \ldots, p$. For an index set $I \subset \{1, \ldots, p\}$, $M_I \in \mathbb{R}^{|I| \times q}$ denotes the submatrix of M composed of rows with indices $i \in I$. Similarly, for a vector $v \in \mathbb{R}^p$ we denote by $v_I \in \mathbb{R}^{|I|}$ a subvector composed of components $v_i, i \in I$.

Most of the vectors used in this paper have dimension $nl+1$ which is the dimension of (x, y) in (1). To simplify the notation of subvectors of $v \in \mathbb{R}^{nl+1}$, we often use $v_{x^i} := v_J$, $J = \{(i-1)l + 1, \ldots, il\}$ for $i = 1, \ldots, n$, and $v_y := v_{nl+1}$. Analogously for a matrix $M \in \mathbb{R}^{nl+1 \times q}$, we often use $M_{x^i} := M_J$, $J = \{(i-1)l + 1, \ldots, il\}$ for $i = 1, \ldots, n$, and $M_y := M_{nl+1}$.

2. PARAMETER-FREE PROBLEM

In this paper, we restrict our attention to the simplest form of EPCC constrained by the lower level problem in the form of a one-dimensional linear complementarity problem

$$\begin{aligned} &\text{for a given vector } x \text{ find } y \\ &\qquad \text{such that } 0 \leq Ax + by + a \perp y \geq 0. \end{aligned} \tag{2}$$

We assume that $b > 0$, which is sufficient for the linear complementarity problem (2) to be uniquely solvable, [4]. Hence, we are interested in the EPCC (associated with n MPCCs (1))

$$\begin{aligned} &\underset{x^i, y}{\text{minimize}} \; \frac{1}{2} \begin{pmatrix} x \\ y \end{pmatrix}^\top Q^i \begin{pmatrix} x \\ y \end{pmatrix} + (c^i)^\top \begin{pmatrix} x \\ y \end{pmatrix} \qquad i = 1, \ldots, n. \\ &\text{subject to } 0 \leq Ax + by + a \perp y \geq 0, \end{aligned} \tag{3}$$

As it is usual in works concerning complementarity problems, for a feasible point $\bar{z}$ of the EPCC (3) we define the active index sets associated with the complementarity problem (2)

$$I^+(\bar{x},\bar{y}) = \begin{cases} \{1\} & \text{if } Ax+by+a>0, \\ \emptyset & \text{otherwise,} \end{cases}$$

$$L(\bar{x},\bar{y}) = \begin{cases} \{1\} & \text{if } y>0, \\ \emptyset & \text{otherwise,} \end{cases}$$

$$I^0(\bar{x},\bar{y}) = \begin{cases} \{1\} & \text{if } Ax+by+a=y=0, \\ \emptyset & \text{otherwise.} \end{cases}$$

If there is no confusion about the reference point, we write only I^+, L and I^0. We denote $a^+ = |I^+(\bar{x},\bar{y})|$ and $a^0 = |I^0(\bar{x},\bar{y})|$.

For $i = 1,\dots,n$, we say that the MPEC linear independence constraint qualification (MPEC-LICQ) holds true at a feasible point $(\bar{x},\bar{y})$ for the ith MPCC (1), if the $(l+1)\times(1+a^0)$ matrix

$$\begin{pmatrix} (A^\top_{L\cup I^0})_{x^i} & 0_{I^+\cup I^0} \\ b_{L\cup I^0} & 1_{I^+\cup I^0} \end{pmatrix}$$

has full column rank. The EPEC linear independence constraint qualification (EPEC-LICQ) is said to hold at $(\bar{x},\bar{y})$, if MPEC-LICQ holds at $(\bar{x},\bar{y})$ for each MPCC (1), $i = 1,\dots,n$.

Denote by λ^i and μ^i the multipliers of the ith mathematical program (1) corresponding to the constraints $Ax+by+a\geq 0$ and $y\geq 0$, respectively. The stationarity concepts of our interest for the ith MPCC (1) differ from the KKT conditions of the related quadratic program

$$\begin{aligned} &\underset{x^i,y}{\text{minimize}}\ \frac{1}{2}\begin{pmatrix} x \\ y \end{pmatrix}^\top Q^i \begin{pmatrix} x \\ y \end{pmatrix} + c^{i\top}\begin{pmatrix} x \\ y \end{pmatrix} \\ &\text{subject to } Ax+by+a\geq 0,\ \ y\geq 0 \end{aligned} \tag{4}$$

only in restrictions imposed on multipliers.

Definition 2.1. Let $(\bar{x},\bar{y})$ be feasible for the ith MPCC (1). Then we call $(\bar{x},\bar{y})$

i) weakly stationary if there exist Lagrange multipliers $\bar{\lambda}^i,\bar{\mu}^i$ such that $(\bar{x},\bar{y},\bar{\lambda}^i,\bar{\mu}^i)$ satisfies the conditions

$$0 = \begin{pmatrix} (Q^i_{xx})_{x^i} & (Q^i_{xy})_{x^i} & -(A^\top_{L\cup I^0})_{x^i} & 0 \\ Q^i_{yx} & Q^i_{yy} & -b_{L\cup I^0} & -1_{I^+\cup I^0} \\ A_{L\cup I^0} & b_{L\cup I^0} & 0 & 0 \\ 0 & 1_{I^+\cup I^0} & 0 & 0 \end{pmatrix}\begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{\lambda}^i_{L\cup I^0} \\ \bar{\mu}^i_{I^+\cup I^0} \end{pmatrix} + \begin{pmatrix} c^i_{x^i} \\ c^i_y \\ a_{L\cup I^0} \\ 0 \end{pmatrix}; \tag{5}$$

ii) C-stationary, if there exist Lagrange multipliers $\bar{\lambda}^i,\bar{\mu}^i$ such that $(\bar{x},\bar{y},\bar{\lambda}^i,\bar{\mu}^i)$ satisfies the conditions (5) and, additionally, $\bar{\lambda}^i_{I^0}\bar{\mu}^i_{I^0}\geq 0$;

iii) M-stationary, if there exist Lagrange multipliers $\bar{\lambda}^i, \bar{\mu}^i$ such that $(\bar{x}, \bar{y}, \bar{\lambda}^i, \bar{\mu}^i)$ satisfies the conditions (5) and, additionally, either $\bar{\lambda}^i_{I^0} > 0$ and $\bar{\mu}^i_{I^0} > 0$ or $\bar{\lambda}^i_{I^0}\bar{\mu}^i_{I^0} = 0$;

iv) strongly stationary, if there exist Lagrange multipliers $\bar{\lambda}^i, \bar{\mu}^i$ such that $(\bar{x}, \bar{y}, \bar{\lambda}^i, \bar{\mu}^i)$ satisfies the conditions (5) and, additionally, $\bar{\lambda}^i_{I^0} \geq 0$ and $\bar{\mu}^i_{I^0} \geq 0$.

Clearly, we have the following chain of implications:

strongly stationarity $\Rightarrow$ M-stationarity $\Rightarrow$ C-stationarity $\Rightarrow$ weakly stationarity.

If MPEC-LICQ holds true, strong stationarity conditions and hence also all other above defined stationarity conditions are the first order necessary optimality conditions [6, Theorem 7(1)]. The names "C-" and "M-" come from the fact that the above statement could be proven when the Clarke and Mordukhovich calculi are applied, respectively.

The collection of conditions (5) for each $i = 1, \ldots, n$, together into one system of conditions produces a non-square system of linear equations. Recall that we assume $b > 0$ and thus the variable y is uniquely determined by the vector x. We can therefore treat the variable y in each MPCC separately, denoting it by y^i. This allows us to work with the following system of linear equation with square system matrix and where, implicitly, variables y^i attain the same value for all $i = 1, \ldots, n$:

$$0 = \begin{pmatrix} Q_{xx} & Q_{xy} & -\tilde{A}_{L\cup I^0} & 0 \\ Q_{yx} & Q_{yy} & -\tilde{B}^\top_{L\cup I^0} & -E^\top_{I^+\cup I^0} \\ \bar{A}_{L\cup I^0} & \tilde{B}_{L\cup I^0} & 0 & 0 \\ 0 & E_{I^+\cup I^0} & 0 & 0 \end{pmatrix} \begin{pmatrix} \bar{x} \\ \tilde{y} \\ \bar{\lambda}_{L\cup I^0} \\ \bar{\mu}_{I^+\cup I^0} \end{pmatrix} + \begin{pmatrix} c_x \\ c_y \\ \tilde{a}_I \\ 0 \end{pmatrix}, \qquad (6)$$

where

$$Q_{xx} := \begin{pmatrix} (Q^1_{xx})_{x^1} \\ \vdots \\ (Q^n_{xx})_{x^n} \end{pmatrix} \in \mathbb{R}^{nl\times nl}, Q_{yx} := \begin{pmatrix} Q^1_{yx} \\ \vdots \\ Q^n_{yx} \end{pmatrix} \in \mathbb{R}^{n\times nl},$$

$$Q_{xy} := \operatorname{diag}((Q^1_{xy})_{x^1}, \ldots, (Q^n_{xy})_{x^n}) \in \mathbb{R}^{nl\times n},\ Q_{yy} := \operatorname{diag}(Q^1_{yy}, \ldots, Q^n_{yy}) \in \mathbb{R}^{n\times n},$$

