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Martin Gavalec; Karel Zimmermann

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SOLVING SYSTEMS OF TWO–SIDED (MAX, MIN)–LINEAR EQUATIONS

Martin Gavalec and Karel Zimmermann

A finite iteration method for solving systems of (max, min)-linear equations is presented. The systems have variables on both sides of the equations. The algorithm has polynomial complexity and may be extended to wider classes of equations with a similar structure.

Keywords: (max, min)-linear equations, two-sided system

Classification: 08A72, 90B35, 90C47

1. INTRODUCTION

Problems on algebraic structures, in which pairs of operations $(\max, +)$ or $(\max, \min)$ replace addition and multiplication of the classical linear algebra, appear in the literature approximately since the sixties of the last century (see e.g. [6], [11]). A systematic theory of such algebraic structures was published probably for the first time in [6]. Systems of so-called $(\max, +)$- or $(\max, \min)$-linear equations with variables on only one side of the equations were investigated, among other problems, in these publications. Since the operation 'max' replacing addition is not a group operation, but only a semigroup one, there is a substantial difference between solving systems with variables on one side of the equations, and systems with variables occurring on both sides. The former systems will be called 'one-sided' and the latter ones 'two-sided'. Special two-sided systems were studied in [4], [5], [6], [9], [11] in connection with the so-called $(\max, +)$- or $(\max, \min)$-eigenvalue problem. General two-sided $(\max, +)$-linear systems were studied in [2], [3], [7]. The general results obtained in these papers are influenced by the fact that the second operation in $(\max, +)$ algebra is a group operation, while in $(\max, \min)$ algebra it is an idempotent semigroup operation. The presented paper uses special properties of operation min and describes a simple polynomial algorithm for solving two-sided (max,min)-linear systems.

Two-sided systems with a more general structure, in which residuated functions occur on both sides of the equations, were investigated in [8], where a general iteration method for solving such systems was proposed. In general case, the method does not work in finite time and an approximation of the solution is only obtained as a result of the computation. If the method is applied to a $(\max, \min)$-linear system, then it works in finite time and gives an exact solution, however, the convergence is rather slow and only pseudopolynomial complexity can be proved. Applications of the problems mentioned above to synchronization of events, system reliability and fuzzy relations can be found e.g. in [1], [6], [9], [10], [11].

In this paper, a polynomial method for solving a general two-sided system of max-min linear equations is presented. The method finds the maximum solution of the system. Computational complexity of the proposed method is $O(mn \cdot \min(m,n))$, where m is the number of equations and n is the number of variables ($O(n^3)$, if $m = n$). The method is demonstrated on a small numerical example. Two other examples showing further application areas are presented.

2. NOTATION AND FORMULATION OF THE PROBLEM

Let us introduce the following notation: $I = \{1,\dots,m\}$, $J = \{1,\dots,n\}$, $R = (-\infty,\infty)$, $\overline{R} = [-\infty,\infty]$, $R^n = R \times \cdots \times R$ (n-times), similarly $\overline{R}^n = \overline{R} \times \cdots \times \overline{R}$. Further, we denote $\alpha \wedge \beta \equiv \min\{\alpha,\beta\}$ for any $\alpha,\beta \in \overline{R}$, and we set per definition $-\infty \wedge \infty = -\infty$.

Let $a_{ij}, b_{ij} \in \overline{R}$, $i \in I$, $j \in J$ be given numbers. Then we define for any vector $x = (x_1,\dots,x_n) \in \overline{R}^n$ and for every $i \in I$

$$\begin{aligned} a_i(x) &\equiv \max_{j\in J}(a_{ij} \wedge x_j) \\ b_i(x) &\equiv \max_{j\in J}(b_{ij} \wedge x_j) \end{aligned}$$

We will consider the following system of (max, min)-linear (i.e. (max, $\wedge$)-linear) equations

$$a_i(x) = b_i(x) \quad \text{for } i \in I \tag{1}$$

The set of all solutions of system (1) will be denoted by M. Further we define sets $M(\overline{x})$, $M_i(\overline{x})$ for any $\overline{x} \in \overline{R}^n$, $i \in I$ as follows

$$M(\overline{x}) \equiv \{x;\, x \in M \;\&\; x \le \overline{x}\} \tag{2}$$

$$M_i(\overline{x}) \equiv \{x;\, a_i(x) = b_i(x) \;\&\; x \le \overline{x}\} \tag{3}$$

The inequalities between vectors in equations (2), (3), as well as in the rest of the paper, are meant componentwise.

