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MAXIMAL SOLUTIONS OF TWO–SIDED LINEAR SYSTEMS IN MAX–MIN ALGEBRA

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Max-min algebra and its various aspects have been intensively studied by many authors [1, 4] because of its applicability to various areas, such as fuzzy system, knowledge management and others. Binary operations of addition and multiplication of real numbers used in classical linear algebra are replaced in max-min algebra by operations of maximum and minimum. We consider two-sided systems of max-min linear equations $A \otimes x = B \otimes x$, with given coefficient matrices A and B. We present a polynomial method for finding maximal solutions to such systems, and also when only solutions with prescribed lower and upper bounds are sought.

Keywords: max-min algebra, two-sided linear systems, lower bound, upper bound

Classification: 15A06, 15A24

1. INTRODUCTION

Max-min algebra $(\bar{\mathbb{R}}, \oplus, \otimes)$ is the set $\bar{\mathbb{R}} = \mathbb{R} \bigcup \{-\infty, \infty\}$ equipped with operations $\otimes = \min$ and $\oplus = \max$. These operations are used instead of the operations of multiplication and addition, respectively. Algebraically, max-min algebra is an idempotent semiring. In many application, the operations maximum and minimum can be considered for fuzzy relations [6]. Fuzzy relation equations are important in dynamic systems, knowledge engineering and other areas.

Max-min algebra belongs to the family of so-called extremal algebras. Interval systems in extremal algebras were investigated by [3], two-sided systems in max-plus algebras were studied by [2]. Two-sided systems of max-min linear equations are also treated in [5].

In this paper an approach to solving the two-sided problem is described. Suppose A, B are two matrices of dimension $m \times n$ and x is a vector of unknown values of dimension $n \times 1$. The main problem is to find a maximal solution of system

$$A \otimes x = B \otimes x, \tag{1}$$

sets of indices $M = \{1, 2, \ldots, m\}$, $N = \{1, 2, \ldots, n\}$. The second section describes how to find the maximal solution for two-sided problem for system (1) of dimension $(1, n)$, as well as solutions with prescribed lower and upper bounds (in a remark).

Section 3 presents an algorithm with complexity $O(m^2n)$, which finds the maximal solution for system (1) with dimension $m\ x\ n$, and also when upper and lower bounds for the solutions are given apriori. The theoretical results are accompanied by numerical examples.

2. MAXIMAL SOLUTION FOR MATRICES A, B OF DIMENSION $(1, N)$

In this section we are finding the maximal solution of the system (1) for given matrices A, B of dimension $(1, n)$ without restriction, and also with lower and upper bound constraint for the solution. In more details, the equation can be written in the form

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \end{pmatrix} \otimes \begin{pmatrix} x_1 \\ \dots \\ x_n \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \end{pmatrix} \otimes \begin{pmatrix} x_1 \\ \dots \\ x_n \end{pmatrix}, \text{ or}$$

$$(a_{11} \otimes x_1) \oplus (a_{12} \otimes x_2) \oplus \ldots \oplus (a_{1n} \otimes x_n) = (b_{11} \otimes x_1) \oplus (b_{12} \otimes x_2) \oplus \ldots \oplus (b_{1n} \otimes x_n)$$

Equivalently, we are looking for maximal values of the unknown vector x satisfying the equation

$$\bigoplus_{j=1}^{n} (a_{1j} \otimes x_j) = \bigoplus_{j=1}^{n} (b_{1j} \otimes x_j).$$

Let us denote

$$a = \max_{j=1\ldots n} (a_{11}, \ldots, a_{1n}) = \bigoplus_{j=1}^{n} a_{1j}, \quad b = \max_{j=1\ldots n} (b_{11}, \ldots, b_{1n}) = \bigoplus_{j=1}^{n} b_{1j}.$$

There are 3th three possibilities: $a = b$, $a < b$, or $a > b$. The following lemmas describe the first two cases; the third is similar to the second one.

Lemma 2.1. If $a{=}b$, then $\bar{x} = (\infty, \infty, \ldots, \infty)^T$ is the greatest solution of linear system $A \otimes x = B \otimes x$.