$$\tilde{A}_{L\cup I^0} := \operatorname{diag}((A^\top_{L\cup I^0})_{x^1}, \ldots, (A^\top_{L\cup I^0})_{x^n}) \in \mathbb{R}^{nl\times(1-a^+)n},$$

$$\bar{A}_{L\cup I^0} := \begin{pmatrix} A_{L\cup I^0} \\ \vdots \\ A_{L\cup I^0} \end{pmatrix} \in \mathbb{R}^{(1-a^+)n\times nl},\ \tilde{B}_{L\cup I^0} := \operatorname{diag}(b_{L\cup I^0}, \ldots, b_{L\cup I^0}) \in \mathbb{R}^{(1-a^+)n\times n},$$

$$E_{I^+\cup I^0} := \operatorname{diag}(1_{I^+\cup I^0}, \ldots, 1_{I^+\cup I^0}) \in \mathbb{R}^{(a^++a^0)n\times n},$$

$$\tilde{y} := \begin{pmatrix} \bar{y}^1 \\ \vdots \\ \bar{y}^n \end{pmatrix} \in \mathbb{R}^n,\ \tilde{\lambda}_{L\cup I^0} := \begin{pmatrix} \bar{\lambda}^1_{L\cup I^0} \\ \vdots \\ \bar{\lambda}^n_{L\cup I^0} \end{pmatrix} \in \mathbb{R}^{(1-a^+)n},$$

$$\tilde{\mu}_{I^+\cup I^0} := \begin{pmatrix} \bar{\mu}^1_{I^+\cup I^0} \\ \vdots \\ \bar{\mu}^n_{I^+\cup I^0} \end{pmatrix} \in \mathbb{R}^{(a^+ + a^0)n}, \; \tilde{a}_{L\cup I^0} := \begin{pmatrix} a_{L\cup I^0} \\ \vdots \\ a_{L\cup I^0} \end{pmatrix} \in \mathbb{R}^{(1-a^+)n},$$

$$c_x := \begin{pmatrix} c^1_{x^1} \\ \vdots \\ c^n_{x^n} \end{pmatrix} \in \mathbb{R}^{nl}, \text{ and } c_y := \begin{pmatrix} c^1_y \\ \vdots \\ c^n_y \end{pmatrix} \in \mathbb{R}^n.$$

We illustrate the structure of the system (6) on a simple academic example.

Example 1. Consider the EPCC consisting of the following two MPCCs with parameters $\alpha, \beta \in \mathbb{R}$:

$$\begin{aligned} \underset{x^1\in\mathbb{R}, y\in\mathbb{R}}{\text{minimize}} \quad & \frac{1}{2}(x^1, x^2, y)^\top \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ y \end{pmatrix} + (1, 0, \alpha)^\top \begin{pmatrix} x^1 \\ x^2 \\ y \end{pmatrix} \\ \text{subject to} \quad & 0 \le 2x^1 + 2x^2 + y - 2 \perp y \ge 0, \end{aligned}$$

$$\begin{aligned} \underset{x^2\in\mathbb{R}, y\in\mathbb{R}}{\text{minimize}} \quad & \frac{1}{2}(x^1, x^2, y)^\top \begin{pmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ y \end{pmatrix} + (0, 1, \beta)^\top \begin{pmatrix} x^1 \\ x^2 \\ y \end{pmatrix} \\ \text{subject to} \quad & 0 \le 2x^1 + 2x^2 + y - 2 \perp y \ge 0, \end{aligned}$$

E.g., at a feasible point $(\bar{x}^1, \bar{x}^2, \bar{y}) = (1, 0, 0)$ both parts of the complementarity constraint are active, hence $I^0 = \{1\}$ and the system (6) becomes

$$0 = \begin{pmatrix} 2 & 1 & 1 & 0 & -2 & 0 & 0 & 0 \\ 2 & 3 & 0 & 2 & 0 & -2 & 0 & 0 \\ 1 & 1 & 1 & 0 & -1 & 0 & -1 & 0 \\ 1 & 2 & 0 & 3 & 0 & -1 & 0 & -1 \\ 2 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \bar{x}^1 \\ \bar{x}^2 \\ \bar{y}^1 \\ \bar{y}^2 \\ \bar{\lambda}^1 \\ \bar{\lambda}^2 \\ \bar{\mu}^1 \\ \bar{\mu}^2 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ \alpha \\ \beta \\ -2 \\ -2 \\ 0 \\ 0 \end{pmatrix}. \qquad (7)$$

Analogously to the definition of EPEC-LICQ, we define the stationarity concepts for EPCCs as follows.

Definition 2.2. Let $(\bar{x}, \bar{y})$ be feasible for the EPCC (3). Then we call the point $(\bar{x}, \bar{y})$

i) weakly stationary if there exist Lagrange multipliers $\bar{\lambda}, \bar{\mu}$ such that $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{\mu})$ satisfies conditions (6);

ii) C-stationary, if there exist Lagrange multipliers $\bar{\lambda}, \bar{\mu}$ such that $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{\mu})$ satisfies conditions (6) and, additionally, $\bar{\lambda}^i_{I^0}\bar{\mu}^i_{I^0} \ge 0, i = 1, \ldots, n$;

iii) M-stationary, if there exist Lagrange multipliers $\bar{\lambda}, \bar{\mu}$ such that $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{\mu})$ satisfies conditions (6) and, additionally, either $\bar{\lambda}^i_{I^0} > 0$ and $\bar{\mu}^i_{I^0} > 0$ or $\bar{\lambda}^i_{I^0}\bar{\mu}^i_{I^0} = 0, i = 1, \ldots, n$;

iv) strongly stationary, if there exist Lagrange multipliers $\bar{\lambda}, \bar{\mu}$ such that $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{\mu})$ satisfies conditions (6) and, additionally, $\bar{\lambda}^i_{I^0} \geq 0$ and $\bar{\mu}^i_{I^0} \geq 0, i = 1, \ldots, n$.

The following theorem shows that under EPEC-LICQ, the set of strongly stationary points of EPCC coincide with the set of solutions to EPCC.

Theorem 2.3. Let $(\bar{x}, \bar{y})$ be a local equilibrium point of EPCC (3). If EPEC-LICQ holds at $(\bar{x}, \bar{y})$ then it is a strongly stationary point with unique multipliers. Conversely, a strongly stationary point $(\bar{x}, \bar{y})$ is a solution to EPCC.

Proof. The first statement of the theorem follows from [6, Theorem 7(1)] applied to each MPCC (1), $i = 1, \ldots, n$. Since the Lagrangian of each MPCC is strictly convex, [6, Theorem 7(2)] implies the second statement. □

The following two assumptions imposed on the data of the EPCC (3) are crucial for the homotopy method to execute each step in a "regular" way.

(A1) EPEC-LICQ holds at each feasible point of the EPCC (3).

(A2) Consider two matrices

$$\begin{pmatrix} Q_{xx} & Q_{xy} & -\tilde{A}_L \\ Q_{yx} & Q_{yy} & -\tilde{B}_L^\top \\ \tilde{A}_L & \tilde{B}_L & 0 \end{pmatrix}, \begin{pmatrix} Q_{xx} & Q_{xy} & 0 \\ Q_{yx} & Q_{yy} & -E_{I^+}^\top \\ 0 & E_{I^+} & 0 \end{pmatrix}$$

and all matrices

$$\begin{pmatrix} Q_{xx} & Q_{xy} & -(\tilde{A}_{I^0}^\top)_I^\top & 0 \\ Q_{yx} & Q_{yy} & -(\tilde{B}_{I^0})_I^\top & -(E_{I^0})_J^\top \\ (\tilde{A}_{I^0})_1 & (\tilde{B}_{I^0})_1 & 0 & 0 \\ 0 & E_{I^0} & 0 & 0 \end{pmatrix},$$

where the index sets $I \subset \{1, \ldots, n\}$ and $J \subset \{1, \ldots, n\}$ fulfill $|I| + |J| = n + 1$. Then we suppose that all these matrices are nonsingular.

When we say that some condition imposed upon data or some property of data holds in *generic* sense, we mean that it holds for all data in an open and dense subset of the data space. This notion of "typical data" is particularly attractive if the data space is endowed with a topology. One of the possibilities how to prove that some condition holds in a generic sense is to show that data which do not satisfy such condition or data with undesired property lie in the union of finitely many smooth manifolds of positive codimensions.

Alternatively, if the data space is endowed with a measure, a property holds in a generic sense whenever it holds for almost all data with respect to this measure, cf. [7].

Although the above assumptions on data of the EPCC might appear too restrictive, both hold true in generic sense.

Theorem 2.4. Assumption (A1) holds for all (A, b, a) from some open and dense subset M^* of $M = \{(A, b, a) \in \mathbb{R}^{1\times(nl+1)} \times \mathbb{R}^1 \times \mathbb{R}^1\}$.