Let us note that sets $M(\overline{x})$, $M_i(\overline{x})$ are always nonempty, since $x(\alpha) \equiv (\alpha,\dots,\alpha)$ belongs to $M(\overline{x})$, if $\alpha \le \min\{a_{ij} \wedge b_{ij} \wedge \overline{x}_j;\, (i,j) \in I \times J\}$. Let us further note that for $\overline{x} \equiv (\infty,\dots,\infty)$ and $\underline{x} = (-\infty,\dots,-\infty)$ we have $M(\overline{x}) = M$ and $\underline{x} \le x$ for any $x \in M$.

Definition 2.1. Let $L \subseteq \overline{R}^n$, $\tilde{x} \in L$, such that for every $x \in L$ the inequality $x \le \tilde{x}$ holds true. Then $\tilde{x}$ is called the *maximum element* of L.

Remark 2.2. The algorithm presented in this paper will find the maximum element in $M(\overline{x})$ for any given $\overline{x} \in \overline{R}^n$.

3. THEORETICAL BACKGROUND

In this section we prepare the theoretical background for an algorithm for finding the maximum element $x^{\max}$ in $M(\overline{x})$.

If $\overline{x} \in M(\overline{x})$, then we evidently have $x^{\max} = \overline{x}$. Therefore we will assume in what follows that $\overline{x} \notin M(\overline{x})$. Further, we can assume w.l.o.g. that the notation has been possibly changed in such a way that $a_i(\overline{x}) \le b_i(\overline{x})$ holds for all $i \in I$. Since we have assumed that $\overline{x} \notin M(\overline{x})$, the set

$$I^{<}(\overline{x}) \equiv \big\{ i \in I;\ a_i(\overline{x}) < b_i(\overline{x}) \big\}$$

is nonempty. Let us set further

$$I^{=}(\overline{x}) \equiv \big\{ i \in I;\ a_i(\overline{x}) = b_i(\overline{x}) \big\}$$

and let us introduce the following notations for any given upper bound $\overline{x}$

$$\begin{aligned}
\alpha(\overline{x}) &\equiv \min\big\{ a_i(\overline{x});\ i \in I^{<}(\overline{x}) \big\} \\
I^{<}(\alpha(\overline{x})) &\equiv \big\{ i \in I^{<}(\overline{x});\ a_i(\overline{x}) = \alpha(\overline{x}) \big\} \\
I^{=}(\alpha(\overline{x})) &\equiv \big\{ i \in I^{=}(\overline{x});\ a_i(\overline{x}) \le \alpha(\overline{x}) \big\} \\
J(\alpha(\overline{x})) &\equiv \big\{ j \in J;\ \big(\exists i \in I^{<}(\alpha(\overline{x}))\big)\big[b_{ij} \wedge \overline{x}_j > \alpha(\overline{x}) \big] \big\}
\end{aligned}$$

Notice that $J(\alpha(\overline{x}))$ is always nonempty if $\overline{x} \notin M$. To simplify the explanations, we will replace in what follows the notation $\alpha(\overline{x})$ with α, if it cannot cause any confusion.

Theorem 3.1. Let us assume that $\overline{x} \notin M(\overline{x})$. Let us define $\tilde{x}$ as follows

$$\tilde{x}_j = \begin{cases} \alpha & \text{for } j \in J(\alpha) \\ \overline{x}_j & \text{for } j \in J \setminus J(\alpha) \ . \end{cases} \tag{4}$$

Then the following assertions are fulfilled

(i) $\tilde{x}$ is the maximum element of the set of all solutions of the system

$$a_i(x) = b_i(x) \quad \text{for } i \in I^{<}(\alpha) \cup I^{=}(\alpha) \tag{5}$$

$$x_j \le \overline{x}_j \quad \text{for } j \in J \ , \tag{6}$$

(ii) for every $i \in I$ with $a_i(\tilde{x}) \ne b_i(\tilde{x})$ the following inequalities hold true

$$\alpha < \min\big(a_i(\overline{x}), b_i(\overline{x})\big) \tag{7}$$

$$\alpha \le \min\big(a_i(\tilde{x}), b_i(\tilde{x})\big) \ , \tag{8}$$

hence $\alpha(\tilde{x}) \ge \alpha(\overline{x})$.