P r o o f. Let $a = b$. Direct computation shows that $\overline{x}$ is the solution of (1):

$$\begin{aligned} L &= A \otimes \overline{x} = \bigoplus_{j=1}^{n} (a_{1j} \otimes \infty) = \bigoplus_{j=1}^{n} (a_{1j}) = a, \\ R &= B \otimes \overline{x} = \bigoplus_{j=1}^{n} (b_{1j} \otimes \infty) = \bigoplus_{j=1}^{n} (b_{1j}) = b. \end{aligned}$$

Thus $L = R$ holds and $\overline{x}$ is the solution of system (1). Obviously $\overline{x}$ is the greatest solution. □

Lemma 2.2. Let us denote $N_1 = \{j \in N : a_{1j} > b\}$. If $a > b$, then $\bar{x} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)^T$ is the greatest solution of linear system $A \otimes x = B \otimes x$, where

$$\bar{x}_j := \begin{cases} b & \text{if } j \in N_1 \\ \infty & \text{otherwise,} \end{cases}$$

for $j = 1, \ldots, n$.

P r o o f. Assumption $a > b$. We can verify by direct computation, whether $\overline{x}$ is a solution:

$$R = B \otimes \overline{x} = \bigoplus_{j=1}^{n} (b_{1j} \otimes \overline{x}_j) \leq b.$$

There exists $k \in N$, for which $b_k = b$ holds true, $R \geq (b \otimes \overline{x}_j) = b$, for $j = 1, \ldots n$, hence $R = b$. Then $\overline{x}_j = b$ for $j \in N_1$ and $\overline{x}_j = \infty$ for $j \in N - N_1$. We have

$$L = A \otimes \overline{x} = \bigoplus_{j=1}^{n} (a_{1j} \otimes \overline{x}_j) = \left(\bigoplus_{j \in N_1} (a_{1j} \otimes b) \right) \oplus \left(\bigoplus_{j \in N - N_1} (a_{1j} \otimes \infty) \right)$$

$$= b \oplus \left(\bigoplus_{j \in N - N_1} a_{1j} \right) = b.$$

Therefore $L = b = R$ and $\overline{x} = (\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n)^T$ is the solution of system (1). It remains to prove that $\overline{x}$ is the greatest solution. The proof will be done by contradiction. Let $x' = (x'_1, x'_2, \ldots, x'_n)^T$ be a solution of (1). Suppose that $\overline{x}$ is not greater than x'. There exists $k \in N$ with $\overline{x}_k < x'_k$. Then necessarily $\overline{x}_k \neq \infty$, hence $k \in N_1$ and $a_k > b = \overline{x}_k$. By substituting x' into the original system (1) we get: $L' \geq a_k \otimes x'_k > b$, because $a_k > b$ and $x'_k > \overline{x}_k = b$. On the other hand $R' \leq b$, hence $L' \neq R'$ and x' is not a solution. □

Example 2.3. $\begin{pmatrix} 3 & 7 & 2 \end{pmatrix} \otimes \begin{pmatrix} \overline{x}_1 \\ \overline{x}_2 \\ \overline{x}_3 \end{pmatrix} = \begin{pmatrix} 4 & 2 & 5 \end{pmatrix} \otimes \begin{pmatrix} \overline{x}_1 \\ \overline{x}_2 \\ \overline{x}_3 \end{pmatrix},$

$$a = \bigoplus_{j=1}^{n} (a_{1j}) = 7, \quad b = \bigoplus_{j=1}^{n} (b_{1j}) = 5, a > b,$$

so according to Lemma 2.2:
$\overline{x}_1 = \infty, \overline{x}_2 = 5$, because $a_2 > b$, $\overline{x}_3 = \infty$, so $\overline{x} = (\infty, 5, \infty)$.

Remark 2.4. Maximal solution with given lower-bound constraint $(l_1, \ldots, l_n)^T \leq (x_1, \ldots, x_n)^T$ is defined analogously as the maximal solution without restriction. If the maximal solution without restriction is not greater than or equal to the given lower bound l, than system $A \otimes x = B \otimes x, l \leq x$ has no maximal solution. Otherwise solution is the same as the solution of the system without any restriction.

2.1. Maximal solution of $A \otimes x = B \otimes x$ with defined upper bound for matrices *A*, *B* of dimension $(1, n)$

We are finding maximal existing solution $\overline{x} = (\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n)$ of linear system $A \otimes \overline{x} = B \otimes \overline{x}$, matrices A, B of dimension $(1, n)$, with constraint condition $(x_1, \ldots, x_n)^T \leq (u_1, \ldots, u_n)^T$.

Denote $a = A \otimes u$, $b = B \otimes u$. There are three possibilities: $a = b$, $a < b$, or $a > b$. The following two lemmas describe the first two cases; the third case can be described analogously as the second case.