Proof. The validity of EPEC-LICQ in generic sense is an immediate consequence of [7, Theorem 3(1)], which states that MPEC-LICQ holds true in generic sense. □

Theorem 2.5. Assumption (A2) holds for all (Q, A, b, a) from some open and dense subset $N^\#$ of $N = \{(Q, A, b, a) \in \mathbb{R}^{(nl+1)\times(nl+1)} \times \mathbb{R}^{1\times(nl+1)} \times \mathbb{R}^1 \times \mathbb{R}^1\}$.

Proof. The statement follows from the fact that the set of all matrices $M \in \mathbb{R}^{m\times n}$ of rank $r \leq \min\{m, n\}$ is a smooth manifold of codimension $(m-r)(n-r)$ in $\mathbb{R}^{m\times n}$, cf. [1]. Thus, each square matrix is nonsingular in generic sense. This completes the proof. □

In view of Theorems 2.4 and 2.5, we presume that from now on assumptions (A1) and (A2) are satisfied.

Similarly to [5] we can define a nondegenerate C-stationary point of EPCC as follows.

Definition 2.6. Let $(\bar{x}, \bar{y})$ be a C-stationary point of the EPCC with multipliers $\bar{\lambda}$ and $\bar{\mu}$. Then we call $(\bar{x}, \bar{y})$ nondegenerate if for each $i = 1 \ldots, n$, and $j \in I^0$ the sign conditions imposed on multipliers are satisfied with strict inequality, i. e., $\lambda^i_j \mu^i_j > 0$.

The above condition is usually called the *upper-level strict complementarity*. Now, for a nondegenerate C-stationary point, we can introduce the following generalization of the concept of a C-index from [5].

Definition 2.7. The C-index of a nondegenerate C-stationary point $(\bar{x}, \bar{y})$ is the sum of negative entries of the vector $\bar{\lambda}_{I^0}$ (or, equivalently, $\bar{\mu}_{I^0}$).

Clearly, a nondegenerate C-stationary point is strongly stationary if and only if its C-index vanishes.

3. A ONE-PARAMETRIC PROBLEM

Let us modify our EPCC such that it will include a one-dimesional real-valued parameter t. The parametric problem EPCC(t) will then consist of n one-parametric MPCCs, where the ith MPCC(t), $i = 1, \ldots, n$, is defined by

$$\begin{aligned} &\underset{x^i, y}{\text{minimize}}\ \frac{1}{2}\begin{pmatrix} x \\ y \end{pmatrix}^\top Q^i \begin{pmatrix} x \\ y \end{pmatrix} + (d^i(t))^\top \begin{pmatrix} x \\ y \end{pmatrix} \\ &\text{subject to } 0 \leq Ax + By + a \perp y \geq 0 \end{aligned} \tag{8}$$

with $d^i(t) := d^i + t(c^i - d^i), i = 1, \ldots, n$, for some vectors $d^i \in \mathbb{R}^{nl+1}$ and $t \in \mathbb{R}$. Later we will describe how the vectors $d^i = d^i(0), i = 1, \ldots, n$, are constructed.

The C-stationary conditions of the EPCC(t) consist of

$$0=\begin{pmatrix} Q_{xx} & Q_{xy} & -\tilde{A}_{L\cup I^0} & 0 \\ Q_{yx} & Q_{yy} & -\tilde{B}^\top_{L\cup I^0} & -E^\top_{I^+\cup I^0} \\ \bar{A}_{L\cup I^0} & \tilde{B}_{L\cup I^0} & 0 & 0 \\ 0 & E_{I^+\cup I^0} & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ \tilde{y} \\ \lambda_{L\cup I^0} \\ \mu_{I^+\cup I^0}\end{pmatrix}+\begin{pmatrix} d_x(t) \\ d_y(t) \\ \tilde{a}_{L\cup I^0} \\ 0\end{pmatrix} \tag{9}$$

$$0\le \lambda^i_{I^0}\mu^i_{I^0},\quad i=1,\dots,n, \tag{10}$$

where the vectors $d_x(t)$ and $d_y(t)$ are composed of components of vectors $d^1(t),\dots$ $\dots,d^n(t)$ in the following way

$$d_x(t)=\begin{pmatrix}(d^1(t))_{x^1}\\ \vdots \\ (d^n(t))_{x^n}\end{pmatrix}\ \text{and}\ d_y(t)=\begin{pmatrix}(d^1(t))_{y}\\ \vdots \\ (d^n(t))_{y}\end{pmatrix}.$$

Note that by choosing $t=1$ we arrive at the original EPCC (3) and its corresponding C-stationarity conditions.

Let us introduce the following sets:

$$\Sigma_{eq}=\{(t,x,y)\in\mathbb{R}\times\mathbb{R}^{nl}\times\mathbb{R}^1|(x,y)\text{ is an equilibrium point of EPCC}(t)\}$$
$$\Sigma_{S-stat}=\{(t,x,y)\in\mathbb{R}\times\mathbb{R}^{nl}\times\mathbb{R}^1|(x,y)\text{ is a strongly stationary point of EPCC}(t)\}$$
$$\Sigma_{C-stat}=\{(t,x,y)\in\mathbb{R}\times\mathbb{R}^{nl}\times\mathbb{R}^1|(x,y)\text{ is a C-stationary point of EPCC}(t)\}$$

As mentioned above, we have the relation $\Sigma_{eq}\subset\Sigma_{S-stat}\subset\Sigma_{C-stat}$ and due to assumption (A1) the first inclusion becomes equality.

For one-parametric as well as parameter-free EPCCs, it does not hold that all C-stationary points are nondegenerate in generic sense, see Figure below. In our analysis, we are particularly interested in the following class of singular points.

Definition 3.1. For $\bar{t}\in\mathbb{R}$ a C-stationary point $(\bar{x},\bar{y})$ of EPCC($\bar{t}$) with multipliers $\bar{\lambda},\bar{\mu}$ is called codimension n singularity (co-n-singularity) if the following conditions hold

i) Exactly n entries of the vector $(\bar{\lambda}_{I^0},\bar{\mu}_{I^0})$ vanish.

ii) If $I\subset\{1,\dots,n\}$ and $J\subset\{1,\dots,n\}$ are index sets such that $\bar{\lambda}_I\neq 0,\bar{\mu}_J\neq 0$ and $|I|+|J|=n$, then the matrix

$$\begin{pmatrix} c_x-d_x & Q_{xx} & Q_{xy} & -(\tilde{A}^\top_{I^0})^\top_I & 0 \\ c_y-d_y & Q_{yx} & Q_{yy} & -(\tilde{B}_{I^0})^\top_I & -(E_{I^0})^\top_J \\ 0 & (\bar{A}_{I^0})_1 & (\tilde{B}_{I^0})_1 & 0 & 0 \\ 0 & 0 & E_{I^0} & 0 & 0\end{pmatrix}$$

is nonsingular.

Further, we call the co-n-singular C-stationary point $(\bar{x},\bar{y})$

i) *0-singularity*, if $I\neq\emptyset, J\neq\emptyset$ and $I\cap J=\emptyset$;

ii) *i-singularity*, if $|I \cap J| = i$;

iii) *exit point*, if either $I = \emptyset$ or $J = \emptyset$.

Note that each co-n-singularity falls to exactly one of the above mentioned categories.

The homotopy method traces the set Σ_{C-stat}, searching for C-stationary points of the original problem. In order to design such algorithm, we have to understand the structure of the set Σ_{C-stat}, in particular its local structure around co-n-singularities. In the following we show that around each type of co-n-singularity, Σ_{C-stat} admits a different structure.

3.1. 0-singularity

Let us fix a $\bar{t} \in [0,1]$ and consider first the 0-singular C-stationary point $(\bar{x}, \bar{y})$ of the EPCC($\bar{t}$). If $I \subset \{1,\dots,n\}$ and $J \subset \{1,\dots,n\}$ are index sets uniquely defined by conditions $\bar{\lambda}_I \neq 0, \bar{\mu}_J \neq 0$ and $|I|+|J| = n$, let I^c and J^c denote the complement of I and J in $\{1,\dots,n\}$, respectively.

Then Σ_{C-stat} can be described locally around $(\bar{x}, \bar{y})$ by means of the following n systems of equations

$$
\begin{aligned}
0 &= H^{\lambda_j}(t,x,y,\lambda_{I\cup\{j\}},\mu_J)\\
&= \begin{pmatrix} Q_{xx} & Q_{xy} & -(\tilde{A}_{I^0}^\top)^\top_{I\cup\{j\}} & 0 \\ Q_{yx} & Q_{yy} & -(\tilde{B}_{I^0})^\top_{I\cup\{j\}} & -(E_{I^0})^\top_J \\ (\bar{A}_{I^0})_1 & (\tilde{B}_{I^0})_1 & 0 & 0 \\ 0 & E_{I^0} & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ \tilde{y} \\ \lambda_{I\cup\{j\}} \\ \mu_J \end{pmatrix} + \begin{pmatrix} d_x(t) \\ d_y(t) \\ \tilde{a}_{I\cup\{j\}} \\ 0 \end{pmatrix},
\end{aligned}
$$

for each $j \in I^c$ and

$$
\begin{aligned}
0 &= H^{\mu_j}(t,x,y,\lambda_I,\mu_{J\cup\{j\}})\\
&= \begin{pmatrix} Q_{xx} & Q_{xy} & -(\tilde{A}_{I^0}^\top)^\top_I & 0 \\ Q_{yx} & Q_{yy} & -(\tilde{B}_{I^0})^\top_I & -(E_{I^0})^\top_{J\cup\{j\}} \\ (\bar{A}_{I^0})_1 & (\tilde{B}_{I^0})_1 & 0 & 0 \\ 0 & E_{I^0} & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ \tilde{y} \\ \lambda_I \\ \mu_{J\cup\{j\}} \end{pmatrix} + \begin{pmatrix} d_x(t) \\ d_y(t) \\ \tilde{a}_I \\ 0 \end{pmatrix},
\end{aligned}
$$

for each $j \in J^c$.

Clearly, for $j \in I^c$ we have $H^{\lambda_j}(\bar{t},\bar{x},\bar{y},\bar{\lambda}_{I\cup\{j\}},\bar{\mu}_J) = 0$ and for $j \in J^c$ we have $H^{\mu_j}(\bar{t},\bar{x},\bar{y},\bar{\lambda}_I,\bar{\mu}_{J\cup\{j\}}) = 0$. Moreover, each system matrix is nonsingular due to the assumption (A2).

Hence, locally around $\bar{t}$ for each $j \in I^c$ there exists a *locally unique linear function* $(x^{\lambda_j}(t), y^{\lambda_j}(t), \lambda^{\lambda_j}(t), \mu^{\lambda_j}(t))$ such that

$$(x^{\lambda_j}(\bar{t}), y^{\lambda_j}(\bar{t}), \lambda^{\lambda_j}(\bar{t}), \mu^{\lambda_j}(\bar{t})) = (\bar{x}, \bar{y}, \bar{\lambda}_{I\cup\{j\}}, \bar{\mu}_J)$$

and

$$H^{\lambda_j}(t, x^{\lambda_j}(t), y^{\lambda_j}(t), \lambda^{\lambda_j}(t), \mu^{\lambda_j}(t)) = 0. \tag{11}$$

Analogously, for each $j \in J^c$ there exist a locally unique linear function $(x^{\mu_j}(t), y^{\mu_j}(t), \lambda^{\mu_j}(t), \mu^{\mu_j}(t))$.

Around the 0-singularity $(\bar{t}, \bar{x}, \bar{y})$, for some $\epsilon > 0$ only a part of the set

$$\Sigma^{\lambda_j} := \{(t, x^{\lambda_j}(t), y^{\lambda_j}(t)) | t - \bar{t} \in (-\epsilon, \epsilon)\}$$

belongs to the set Σ_{C-stat}. This is that part of Σ^{λ_j}, denoted by $\Sigma_+^{\lambda_j}$, where the sign of multiplier $\lambda_j^{\lambda_j}(t)$ is the same as the sign of multiplier $\mu_j^{\lambda_j}(t)$. Analogously, the feasible part of the set

$$\Sigma^{\mu_j} := \{(t, x^{\mu_j}(t), y^{\mu_j}(t)) | t - \bar{t} \in (-\epsilon, \epsilon)\},$$

which belongs to Σ_{C-stat} is denoted by $\Sigma_+^{\mu_j}$.

Theorem 3.2. At a 0-singular C-stationary point $(\bar{x}, \bar{y})$ of the EPCC$(\bar{t})$ with multipliers $\bar{\lambda}, \bar{\mu}$, for each $j \in I$ the linear function $(x^{\lambda_j}(t), y^{\lambda_j}(t), \lambda^{\lambda_j}(t), \mu^{\lambda_j}(t))$ intersects transversally at $(\bar{t}, \bar{x}, \bar{y}, \bar{\lambda}, \bar{\mu})$ with each linear function $(x^{\lambda_k}(t), y^{\lambda_k}(t), \lambda^{\lambda_k}(t), \mu^{\lambda_k}(t))$, $k \in I \setminus \{j\}$ and $(x^{\mu_k}(t), y^{\mu_k}(t), \lambda^{\mu_k}(t), \mu^{\mu_k}(t))$, $k \in J$.

Also for each $j \in J$ the linear function $(x^{\mu_j}(t), y^{\mu_j}(t), \lambda^{\mu_j}(t), \mu^{\mu_j}(t))$ intersects transversally at $(\bar{t}, \bar{x}, \bar{y}, \bar{\lambda}, \bar{\mu})$ with each linear function $(x^{\lambda_k}(t), y^{\lambda_k}(t), \lambda^{\lambda_k}(t), \mu^{\lambda_k}(t))$, $k \in I$ and $(x^{\mu_k}(t), y^{\mu_k}(t), \lambda^{\mu_k}(t), \mu^{\mu_k}(t))$, $k \in J \setminus \{j\}$.

Proof. It is sufficient to show that $\dot{\lambda}_j^{\lambda_j}(\bar{t}) := \frac{d}{dt}\lambda_j^{\lambda_j}(\bar{t}) \neq 0$. Since $\dot{\lambda}_j^{\lambda_k}(\bar{t}) = 0$, $k \in I \setminus \{j\}$ and $\dot{\lambda}_j^{\mu_k}(\bar{t}) = 0$, $k \in J$, this would mean that the linear function $(x^{\lambda_j}(t), y^{\lambda_j}(t), \lambda^{\lambda_j}(t), \mu^{\lambda_j}(t))$ does not point into the same direction as any of the other linear functions.

Take derivatives with respect to t in (11). This yields

$$0 = \begin{pmatrix} Q_{xx} & Q_{xy} & -(\tilde{A}_{I^0}^\top)_{I\cup\{j\}}^\top & 0 \\ Q_{yx} & Q_{yy} & -(\tilde{B}_{I^0})_{I\cup\{j\}}^\top & -(E_{I^0})_J^\top \\ (\bar{A}_{I^0})_1 & (\tilde{B}_{I^0})_1 & 0 & 0 \\ 0 & E_{I^0} & 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{x}^{\lambda_j}(\bar{t}) \\ \dot{y}^{\lambda_j}(\bar{t}) \\ \dot{\lambda}_{I\cup\{j\}}^{\lambda_j}(\bar{t}) \\ \dot{\mu}_J^{\lambda_j}(\bar{t}) \end{pmatrix} + \begin{pmatrix} c_x - d_x \\ c_y - d_y \\ 0 \\ 0 \end{pmatrix}.$$

This system of linear equations can equivalently be rewritten to

$$\begin{pmatrix} c_x - d_x & Q_{xx} & Q_{xy} & -(\tilde{A}_{I^0}^\top)_{I\cup\{j\}}^\top & 0 \\ c_y - d_y & Q_{yx} & Q_{yy} & -(\tilde{B}_{I^0})_{I\cup\{j\}}^\top & -(E_{I^0})_J^\top \\ 0 & (\bar{A}_{I^0})_1 & (\tilde{B}_{I^0})_1 & 0 & 0 \\ 0 & 0 & E_{I^0} & 0 & 0 \\ 0 & 0 & 0 & (e^j)_{I\cup\{j\}}^\top & 0 \end{pmatrix} \begin{pmatrix} 1 \\ \dot{x}^{\lambda_j}(\bar{t}) \\ \dot{y}^{\lambda_j}(\bar{t}) \\ \dot{\lambda}_{I\cup\{j\}}^{\lambda_j}(\bar{t}) \\ \dot{\mu}_J^{\lambda_j}(\bar{t}) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \dot{\lambda}_j^{\lambda_j}(\bar{t}) \end{pmatrix},$$

where e^j denotes the jth unit vector of basis in $\mathbb{R}^n$.

By Laplace formula applied to the last row, the latter system matrix is nonsingular, since the matrix

$$\begin{pmatrix} c_x - d_x & Q_{xx} & Q_{xy} & -(\tilde{A}_{I^0}^\top)_I^\top & 0 \\ c_y - d_y & Q_{yx} & Q_{yy} & -(\tilde{B}_{I^0})_I^\top & -(E_{I^0})_J^\top \\ 0 & (\bar{A}_{I^0})_1 & (\tilde{B}_{I^0})_1 & 0 & 0 \\ 0 & 0 & E_{I^0} & 0 & 0 \end{pmatrix}$$

is nonsingular for a co-n-singularity. Hence, $\dot{\lambda}_j^{\lambda_j}(\bar{t})$ cannot vanish. This proves the first part.

The proof of the second statement is analogous. □

Theorem 3.3. On a neighborhood of a 0-singular C-stationary point $(\bar{x}, \bar{y})$ of the EPCC($\bar{t}$) with multipliers $\bar{\lambda}, \bar{\mu}$, the set Σ_{C-stat} coincides with a convex hull of the sets $\Sigma_+^{\lambda_j}, j \in I^c$, and $\Sigma_+^{\mu_j}, j \in J^c$. Moreover, all interior points of such a convex hull share the same value of C-index.

Proof. Without loss of generality, it suffices to show that for $j, k \in J$, a convex hull of $\Sigma_+^{\lambda_j}$ and $\Sigma_+^{\lambda_k}$ belongs to Σ_{C-stat}.