Lemma 2.5. If $a = b$, then vector $u = (u_1, u_2, \ldots, u_n)^T$ is the greatest solution of linear system $A \otimes x = B \otimes x$ with upper bound u.

Proof. Let $a = b$. Direct computation shows, that u is the solution of (1):

$$L = a \otimes u = \bigoplus_{j=1}^{n} (a_{1j} \otimes u_j) = a$$

and

$$R = b \otimes u = \bigoplus_{j=1}^{n} (b_{1j} \otimes u_j) = b$$

and because u is the upper boundary, u is the greatest solution. □

Lemma 2.6. Let us denote $N_2 = \{j \in N : a_{1j} \otimes u_j > b\}$. If $a > b$, then the greatest solution of linear system $A \otimes x = B \otimes x$ with the upper bound u is the vector $\bar{x} = (\bar{x}_1, \bar{x}_2, .., \bar{x}_n)^T$ defined as follows:

$$\bar{x}_j := \begin{cases} b & \text{if } j \in N_1 \\ u_j & \text{otherwise,} \end{cases}$$

for $j = 1, \ldots, n$.

By definition of $\bar{x}$, we have $\bar{x}_j = b < a_{1j} \otimes u_j \leq b$ or $\bar{x}_j = u_j \leq u_j$, for $j = 1, \ldots, n$.

Proof. Let $a > b$. We verify by direct computation that $\overline{x}$ is a solution of (1):

$$R = B \otimes \overline{x} = \bigoplus_{j=1}^{n} (b_{1j} \otimes \overline{x}_j) \leq b.$$

There exists $k \in N$, for which $b_k \otimes u_k = b$ holds true, $R \geq (b \otimes \overline{x}_j) = b$, for $j \in N$, hence $R = b$. We have

$$L = A \otimes \overline{x} = \bigoplus_{j=1}^{n} (a_{1j} \otimes \overline{x}_j) = \left(\bigoplus_{j \in N_1} (a_{1j} \otimes b) \right) \oplus \left(\bigoplus_{j \in N - N_1} (a_{1j} \otimes u_j) \right)$$

$$= b \oplus \left(\bigoplus_{j \in N - N_1} (a_{1j} \otimes u_j) \right) = b.$$

Consequently $L = b = R$ and $\overline{x} = (\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n)^T$ is the solution of system (1).

It remains to prove that $\overline{x}$ is the greatest solution. The proof will be done by contradiction. Let $x' = (x'_1, x'_2, \ldots, x'_n)^T$ be a solution of (1). Suppose that $\overline{x}$ is not greater than x'. There exists $k \in N$ with $\overline{x}_k < x'_k$. Then necessarily $\overline{x}_k \neq u_k$, hence $k \in N_1$ and $a_{1k} \otimes u_k > b = \overline{x}_k$. By substituting x' into the original system (1) we get: $L' \geq (a_{1k} \otimes u_k) \otimes x'_k > b$, because $a_{1k} \otimes u_k > b$ and $x'_k > \overline{x}_k = b$. On the other hand $R' \leq b$, hence $L' \neq R'$ and x' is not a solution. □

Example 2.7. $\begin{pmatrix} 10 & 8 & 10 \end{pmatrix} \otimes \begin{pmatrix} \overline{x}_1 \\ \overline{x}_2 \\ \overline{x}_3 \end{pmatrix} = \begin{pmatrix} 9 & 9 & 9 \end{pmatrix} \otimes \begin{pmatrix} \overline{x}_1 \\ \overline{x}_2 \\ \overline{x}_3 \end{pmatrix}, u = \begin{pmatrix} 11 \\ 11 \\ 11 \end{pmatrix}.$

Set $x = u, a(x) = A \otimes u = 10, b(x) = B \otimes u = 9, a(x) > b(x)$, so according to Lemma 2.6: $\overline{x}_1 = 9$, because $a_{11} \otimes x_1 = 10 \otimes 11 > 9, \overline{x}_3 = 9$, because $a_{13} \otimes x_3 = 10 \otimes 11 > 9$. There is the maximal solution $\overline{x} = (9, 11, 9)$.

Lemma 2.8. Assume $a > b$ and denote $N_2 = \{j \in N : a_{1j} \otimes u_j > b\}$. Suppose the interval solution $\hat{x}$:

$$\hat{x}_j := \begin{cases} < b, b > & \text{if } j \in N_2 \\ < b, u_j > & \text{otherwise,} \end{cases}$$

for $j = 1, \ldots, n$.