Take $\alpha \in (0,1)$ and points $(t_1, x^{(1)}, y^{(1)}) \in \Sigma_+^{\lambda_j}, (t_2, x^{(2)}, y^{(2)}) \in \Sigma_+^{\lambda_k}$. Then we need to show that also

$$(t_\alpha, x^\alpha, y^\alpha) := \alpha(t_1, x^{(1)}, y^{(1)}) + (1-\alpha)(t_2, x^{(2)}, y^{(2)}) \in \Sigma_{C-stat}.$$

The point $(t_1, x^{(1)}, y^{(1)}, \lambda^{(1)}, \mu^{(1)})$ solves (9), where multipliers $\lambda^{(1)}, \mu^{(1)}$, uniquely determined by nonvanishing entries given by $\lambda_{I\cup\{j\}}^{\lambda^j}(t_1), \mu_J^{\lambda^j}(t_1)$, respectively, satisfy conditions (10). Analogously, the point $(t_2, x^{(2)}, y^{(2)}, \lambda^{(2)}, \mu^{(2)})$ solves (9), where multipliers $\lambda^{(2)}, \mu^{(2)}$, uniquely determined by nonvanishing entries given by $\lambda_{I\cup\{k\}}^{\lambda^k}(t_2), \mu_J^{\lambda^k}(t_2)$, respectively, satisfy conditions (10).

Then, clearly, conditions (10) are satisfied for $\lambda^\alpha = \alpha\lambda^{(1)} + (1-\alpha)\lambda^{(2)}$ and $\mu^\alpha = \alpha\mu^{(1)} + (1-\alpha)\mu^{(2)}$. It remains to show that also $(t_\alpha, x^\alpha, y^\alpha, \lambda^\alpha, \mu^\alpha)$ solves (9). To prove the latter statement, it suffices to recall that $d_x(t)$ and $d_y(t)$ is linear in t.

Taking any $\alpha \notin [0,1]$, conditions (10) are violated for $\lambda^\alpha = \alpha\lambda^{(1)} + (1-\alpha)\lambda^{(2)}$.

This finishes the proof of both parts of the theorem. □

3.2. i-singularity

At the i-singularity, let k be an index such that $\bar{\lambda}_k = \bar{\mu}_k = 0$. Then locally around $(\bar{t}, \bar{x}, \bar{y})$ the whole sets Σ^{λ_k} and Σ^{μ_k} belong to Σ_{C-stat}. This is due to the fact that $\mu_k^{\lambda_k}(t) = 0$ and $\lambda_k^{\mu_k}(t) = 0$ for each $t \in (-\epsilon, \epsilon)$ and the respective kth sign condition on biactive multipliers is thus satisfied regardless of the signs of $\lambda_k^{\lambda_k}(t)$ and $\mu_k^{\mu_k}(t)$, respectively.

Theorem 3.2 clearly holds also for i-singularity. Then a convex hull of the sets $\Sigma_+^{\lambda_j}, j \in I^c \setminus \{k\}, \Sigma_+^{\mu_j}, j \in J^c \setminus \{k\}, \{(t, x^{\lambda_k}(t), y^{\lambda_k}(t)) | t - \bar{t} \in [0, \epsilon)\}$ and

$\{(t, x^{\mu_k}(t), y^{\mu_k}(t)) | t - \bar{t} \in [0, \epsilon)\}$ as well as a convex hull of the sets $\Sigma_+^{\lambda_j}, j \in I^c \setminus \{k\}, \Sigma_+^{\mu_j}, j \in J^c \setminus \{k\}, \{(t, x^{\lambda_k}(t), y^{\lambda_k}(t)) | t - \bar{t} \in (-\epsilon, 0]\}$ and $\{(t, x^{\mu_k}(t), y^{\mu_k}(t)) | t - \bar{t} \in (-\epsilon, 0]\}$ belongs to the set Σ_{C-stat}. We summarize this in the following theorem.

Theorem 3.4. On a neighborhood of an i-singular C-stationary point $(\bar{x}, \bar{y})$ of the EPCC($\bar{t}$) with multipliers $\bar{\lambda}, \bar{\mu}$, the set Σ_{C-stat} coincides with a union of 2^i convex hulls of parts of sets $\Sigma^{\lambda_j}, j \in I^c$ and $\Sigma^{\mu_j}, j \in J^c$ specified above. Moreover, all interior points of each such convex hull share the same value of C-index.

Proof. The proof follows from the same arguments used in the proof of Theorem 3.3 and the observations above. □

3.3. Exit point

Note that there are only two possible exit points $(\bar{t}, \bar{x}, \bar{y})$. At the first one with $\bar{\lambda} = 0$, all sets $\Sigma_+^{\lambda_j}, j = 1, \ldots, n$, belong to the set Σ_{C-stat}.

Moreover, the same is true for the feasible part of the set

$$\Sigma^{I^+} = \{(t, x^{I^+}(t), y^{I^+}(t)) | t - \bar{t} \in (-\epsilon, \epsilon)\}$$

for some $\epsilon > 0$, where the locally unique linear function $(x^{I^+}(t), y^{I^+}(t), 0, \mu^{I^+}(t))$ is defined by the regular system of equations

$$0 = \begin{pmatrix} Q_{xx} & Q_{xy} & 0 \\ Q_{yx} & Q_{yy} & -E_{I^+}^\top \\ 0 & E_{I^+} & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ \mu_{I^+} \end{pmatrix} + \begin{pmatrix} d_x(t) \\ d_y(t) \\ 0 \end{pmatrix}.$$

Analogously, at the other exit point with $\bar{\mu} = 0$, the sets $\Sigma_+^{\mu_j}, j = 1, \ldots, n$, and the feasible part of the set Σ^L belong to the set Σ_{C-stat}.

Theorem 3.5. At an exit point $(\bar{x}, \bar{y})$ of the EPCC($\bar{t}$) with multipliers $\bar{\lambda}, \bar{\mu}$, the statement of Theorem 3.2 holds true. Moreover, either the linear function $(x^{I^+}(t)$, $y^{I^+}(t)$, 0, $\mu^{I^+}(t))$ intersects at $(\bar{t}, \bar{x}, \bar{y}, \bar{\lambda}, \bar{\mu})$ transversally with the linear functions $(x^{\lambda_j}(t), y^{\lambda_j}(t), \lambda^{\lambda_j}(t), \mu^{\lambda_j}(t)), j = 1, \ldots, n$, or the linear function $(x^L(t), y^L(t), \lambda^L(t)$, 0) intersects at $(\bar{t}, \bar{x}, \bar{y}, \bar{\lambda}, \bar{\mu})$ transversally with $(x^{\mu_j}(t), y^{\mu_j}(t), \lambda^{\mu_j}(t), \mu^{\mu_j}(t)), j = 1, \ldots, n$.

Proof. Using the arguments from the proof of Theorem 3.2, we can prove that at the exit point with $\bar{\lambda} = 0$, for each $j = 1, \ldots n$, the derivative $\dot{\lambda}_j^{\lambda_j}(\bar{t}) \neq 0$ while $\dot{\lambda}_j^{I^+} = 0$. Similarly, at the second exit point for each $j = 1, \ldots n$, the derivative $\dot{\mu}_j^{\mu_j}(\bar{t}) \neq 0$ while $\dot{\mu}_j^L = 0$. □

Theorem 3.6. On a neighborhood of the exit point $(\bar{x}, \bar{y})$ of the EPCC($\bar{t}$) with multipliers $\bar{\lambda} = 0$ and $\bar{\mu} \neq 0$, the set Σ_{C-stat} coincides with a union of the feasible part of Σ^{I^+} and a convex hull of the sets $\Sigma_+^{\lambda_j}, j = 1, \ldots, n$.

On a neighborhood of the exit point $(\bar{x}, \bar{y})$ of the EPCC($\bar{t}$) with multipliers $\bar{\lambda} \neq 0$ and $\bar{\mu} = 0$, the set Σ_{C-stat} coincides with a union of the feasible part of Σ^L and a convex hull of the sets $\Sigma_+^{\mu_j}, j = 1, \ldots, n$.

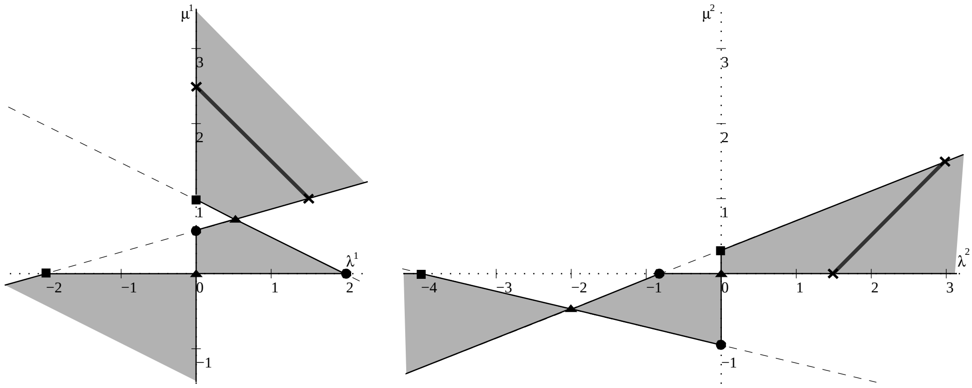

Fig. Co-2-singularities of the EPCC(t).

P r o o f. The proof follows from the same arguments used in the proof of Theorem 3.3 and observations above. □

Clearly, the C-index of nondegenerate C-stationary points can change only at co-n-singularities which are not 0-singular. We show the change of C-index on the EPCC from Example 1 with one particular setting of parameters α, β.

Example 1. (continued) Consider the EPCC from Example 1 with $\alpha = 3/2$ and $\beta = 1/2$ and suppose that $d_x(t) = (-6, -10)^\top$ and $d_y(t) = (-3, -5)^\top$.

Then one can find exactly six co-2-singularities of the EPCC(t): two exit points, $(1/3, 2/3, 0)$ at $t = 2/3$ and $(1, 0, 0)$ at $t = 0$ with multipliers (λ, μ) equal (0,0,1, 1/3) and $(-2, -4, 0, 0)$, respectively; two 0-singularities, $(25/9, -16/9, 0)$ at $t = 8/9$ and (1,0,0) at $t = 4/7$ with multipliers $(2, 0, 0, -8/9)$ and (0,-6/7,4/7,0), respectively; and two 1-singularities, (1, 0, 0) at $t = 8/11$ and $(17/9, -8/9, 0)$ at $t = 4/9$ with multipliers (6/11, 0, 8/11, 0) and $(0, -2, 0, -4/9)$, respectively.

All co-2-singularities are depicted on Figure in multiplier spaces; exit points as boxes, 0-singularities as bullets and 1-singularities as triangles. The shaded area is the set of all biactive multipliers corresponding to C-stationary points of the EPCC(t).

The interior points of the bounded piece correspond to multipliers of C-stationary points with C-index 1. The 1-singularity (1, 0, 0) at $t = 8/11$ connects this piece with the one with interior points with vanishing C-index. The other 1-singularity connects it with the piece with interior points with C-index 2. The latter two pieces are connected to the parts of the set Σ_{C-stat} of points with vanishing C-index by exit points.

Notice that slight shifts of the dashed lines in Figure due to small perturbations of the data eliminate neither the co-2-singularities nor singular C-stationary points on the border of the shaded area.

4. HOMOTOPY METHOD

The basic idea of the homotopy method we are about to describe in detail is to formulate an artificial EPCC by modifying (jointly) objective functions of all MPCCs in (3) such that a chosen feasible point $(\bar{x}, \bar{y})$ becomes strongly stationary. The parameter t then creates a connection between the original and the artificial problem.

Let $(\bar{x}, \bar{y})$ be a feasible point of the EPCC (3) and L, I^+ and I^0 be the associated index sets. Based on the structure of the index sets we construct the vector $d(0) = (d_x(0), d_y(0))$.

If $L = \{1\}$, then we set $\bar{\mu} := 0$ and choose a vector $\bar{\lambda}$ with arbitrary strictly positive components. If $I^+ = \{1\}$, then we set $\bar{\lambda} := 0$ and $\bar{\mu}$ with arbitrary strictly positive components. If $I^0 = \{1\}$, we set either $\bar{\mu} := 0$ and choose a multiplier vector $\bar{\lambda}$ with arbitrary strictly positive components or vice versa. In either case, we use the following formula to compute the vector $d(0)$.

$$d(0) := -\begin{pmatrix} Q_{xx} & Q_{xy} \\ Q_{yx} & Q_{yy} \end{pmatrix} \begin{pmatrix} \bar{x} \\ \tilde{y} \end{pmatrix} + \begin{pmatrix} \tilde{A}_{L\cup I^0} \\ \tilde{B}^\top_{L\cup I^0} \end{pmatrix} \bar{\lambda} + \begin{pmatrix} 0 \\ E^\top_{I^+\cup I^0} \end{pmatrix} \bar{\mu}. \tag{12}$$

Then $(\bar{x}, \bar{y})$ is a solution of EPCC(0). To obtain vector $d(t)$, we set

$$d(t) = \begin{pmatrix} d_x(t) \\ d_y(t) \end{pmatrix} := \begin{pmatrix} d_x(0) \\ d_y(0) \end{pmatrix} + t\left(\begin{pmatrix} c_x \\ c_y \end{pmatrix} - \begin{pmatrix} d_x(0) \\ d_y(0) \end{pmatrix}\right). \tag{13}$$

The homotopy method traces the set Σ_{C-stat} starting at $t = 0$. Note that if for the initial feasible point the complementarity constraint is biactive, the method starts at one of the two exit points.

4.1. Overview of the homotopy method I for MPCCs

Before we proceed to the homotopy method in detail, let us summarize the homotopy method I from [5] which searches for C-stationary points of convex-quadratic mathematical programs with linear complementarity constraints.

The program (3) can be converted to the following convex-quadratic MPCC in variable $z = \begin{pmatrix} x^i \\ y \end{pmatrix}$

$$\begin{aligned} &\text{minimize } \frac{1}{2} z^\top \bar{Q} z + \bar{c}^\top z \\ &\text{subject to } 0 \le \bar{A}z + \bar{a} \perp \bar{B}z + \bar{b} \ge 0, \end{aligned} \tag{14}$$

where

$$\bar{Q} = \begin{pmatrix} Q^i_{x^ix^i} & Q^i_{x^iy} \\ Q^i_{yx^i} & Q^i_{yy} \end{pmatrix}, \quad \bar{c} = \begin{pmatrix} c^i_{x^i} \\ c^i_y \end{pmatrix} + 2\begin{pmatrix} Q_{x^i,x^{-i}} \\ Q_{y,x^{-i}} \end{pmatrix} \bar{x}^{-i},$$

$$\bar{A} = (A_{x^i},\, b), \quad \bar{B} = (0,\, 1), \quad \bar{a} = (A^\top_{x^{-i}})^\top \bar{x}^{-i} + a, \quad \bar{b} = 0.$$

For the purpose of this overview, consider the general problem (14) with matrices $\bar{A}, \bar{B} \in \mathbb{R}^{m\times(l+m)}$.

Let MPEC-LICQ be satisfied at each feasible point of (14). Given a feasible point $\bar{z}$ of the MPCC (14), put $\bar{\lambda}_{I^+} = 0, \bar{\mu}_L = 0$ and

$$\bar{d} = -\bar{Q}\bar{z} + \bar{A}^\top_{L\cup I^0}\bar{\lambda}_{L\cup I^0} + \bar{B}^\top_{I^+\cup I^0}\bar{\mu}_{I^+\cup I^0}$$

with some strictly positive values of components of vectors $\bar{\lambda}_{L\cup I^0}, \bar{\mu}_{I^+\cup I^0}$.

Then $\bar{z}$ is a local minimizer of the program

$$\begin{aligned} &\text{minimize } \frac{1}{2}z^\top \bar{Q}z + (\bar{d} + t(\bar{c} - \bar{d}))^\top z \\ &\text{subject to } 0 \le \bar{A}z + \bar{a} \perp \bar{B}z + \bar{b} \ge 0 \end{aligned} \tag{15}$$

for $t = 0$. Locally around the point (t, z, λ, μ), C-stationary points of MPCC$(t+\tau)$ and their corresponding multipliers are given by

$$\begin{pmatrix} z(\tau) \\ \lambda(\tau) \\ \mu(\tau) \end{pmatrix} = \begin{pmatrix} z \\ \lambda \\ \mu \end{pmatrix} + \tau \begin{pmatrix} \dot{z} \\ \dot{\lambda} \\ \dot{\mu} \end{pmatrix}$$

with

$$\begin{pmatrix} \bar{Q} & -\bar{A}^\top_{L\cup I^0} & -\bar{B}^\top_{I^+\cup I^0} \\ \bar{A}_{L\cup I^0} & 0 & 0 \\ \bar{B}_{I^+\cup I^0} & 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{z} \\ \dot{\lambda} \\ \dot{\mu} \end{pmatrix} = \begin{pmatrix} \bar{c} - \bar{d} \\ 0 \\ 0 \end{pmatrix}.$$

At the start, t is set to zero and the method traces the homotopy path in the direction of increasing t. The steplength is then determined as the minimal positive value of $\bar{\tau}$ for which one of the following inequalities vanishes

$$\begin{aligned} \bar{A}_i z(\tau) + \bar{a}_i > 0,\ i \in I^+, \\ \bar{B}_j z(\tau) + \bar{b}_j > 0,\ j \in L, \\ \lambda_i(\tau) \ne 0,\ i \in I^0, \\ \mu_j(\tau) \ne 0,\ j \in I^0. \end{aligned}$$

This value can be easily determined using the ratios

$$\begin{aligned} q_i &= -\frac{\bar{A}_i z + \bar{a}_i}{\bar{A}_i \dot{z}},\ \ i \in I^+, \\ q_i &= -\frac{\lambda_i}{\dot{\lambda}_i},\ \ i \in I^0, \\ r_j &= -\frac{\bar{B}_j z + \bar{b}_j}{\bar{B}_j \dot{z}},\ \ j \in L, \\ r_j &= -\frac{\mu_j}{\dot{\mu}_j},\ \ j \in I^0. \end{aligned}$$

Then, if moving forward in t, the method takes the steplength

$$\bar{\tau} = \min(\{q_i \cap (0, 1-t), i \in I^+ \cup I^0, r_j \cap (0, 1-t), j \in L \cup I^0\}).$$

If this minimum is taken over the empty set, the value $t = 1$ can be reached directly and the method terminates with a C-stationary point of the MPCC (14).