Interval solution $\hat{x}$ means that value $\bar{x}_j$, $j \in N$, defined by Lemma 2.6 can be changed to a value from the interval $\hat{x}_j$. Then $\forall \bar{x} \in \hat{x}$ holds $A \otimes \hat{x} = B \otimes \hat{x}$ and moreover:

$$A \otimes \hat{x} = A \otimes \bar{x}, \quad B \otimes \hat{x} = B \otimes \bar{x}.$$

P r o o f. We can verify by direct computation that $\hat{x}$ is a solution of (1), then:

a) $\hat{x} = b$ for $j \in N - N_2$

$$R = B \otimes \hat{x} = \bigoplus_{j=1}^{n} (b_{1j} \otimes \hat{x}_j) = \left(\bigoplus_{j \in N_2} (b_{1j} \otimes b) \right) \oplus \left(\bigoplus_{j \in N - N_2} (b_{1j} \otimes b) \right) = b \oplus b = b.$$

$$L = A \otimes \hat{x} = \bigoplus_{j=1}^{n} (a_{1j} \otimes \hat{x}_j) = \left(\bigoplus_{j \in N_2} (a_{1j} \otimes b) \right) \oplus \left(\bigoplus_{j \in N - N_2} (a_{1j} \otimes b) \right) = b.$$

b) $\hat{x} = u_j$ for $j \in N - N_2$

$$R = B \otimes \hat{x} = \bigoplus_{j=1}^{n} (b_{1j} \otimes \hat{x}_j) = \left(\bigoplus_{j \in N_2} (b_{1j} \otimes b) \right) \oplus \left(\bigoplus_{j \in N - N_2} (b_{1j} \otimes u_j) \right) = b \oplus b = b.$$

$$L = A \otimes \hat{x} = \bigoplus_{j=1}^{n} (a_{1j} \otimes \hat{x}_j) = \left(\bigoplus_{j \in N_2} (a_{1j} \otimes b) \right) \oplus \left(\bigoplus_{j \in N - N_2} (a_{1j} \otimes u_j) \right) = b.$$

Hence holds $L = b = R$ and $\hat{x}$ is the solution of system. □

Example 2.9.

$$\begin{pmatrix} 9 & 18 & 7 & 11 \end{pmatrix} \otimes \begin{pmatrix} \overline{x}_1 \\ \overline{x}_2 \\ \overline{x}_3 \\ \overline{x}_4 \end{pmatrix} = \begin{pmatrix} 10 & 10 & 4 & 9 \end{pmatrix} \otimes \begin{pmatrix} \overline{x}_1 \\ \overline{x}_2 \\ \overline{x}_3 \\ \overline{x}_4 \end{pmatrix}, u = \begin{pmatrix} 20 \\ 20 \\ 20 \\ 20 \end{pmatrix}.$$

According to Lemma 2.8 there is interval: $\hat{x}_1 = \langle b, u_1 \rangle = \langle 10, 20 \rangle$, $\hat{x}_2 = \langle b \rangle = \langle 10 \rangle, \hat{x}_3 = \langle b, u_3 \rangle = \langle 10, 20 \rangle, \hat{x}_4 = \langle b \rangle = \langle 10 \rangle$.

Lemma 2.8 about interval vector $\hat{x}$ will be used in the proof of theorem 3.7 in the next section.

3. MAXIMAL SOLUTION FOR MATRICES A, B OF DIMENSION (M, N)

Algorithm. Maximal solution of linear systems equation $\bar{x} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)^T$ is establihed by the algorithm with four steps. Let us define system $A \otimes x = B \otimes x$ and matrices A, B of dimension (m, n). Algorithm works in cycles and identifies $\bar{x}^k$ in cycle k and computes with

$$a_i(\bar{x}^k) = \bigoplus_{j=1}^{n} a_{ij} \otimes x_j^k, \quad b_i(\bar{x}^k) = \bigoplus_{j=1}^{n} b_{ij} \otimes x_j^k$$

in ith row. Variable k is increasing during the algorithm. Each row of system is chosen only once. The maximal possible number of cycles in the algorithm for (m, n) equals to number of rows in system, so $k \in \{0, \ldots, K\}, K \leq M, K$ is number of cycles.

Initial values before first cycle are k=0, $u = (\infty, \infty, \ldots, \infty)^T$, $\bar{x}^0 = u$.