If the minimum is attained at some $q_i, i \in I^0$, then $\lambda_i(t+\bar{\tau})$ vanishes and biactivity of constraint i is dropped (i. e., we put the index i to the set I^+). The sign of $\mu_i(t+\bar{\tau})$ then decides about the direction in t for the next step: if $\mu_i(t+\bar{\tau}) < 0$, the direction changes. If the minimum is attained at some $q_i, i \in I^+$, then we add the biactivity of the constraint i (i. e., we put index i to the set I^0) and the sign of multiplier $\mu_i(t+\bar{\tau})$ determines the direction of the next step. For ratios r_j we proceed analogously.

If the method currently proceeds in t backwards, the next step is the maximal negative value of the ratios

$$\bar{\tau} = \max(\{q_i \cap (-\infty, 0), i \in I^+ \cup I^0, r_j \cap (-\infty, 0), j \in L \cup I^0\}).$$

If this maximum is taken over the empty set, an infinite step could be taken to $t \searrow -\infty$ and the method thus terminates without a solution. Else, analogous changes in activities are performed.

The method described above depends on the knowledge of initial feasible point $\bar{z}$ of MPCC. The following Phase I.a approach uses the homotopy method itself to provide a feasible point.

Consider the following auxiliary problem in variables $z \in \mathbb{R}^{l+m}$ and $s \in \mathbb{R}$

$$\begin{aligned} &\text{minimize } \frac{1}{2}s^2 \\ &\text{subject to } 0 \le (\bar{A},\ \ u - \bar{a}) \begin{pmatrix} z \\ s \end{pmatrix} + \bar{a} \perp (\bar{B},\ \ v - \bar{b}) \begin{pmatrix} z \\ s \end{pmatrix} + \bar{b} \ge 0 \end{aligned} \tag{16}$$

for some chosen vectors $u, v \in \mathbb{R}^m$ with $0 \le u \perp v \ge 0$. Note that the point $(z, s) = (0, 1)$ is always feasible. Hence we can try to apply the homotopy method I to (16). If a solution point $(\bar{z}, 0)$ is found, $\bar{z}$ is a feasible point for MPCC.

If $l + 1 \le m$, the first $l + m + 1$ components of $(u^\top, v^\top)$ are set to zero and the remaining components are set to one. However, the Hessian of the objective is only positive semidefinite and thus the method may not succeed in some cases. Then, Phase I.b approach is guaranteed to provide either a feasible point or verification of inconsistency.

In Phase I.b, sometimes called the *disjunctive approach* [3], we check, using the Phase I of the simplex method, all 2^m polyhedral pieces of the feasible region. Each such piece is determined by an index set $I \subset \{1, \dots, m\}$ and conditions

$$\bar{A}_I z + \bar{a}_I = 0, \quad \bar{A}_{I^c} z + \bar{a}_{I^c} \ge 0, \tag{17}$$

$$\bar{B}_{I^c} z + \bar{b}_{I^c} = 0, \quad \bar{B}_I z + \bar{b}_I \ge 0. \tag{18}$$

If all polyhedral pieces are inconsistent, then the considered MPCC is also inconsistent.

Now, we modify this homotopy method I to the EPCC composed of MPCCs (3), using the knowledge about the structure of the set Σ_{C-stat} around co-n-singular points.

4.2. Phase I for EPCC

Analogously to Phase I procedure for MPCCs, we can compute an initial feasible point of the EPCC either via application of the homotopy method I for MPCCs to an auxiliary program or via checking each polyhedral piece of the feasible region of EPCC.

The problem (16) now takes the form of an MPCC in variables x, y and s

$$\begin{aligned} &\text{minimize } \frac{1}{2}s^2 \\ &\text{subject to } 0 \le (A,\, b,\, u-\bar{a}) \begin{pmatrix} x \\ y \\ s \end{pmatrix} + \bar{a} \perp (0,\, 1,\, v) \begin{pmatrix} x \\ y \\ s \end{pmatrix} \ge 0 \end{aligned} \tag{19}$$

for some chosen scalars $u, v \in \mathbb{R}$ with $0 \le u \perp v \ge 0$. Again, the point $(x, y, s) = (0, 0, 1)$ is always feasible. Hence, we can try to apply the homotopy method I to (19). If a solution $(\bar{x}, \bar{y}, 0)$ is found, $(\bar{x}, \bar{y})$ is a feasible point of the EPCC (3).

Similarly, if Phase I.a fails to provide a feasible point, we can apply Phase I.b. In our case it is enough to check, using the Phase I of the simplex method, just 2 polyhedral pieces of the feasible region. The first one is determined by conditions

$$Ax + by + a = 0, \quad y \ge 0,$$

while the second one by conditions

$$Ax + by + a \ge 0, \quad y = 0.$$

4.3. Overview of the algorithm

From the analysis of the structure of the set Σ_{C-stat} around co-n-singularities, it is clear that the set Σ_{C-stat} consists of finitely many convex polyhedral pieces: (one-dimensional) halflines corresponding to index sets I^+ and L and n-dimensional polyhedral sets corresponding to index set I^0. It is thus sufficient to design an algorithm which traces all one-dimensional faces of each such convex polyhedral piece; such procedure would give us full information about the set Σ_{C-stat}, see Example 1.

The description of the algorithm to trace the biactive part of the set Σ_{C-stat} is significantly more complicated then in the homotopy method I for MPCCs. We make use of the following lists of points or vectors:

"untreated exit points":	the list of visited exit points for which the corresponding set Σ_+^L or $\Sigma_+^{I^+}$ was not yet traced
"multiplier signs":	the list of vectors of signs of biactive multipliers, uniquely determining each convex polyhedral piece of biactive part of the set Σ_{C-stat}
"co-n-singularities":	the list of visited co-n-singularities
"i-singularities":	the list of visited i-singularities

"biactive C-stationary points":	the list of found C-stationary points in the biactive part of the set Σ_{C-stat}
"new directions":	the list of directions in which the next step can be made from the current iterate
"new multiplier signs":	the list of vectors of signs of biactive multipliers, uniquely determining polyhedral pieces connected by i-singularity to previously traced polyhedral piece of the set Σ_{C-stat}.

At the start of the method, all lists above are empty.

First, we describe the steps of the method based on the initial structure of the index sets.

Starting the method at $(\bar{x}, \bar{y})$ and $t = 0$ with $L = \{1\}$ or $I^+ = \{1\}$, the method traces the set Σ_+^L or $\Sigma_+^{I^+}$ in the direction of increasing t up to the respective exit point. In the former case, we compute the ratio

$$r = -\frac{y}{\dot{y}}$$

with

$$\begin{pmatrix} Q_{xx} & Q_{xy} & -\tilde{A}_L \\ Q_{yx} & Q_{yy} & -\tilde{B}_L^\top \\ \tilde{A}_L & \tilde{B}_L & 0 \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\lambda} \end{pmatrix} = \begin{pmatrix} c_x - d_x(0) \\ c_y - d_y(0) \\ 0 \end{pmatrix}.$$

For $r \leq 0$ or $r \geq 1$ we then make a step into $\bar{t} = 1$ and terminate with the solution, else take a step into $\bar{t} = r$ and add activity of the constraint $y \geq 0$. In the latter case we compute the ratio

$$q = -\frac{Ax + by + a}{A\dot{x} + b\dot{y}}$$

with

$$\begin{pmatrix} Q_{xx} & Q_{xy} & 0 \\ Q_{yx} & Q_{yy} & -E_{I^+}^\top \\ 0 & E_{I^+} & 0 \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\mu} \end{pmatrix} = \begin{pmatrix} c_x - d_x(0) \\ c_y - d_y(0) \\ 0 \end{pmatrix}.$$

For $q \leq 0$ or $r \geq 1$ we then make a step into $t = 1$ and terminate with the solution, else take a step into $t = q$ and add activity of the constraint $Ax + by + a \geq 0$.

Starting the method at $(\bar{x}, \bar{y})$ and $t = 0$ with $I^0 = \{1\}$, we add the point $(\bar{x}, \bar{y})$ to the list "untreated exit points", otherwise proceed in the same way as if we got to one of the exit points by a step described above. The reason for this is that the method traces the set Σ_+^L or $\Sigma_+^{I^+}$ at the end of the procedure unless it was already traced in the step described above.