Step 1. Set $k = k + 1$. Exchange left and right hand side if necessary according to inequality

$$a_i(\bar{x}^{k-1}) \geq b_i(\bar{x}^{k-1}) \tag{2}$$

at the beginning of each cycle.

Step 2. Select ith row of system (1) for next computing such that

$$M_k = \left\{ r \in M \colon a_r(\bar{x}^{k-1}) \neq b_r(\bar{x}^{k-1}) \right\} \tag{3}$$

$i \in I_k$,

$$I_k = \left\{ i \in M_k : b_i(\bar{x}^{k-1}) = \min_{i \in M_k} \ b_i(\bar{x}^{k-1}) \right\}. \tag{4}$$

Step 3. Find solution in the ith row, $i_k = i \in I_k$. The solution in the ith row is denoted as $\bar{x}^k$ and will be formulated according to Lemma 2.6 with upper bound $\bar{x}^{k-1}$,

$$b_k = \bigoplus_{j=1}^{n} \left(b_{i_k j} \otimes \bar{x}_j^{k-1}\right) \tag{5}$$

$$J_k = \{j \in N : a_{i_k j} \otimes \bar{x}_j^{k-1} > b_k\} \tag{6}$$

$$\bar{x}_j^k := \begin{cases} b_k & \text{if } j \in J_k \\ \bar{x}_j^{k-1} & \text{otherwise,} \end{cases}$$

for $j = 1, \ldots, n$.

Step 4. Check whether $\bar{x}^k$ is the solution to $A \otimes \bar{x}^k = B \otimes \bar{x}^k$. If yes, $\bar{x}^k$ is the maximal solution and stop the cycle, else continue with Step 1.

Lemma 3.1. Assume $\bar{x}^k$ is defined in the i_kth row according to the Step 3 in the k-cycle. Then $\bar{x}^k$ can be changed to a value from the interval $\hat{x}^k$:

$$\hat{x}_j^k := \begin{cases} \langle b_k, b_k \rangle & \text{if } j \in J_k \\ \langle b_k, \bar{x}_j^{k-1} \rangle & \text{otherwise,} \end{cases}$$

for $j = 1, \ldots, n$, and

$$a_{i_k}(\hat{x}^k) = b_{i_k}(\hat{x}^k).$$

Proof. Lemma 2.8 defines the interval solution $\hat{x}$ to the maximal solution $\bar{x}$ with defined upper bound u for matrices of dimension $(1, n)$:

$$\hat{x}_j := \begin{cases} \langle b, b \rangle & \text{if } j \in N_2 \\ \langle b, u_j \rangle & \text{otherwise,} \end{cases}$$

for $j = 1, \ldots, n$.

Analogically Step 3 defines the maximal solution $\bar{x}^k$ with defined upper bound $\bar{x}^{k-1}$ for the i_kth row, then:

$$\hat{x}_j^k := \begin{cases} \langle b_k, b_k \rangle & \text{if } j \in J_k \\ \langle b_k, \bar{x}_j^{k-1} \rangle & \text{otherwise,} \end{cases}$$

for $j = 1, \ldots, n$. □

Lemma 3.2. Assume $\bar{x}^k$ is defined according to the Step 3 in the k-cycle and $\bar{x}^{k-1}$ is the maximal solution in the $(k-1)$-cycle then

$$\bar{x}^k \leq \bar{x}^{k-1}.$$

Proof. According to the algorithm $\bar{x}_j^{k-1} \geq a_{i_k j} \otimes \bar{x}_j^{k-1} > b_k = \bar{x}_j^k$ if $j \in J_k$, hence $\bar{x}^k \leq \bar{x}^{k-1}$. □

Let us denote

$$S_1 = \{s \in M - I_k : b_s(\bar{x}^{k-1}) > b_k\}, S_2 = \{s \in M - I_k : b_s(\bar{x}^{k-1}) \leq b_k\}.$$

Lemma 3.3. Assume b_k is defined in the k-cycle (computed with the upper bound $\bar{x}^{k-1}$) and b_{k+1} is defined in the $(k+1)$-cycle (computed with the upper bound $\bar{x}^k$). Then

$$b_{k+1} \geq b_k.$$

Proof. According to the minimal $b_i = b_k$ (4), $\bar{x}^k$ results from lowering certain coordinates of $\bar{x}^{k-1}$ to the level b_k then such $s \in S_1$ that $a_s(\bar{x}^k) > b_s(\bar{x}^k)$ and $b_s(\bar{x}^k) \geq b_k$. Then for minimal b_s, $s \in S_1$ in the next cycle holds $b_s = b_{k+1}$ hence $b_{k+1} \geq b_k$. □