Each step of the algorithm in the biactive case proceeds by tracing line segments between two neighboring co-n-singularities or half lines emanating from each co-n-singularity. Each such line can be generated by fixing $n-1$ vanishing multipliers. In the former case these fixed vanishing multipliers are common to both co-n-singularities. If the index sets of free multipliers are I and J, cf. assumption (A2), and we are moving in t forward, the method takes the steplength

$$\bar{\tau} = \min(\{-\frac{\lambda^j}{\dot{\lambda}^j} \cap (0, 1-t), j \in I, -\frac{\lambda^i}{\dot{\lambda}^i} \cap (0, 1-t), j \in J\}),$$

where the vectors $\dot{\lambda}_I$ and $\dot{\mu}_I$ are given by the solution of

$$\begin{pmatrix} Q_{xx} & Q_{xy} & -(\tilde{A}_{I^0}^\top)_I^\top & 0 \\ Q_{yx} & Q_{yy} & -(\tilde{B}_{I^0})_I^\top & -(E_{I^0})_J^\top \\ (\bar{A}_{I^0})_1 & (\tilde{B}_{I^0})_1 & 0 & 0 \\ 0 & E_{I^0} & 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\lambda}_I \\ \dot{\mu}_J \end{pmatrix} = \begin{pmatrix} c_x - d_x(0) \\ c_y - d_y(0) \\ 0 \\ 0 \end{pmatrix}. \tag{20}$$

If the minimum is taken over the empty set, $t = 1$ can be reached directly.

If we are moving in t backwards, the steplength is determined by

$$\bar{\tau} = \max(\{-\frac{\lambda^j}{\dot{\lambda}^j} \cap (-\infty, 0), j \in I, -\frac{\lambda^j}{\dot{\lambda}^j} \cap (-\infty, 0), j \in J\}).$$

Now we describe how the algorithm proceeds in the biactive case. First, we add the vector of signs of the nonzero multiplier vector to the list "multiplier signs". We label the exit point to be the "starting point" and initiate the following recursive procedure called "SearchStep":

1) If the current iterate is already in the list "co-n-singularities", terminate "SearchStep", else add the iterate to the list.

2) If $t = 1$, and the iterate for the current vector of multiplier signs is not on the list "biactive C-stationary points", add it to the list with information about the multiplier signs and terminate "SearchStep".

3) If the current iterate is an i-singularity, and not in the list "i-singularities", add it to the list.

4) If the current iterate is an exit point not in the list "untreated exit points" and is not labeled as "starting point", add it to the list.

5) Put all possible n directions, determined by the index sets I and J, from the current iterate to the list "new directions". As long as the list is nonempty, execute step 6).

6) For the first direction in the list, determine the direction of the step in t by the sign of the derivative in variable t of that free multiplier which is vanishing at the current iterate and the corresponding component of the vector of sings of multipliers. If they coincide, the method proceeds with the step forward in t, else we proceed backward in t. Find the steplength. If the next step has a finite length, initiate the procedure "SearchStep" for the new iterate. Delete the first entry from the list "new directions".

When the first call of "SearchStep" terminates, we have successfully finished the analysis of the first convex polyhedral patch of the set Σ_{C-stat}. Then, until the list "i-singularities" is empty, we repeat the following steps:

1) Determine the list "new multiplier signs".

2) Until the list "new multiplier signs" is empty, repeat the following. If its first entry is not in the list "multiplier signs", add it to the list "multiplier signs", label the first entry in the list "i-singularities" to be the "starting point", empty the list "co-n-singularities" and initiate "SearchStep". Delete the first entry in the list "new multiplier signs".

3) Delete the first entry in the list "i-singularities".

Now, if the list "untreated exit points" is nonempty, it suffices to check Σ_+^L and $\Sigma_+^{I^+}$ not yet investigated .

Following the set of rules above, the algorithm clearly never traces the same convex polyhedral piece of the set Σ_{C-stat} twice. However, it either terminates after one step at a nonbiactive C-stationary point or traces only polyhedral pieces of the set Σ_{C-stat} connected by i-singular and exit points with $t < 1$.

Theorem 4.1. Let the data of the EPCC (3) satisfy both assumptions (A1) and (A2). Then the following assertions hold:

i) The algorithm terminates after finitely many steps.

ii) If the list "biactive C-stationary points" is nonempty, the set of all detected biactive C-stationary points consists of the union of convex hulls of points from the list "biactive C-stationary points" with the same corresponding vector of signs of multipliers. Moreover, interior points of each such convex hull consist of nondegenerate C-stationary points with the same C-index.

Proof. The statement of part i) follows from the rules described above. There are only finitely many co-n-singularities and each convex polyhedral piece of the set Σ_{C-stat} is traced at most once.

The second statement follows from Theorems 3.3, 3.4 and 3.6. □

Example 1. (continued) Let us choose the initial feasible point $(\bar{x}^1, \bar{x}^2, \bar{y}, \bar{\lambda}^1, \bar{\lambda}^2, \bar{\mu}^1, \bar{\mu}^2) = (2, 2, 0, 0, 0, 1, 1)$. Then the computation of $d(0)$ according to (12) yields $(-6, -10, -3, -5)^\top$. The application of the homotopy method described above results in the following three C-stationary points (x^1, x^2, y): within the biactive case the algorithm finds points $(-2, 3, 0)$ and $(1,\ 0,\ 0)$ with multipliers $(\lambda^1, \lambda^2, \mu^1, \mu^2)$ equal to $(0,\ 3,\ 5/2,\ 3/2)$ and $(3/2,\ 3/2,\ 1,\ 0)$, respectively, and a nondegenerate C-stationary point $(10/3, -8/3, 2/3)$ with the multiplier vector $(17/6,\ 1/2,\ 0,\ 0)$.

The set of C-stationary points then consists of the union of the point $(10/3, -8/3, 2/3)$ and convex hull of points $(-2, 3, 0)$ and $(1,\ 0,\ 0)$. Note that since each point is even strongly stationary, the set of C-stationary points coincides with the set of solutions of the EPCC.

On Figure, the crosses and all points on the dark grey line correspond to multipliers of C-stationary points of the EPCC within the biactive case.

Table. Numerical results for homotopy method.

n	l	I.a	C	M	S	biac	#C-s	#M-s	#S-s	#n-biac	∅cpu	∅biac C-s
2	1	86	62	60	50	53	93	83	59	44	0.065	0.925
2	10	85	56	56	43	51	94	79	51	31	0.075	1.235
2	50	78	63	62	46	52	105	89	61	29	0.158	1.462
3	1	76	66	64	53	52	197	141	83	45	0.116	2.923
3	10	85	67	63	40	59	292	177	75	28	0.151	4.475
3	50	89	71	67	45	65	320	191	82	30	0.528	4.462
4	1	83	65	61	44	57	551	298	83	38	0.309	9.000
4	10	80	87	84	56	76	1191	596	163	39	0.732	15.158
4	50	81	83	80	47	74	1369	580	96	38	4.722	17.987
5	1	81	76	72	53	70	1718	817	117	48	0.949	23.857
5	10	85	85	83	39	79	3657	1277	106	35	5.331	45.848
5	50	82	93	92	54	82	5601	1849	187	44	20.018	67.7683
6	1	80	80	73	49	68	7937	1920	138	43	6.136	116.088
6	10	78	92	89	55	81	15962	4818	182	49	30.147	196.457
6	50	92	97	96	55	91	26650	6478	370	43	130.664	292.385
7	1	81	89	84	52	82	57354	9112	533	43	109.848	698.915
7	10	85	98	97	48	91	111419	17564	493	41	353.154	1223.934
7	50	89	98	98	56	94	178385	23136	727	46	1196.385	1897.223

5. NUMERICAL RESULTS

We have tested the performance of the homotopy method for EPCCs associated with $n = 2, \dots, 7$, MPCCs with convex-quadratic objective functions and with one linear complementarity constraint. For each such problem we considered $l = 1, 10$ and 50 variables on on the upper-level. For each combination of (n, l) we run the method on hundred randomly generated test problems. The algorithm was implemented in Matlab 6.5 and tests were performed on a 2.8GHz PC with 1GB RAM. The results are summarized in Table.

The columns in Table denote the following:

I.a:	number of problems, for which Phase I.a succeeded
C:	number of problems, for which at least one C-stationary point was found
M:	number of problems, for which at least one M-stationary point was found
S:	number of problems, for which at least one solution was found
biac:	number of problems, for which the method entered the biactive case
#C-s:	total number of detected C-stationary points
#M-s:	total number of detected M-stationary points
#S-s:	total number of detected solutions
#n-biact:	total number of detected nonbiactive stationary points
∅cpu:	average CPU-time for solved problems in seconds
∅biac C-s:	average number of computed C-stationary points in the biactive case.

We conclude the paper with several remarks.

For each tested problem we applied first the Phase I.a. If it failed to produce a feasible point of EPCC, the first polyhedral piece in Phase I.b yielded a starting point for our homotopy method. The first piece corresponds in our case to that part of feasible set for which the constraint $y \geq 0$ is active. We could have, of course, started immediately with Phase I.b, since for $m = 1$ this procedure involves checking only 2 pieces and is thus not that costly as in the case of a high number of complementarity constraints.

With higher values of n, the method is more likely to find a C-stationary point. Moreover, only for a very small number of test problems for which a C-stationary point was found the method failed to find also an M-stationary point. The strongly stationary points, in our case already the solutions to EPCCs, were found for each tested combination of (n, l_1) roughly for 50 percent of randomly generated test problems.

The obtained results indicate an interesting fact that EPCCs may possess *huge number of solutions*. This brings up several important issues. The most serious one is the impact of this large cardinality of the solution set on concrete decision making processes and interpretation of these solutions with respect to the input data.

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