Lemma 3.4. Assume $\bar{x}^k$ is defined according to the Step 3 then for such $s \in S_2$, holds that:

$$a_s(\bar{x}^k) = b_s(\bar{x}^k).$$

Proof. According to the Lemma 3.1 solution $\bar{x}^k$ for sth equation can be changed to a value from the interval $\hat{x}^k := \langle b_k, \bar{x}^{k-1} \rangle$, in the concrete such value is b_{k+1} in $(k+1)$th cycle and according to Lemma 3.3 $b_{k+1} \geq b_k$ is in the interval. □

Lemma 3.5. According to the J_k (6) certain $\bar{x}_j^k$ is specified then this coordinates is not changed in l-cycle, $l > k, l \leq K$.

Proof. Suppose $\bar{x}_j^k$ can change in l-cycle, $l > k, l \leq K$ then $a_{i_l j} \otimes \bar{x}_j^{l-1} > b_l$, but $\bar{x}_j^{l-1} = b_{l-1}$ and $b_{l-1} \leq b_l$ according to Lemma 3.3. Hence $b_{l-1} \leq b_l < a_{i_l j} \otimes \bar{x}_j^{l-1}$, a contradiction. □

Lemma 3.6. A maximal solution to the system (1) does not exceed the maximal solution to any equation computed in the Step 3 of the algorithm.

P r o o f . By induction, $\bar{x}^1$ is computed with the upper bound $\bar{x}^0$, that is the maximal solution according to Lemma 2.6. Then there exists k such that maximal solution does not exceed $\bar{x}^k$. The proof will be done by contradiction. Let $\tilde{x} > \bar{x}^k$ be a solution. Then necessarily $\tilde{x}$ cannot be greater than upper bound $\bar{x}^{k-1}$, hence $\tilde{x}_j$, $j \in J_k$. We get $\tilde{x}_j > a_{i_k j} \otimes \bar{x}_j^{k-1} > b_k$, then $a_k(\tilde{x}) > b_k(\tilde{x})$ and $\tilde{x}$ is not a solution. Hence $\bar{x}^k$ does not exceed the greatest solution then it does not exceed $\bar{x}^{k+1}$. □

Theorem 3.7. The maximal solution $\bar{x}$ for system $A \otimes x = B \otimes x$ and matrices A, B of type (m, n), sets of indexes $M = \{1, 2, \ldots, m\}$, $N = \{1, 2, \ldots, n\}$ is defined by the polynomial algorithm defined above. Then $\bar{x}^K$ is the maximal solution according to the polynomial algorithm defined above. The complexity of the algorithm is $O(m^2 n)$.

P r o o f . From Lemma 3.6, it follows that the greatest solution to $A \otimes x = B \otimes x$ does not exceed the vector found by the algorithm. As this vector is itself a solution, Theorem 3.7 follows.

The algorithm complexity is stated as a function relating the input equation system to the number of inner steps. The first step of the algorithm computes maximal solution of each equation side for n coordinates. This is for all m rows hence complexity of each cycle is mn. The next limit for the run-time of the algorithm is the steps count that is determined by rows count m. Hence the presented algorithm can be said to be of order $O(m^2 n)$ that express the worst-case scenario for the given algorithm. □

Remark 3.8. Maximal solution with given lower-bound constraint $(l_1, \ldots, l_n)^T \leq (x_1, \ldots, x_n)^T$ is defined analogously as the maximal solution for system of type (m, n) without restriction. If maximal solution without restriction is not greater or equal than the given lower bound l, than system $A \otimes \bar{x} = B \otimes \bar{x}, l \leq \bar{x}$ has no maximal solution. In other cases maximal solution is the same as the solution of the system without any restriction.

Example 3.9. Example of using algorithm for matrices of type (m, n):

$$\begin{pmatrix} 5 & 1 & 3 \\ 0 & 8 & 6 \\ 1 & 1 & 5 \\ 5 & 4 & 1 \end{pmatrix} \otimes \begin{pmatrix} \infty \\ \infty \\ \infty \end{pmatrix} = \begin{pmatrix} 5 & 2 & 3 \\ 3 & 9 & 7 \\ 8 & 2 & 3 \\ 7 & 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} \infty \\ \infty \\ \infty \end{pmatrix}$$

Following four steps will be repeated in cycle, until maximal solution will be found by using algorithm defined above. The elements in matrices is lined through if equality of left and right hands side holds true.

1. To order left and right sides of equations
2. To select ith row in the k-cycle
3. To find solution in selected ith row

4. To check whether found solution is also solution of system 1

First cycle, first step:
x^0 is the maximal solution for the 1st row of the system. Non-ordered set:

$$\begin{pmatrix} \not{5} & \not{1} & \not{3} \\ 0 & 8 & 6 \\ 1 & 1 & 5 \\ 5 & 4 & 1 \end{pmatrix} \otimes \begin{pmatrix} \infty \\ \infty \\ \infty \end{pmatrix} = \begin{pmatrix} \not{5} & \not{2} & \not{3} \\ 3 & 9 & 7 \\ 8 & 2 & 3 \\ 7 & 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} \infty \\ \infty \\ \infty \end{pmatrix}, \quad \begin{matrix} a_1(\overline{x}^1) = \not{5} & b_1(\overline{x}^1) = \not{5} \\ a_2(\overline{x}^1) = 8 & b_2(\overline{x}^1) = 9 \\ a_3(\overline{x}^1) = 5 & b_3(\overline{x}^1) = 8 \\ a_4(\overline{x}^1) = 5 & b_4(\overline{x}^1) = 7 \end{matrix}$$

Ordered set:

$$\begin{pmatrix} \not{5} & \not{1} & \not{3} \\ 3 & 9 & 7 \\ 8 & 2 & 3 \\ 7 & 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} \infty \\ \infty \\ \infty \end{pmatrix} = \begin{pmatrix} \not{5} & \not{2} & \not{3} \\ 0 & 8 & 6 \\ 1 & 1 & 5 \\ 5 & 4 & 1 \end{pmatrix} \otimes \begin{pmatrix} \infty \\ \infty \\ \infty \end{pmatrix}, \quad \begin{matrix} a_1(\overline{x}^1) = \not{5} & b_1(\overline{x}^1) = \not{5} \\ a_2(\overline{x}^1) = 9 & b_2(\overline{x}^1) = 8 \\ a_3(\overline{x}^1) = 8 & b_3(\overline{x}^1) = 5 \\ a_4(\overline{x}^1) = 7 & b_4(\overline{x}^1) = 5 \end{matrix}$$

First cycle, second step:
$I_1 = \{3, 4\}$. Ordered set:

$$\begin{pmatrix} \not{5} & \not{1} & \not{3} \\ 3 & 9 & 7 \\ 8 & 2 & 3 \\ 7 & 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} \infty \\ \infty \\ \infty \end{pmatrix} = \begin{pmatrix} \not{5} & \not{2} & \not{3} \\ 0 & 8 & 6 \\ 1 & 1 & 5 \\ 5 & 4 & 1 \end{pmatrix} \otimes \begin{pmatrix} \infty \\ \infty \\ \infty \end{pmatrix}, \quad \begin{matrix} a_1(\overline{x}^1) = \not{5} & b_1(\overline{x}^1) = \not{5} \\ a_2(\overline{x}^1) = 9 & b_2(\overline{x}^1) = 8 \\ a_3(\overline{x}^1) = 8 & b_3(\overline{x}^1) = 5 \\ a_4(\overline{x}^1) = 7 & b_4(\overline{x}^1) = 5 \end{matrix}$$

First cycle, third step:
The solution for the 3rd row is computed, 4th row holds equality too. Ordered set:

$$\begin{pmatrix} \not{5} & \not{1} & \not{3} \\ 3 & 9 & 7 \\ \not{8} & \not{2} & \not{3} \\ \not{7} & \not{1} & \not{0} \end{pmatrix} \otimes \begin{pmatrix} 5 \\ \infty \\ \infty \end{pmatrix} = \begin{pmatrix} \not{5} & \not{2} & \not{3} \\ 0 & 8 & 6 \\ \not{1} & \not{1} & \not{5} \\ \not{5} & \not{4} & \not{1} \end{pmatrix} \otimes \begin{pmatrix} 5 \\ \infty \\ \infty \end{pmatrix}, \quad \begin{matrix} a_1(\overline{x}^1) = \not{5} & b_1(\overline{x}^1) = \not{5} \\ a_2(\overline{x}^1) = 9 & b_2(\overline{x}^1) = 8 \\ a_3(\overline{x}^1) = \not{5} & b_3(\overline{x}^1) = \not{5} \\ a_4(\overline{x}^1) = \not{5} & b_4(\overline{x}^1) = \not{5} \end{matrix}$$

First cycle, fourth step:

$$\begin{pmatrix} 5 \\ 9 \\ 5 \\ 5 \end{pmatrix} \neq \begin{pmatrix} 5 \\ 8 \\ 5 \\ 5 \end{pmatrix}$$

Second cycle, first step:
Ordered set:

$$\begin{pmatrix} \not{5} & \not{1} & \not{3} \\ 3 & 9 & 7 \\ \not{8} & \not{2} & \not{3} \\ \not{7} & \not{1} & \not{0} \end{pmatrix} \otimes \begin{pmatrix} 5 \\ \infty \\ \infty \end{pmatrix} = \begin{pmatrix} \not{5} & \not{2} & \not{3} \\ 0 & 8 & 6 \\ \not{1} & \not{1} & \not{5} \\ \not{5} & \not{4} & \not{1} \end{pmatrix} \otimes \begin{pmatrix} 5 \\ \infty \\ \infty \end{pmatrix}, \quad \begin{matrix} a_1(\overline{x}^2) = \not{5} & b_1(\overline{x}^2) = \not{5} \\ a_2(\overline{x}^2) = 9 & b_2(\overline{x}^2) = 8 \\ a_3(\overline{x}^2) = \not{5} & b_3(\overline{x}^2) = \not{5} \\ a_4(\overline{x}^2) = \not{5} & b_4(\overline{x}^2) = \not{5} \end{matrix}$$

Second cycle, second step:
$I_2 = \{2\}$. Ordered set:

$$\begin{pmatrix} \not{5} & \not{1} & \not{3} \\ 3 & 9 & 7 \\ \not{8} & \not{2} & \not{3} \\ \not{7} & \not{1} & \not{0} \end{pmatrix} \otimes \begin{pmatrix} 5 \\ \infty \\ \infty \end{pmatrix} = \begin{pmatrix} \not{5} & \not{2} & \not{3} \\ 0 & 8 & 6 \\ \not{1} & \not{1} & \not{5} \\ \not{5} & \not{4} & \not{1} \end{pmatrix} \otimes \begin{pmatrix} 5 \\ \infty \\ \infty \end{pmatrix}, \quad \begin{array}{ll} a_1(\overline{x}^2) = \not{5} & b_1(\overline{x}^2) = \not{5} \\ a_2(\overline{x}^2) = 9 & b_2(\overline{x}^2) = 8 \\ a_3(\overline{x}^2) = \not{5} & b_3(\overline{x}^2) = \not{5} \\ a_4(\overline{x}^2) = \not{5} & b_4(\overline{x}^2) = \not{5} \end{array}$$

Second cycle, third step:
The maximal solution for the 2nd row is computed. Ordered set:

$$\begin{pmatrix} \not{5} & \not{1} & \not{3} \\ \not{3} & \not{9} & \not{7} \\ \not{8} & \not{2} & \not{3} \\ \not{7} & \not{1} & \not{0} \end{pmatrix} \otimes \begin{pmatrix} 5 \\ 8 \\ \infty \end{pmatrix} = \begin{pmatrix} \not{5} & \not{2} & \not{3} \\ \not{0} & \not{8} & \not{6} \\ \not{1} & \not{1} & \not{5} \\ \not{5} & \not{4} & \not{1} \end{pmatrix} \otimes \begin{pmatrix} 5 \\ 8 \\ \infty \end{pmatrix}, \quad \begin{array}{ll} a_1(\overline{x}^2) = \not{5} & b_1(\overline{x}^2) = \not{5} \\ a_2(\overline{x}^2) = \not{8} & b_2(\overline{x}^2) = \not{8} \\ a_3(\overline{x}^2) = \not{5} & b_3(\overline{x}^2) = \not{5} \\ a_4(\overline{x}^2) = \not{5} & b_4(\overline{x}^2) = \not{5} \end{array}$$

Second cycle, fourth step:

$$\begin{pmatrix} 5 \\ 8 \\ 5 \\ 5 \end{pmatrix} = \begin{pmatrix} 5 \\ 8 \\ 5 \\ 5 \end{pmatrix}.$$ The maximal solution of the example system is $\overline{x} = \begin{pmatrix} 5 \\ 8 \\ \infty \end{pmatrix}$.

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