P r o o f. (i) Let $k \in I^{<}(\alpha)$ be chosen arbitrarily. Then $a_k(\overline{x}) = \alpha < b_k(\overline{x})$. Let us set

$$J_k(\alpha) \equiv \big\{ j \in J;\ b_{kj} \wedge \overline{x}_j \ > \ \alpha \big\} \ .$$

Then $J_k(\alpha) \neq \emptyset$, $J_k(\alpha) \subseteq J(\alpha)$. Note that for any $j \in J_k(\alpha)$ both $b_{kj} > \alpha$ and $\overline{x}_j > \alpha$ hold true. It follows immediately from the definition of $\tilde{x}$ that

$$\begin{aligned} b_{kj} \wedge \tilde{x}_j \leq \alpha \quad & \text{for } \ j \in J \\ b_{kj} \wedge \tilde{x}_j = \alpha \quad & \text{for } \ j \in J_k(\alpha) \end{aligned}$$

hence $b_k(\tilde{x}) = \alpha$ (we remind that $a_k(\overline{x}) = \alpha$).

Let p be any index of J such that $a_k(\overline{x}) = a_{kp} \wedge \overline{x}_p$. Then $a_{kp} \wedge \overline{x}_p = \alpha$, hence $a_{kp} \geq \alpha$ and we have according to (4)

$$\begin{aligned} a_{kp} \wedge \tilde{x}_p = a_{kp} \wedge \alpha = \alpha \qquad & \text{if } \ p \in J(\alpha) \ , \\ a_{kp} \wedge \tilde{x}_p = a_{kp} \wedge \overline{x}_p = a_k(\overline{x}) = \alpha \qquad & \text{if } \ p \notin J(\alpha) \ . \end{aligned}$$

Further, since $a_{kj} \wedge \tilde{x}_j \leq a_{kj} \wedge \overline{x}_j$ for all $j \in J$, we obtain that $a_k(\tilde{x}) = \alpha = b_k(\tilde{x})$.

Let us assume now that k is an arbitrary index of $I^=(\alpha)$ so that we have $a_k(\overline{x}) \leq \alpha$ and $a_k(\overline{x}) = b_k(\overline{x}) = \beta_k \leq \alpha$. Let $s \in J$ be an index such that $a_k(\overline{x}) = a_{ks} \wedge \overline{x}_s$. If $s \notin J(\alpha)$, then $\tilde{x}_s = \overline{x}_s$ by (4) and thus $a_{ks} \wedge \tilde{x}_s = a_{ks} \wedge \overline{x}_s = \beta_k$. Let us assume, on the other hand, that $s \in J(\alpha)$. Then there exists index $i \in I$ such that $b_{is} \wedge \overline{x}_s > \alpha$ and therefore it must be $\overline{x}_s > \alpha$. Since we have assumed that $a_k(\overline{x}) = a_{ks} \wedge \overline{x}_s = \beta_k \leq \alpha$, and we have $\overline{x}_s > \alpha$, it must be $a_{ks} = \beta_k$. Since $s \in J(\alpha)$, we have $\tilde{x}_s = \alpha \geq \beta_k$, and therefore we have $a_{ks} \wedge \tilde{x}_s = \beta_k$. By the inequality $a_{kj} \wedge \tilde{x}_j \leq a_{kj} \wedge \overline{x}_j \leq \beta_k$ which holds for all $j \in J$, we obtain $a_k(\tilde{x}) = \beta_k$.

Next we compute the value $b_k(\tilde{x})$. We have assumed that $b_k(\overline{x}) = \beta_k \leq \alpha$. Let us assume that $b_k(\overline{x}) = b_{ks} \wedge \overline{x}_s$ for some $s \in J$. Similarly as above, we have $b_{ks} \wedge \tilde{x}_s = b_{ks} \wedge \overline{x}_s$, if $s \notin J(\alpha)$. If $s \in J(\alpha)$, then we have $\overline{x}_s > \alpha \geq \beta_k$, $\tilde{x}_s = \alpha \geq \beta_k$ and therefore it must be $b_{ks} = \beta_k$, so that $b_{ks} \wedge \tilde{x}_s = b_{ks} \wedge \alpha = \beta_k$. By the inequality $b_{kj} \wedge \tilde{x}_j \leq b_{kj} \wedge \overline{x}_j$, which holds for all $j \in J$, we obtain $b_k(\tilde{x}) = \beta_k = a_k(\tilde{x})$. In other words, the equality with index $k \in I^=(\alpha)$, which holds for $\overline{x}$, remains satisfied also for $\tilde{x}$.

For the proof of assertions (i) it remains to prove that $\tilde{x}$ is the maximum element satisfying the system (5), (6). Let us assume for this purpose that x is a vector such that $x \not\leq \tilde{x}$, so that there exists an index $r \in J$ such that $\tilde{x}_r < x_r \leq \overline{x}_r$. Therefore it must be $r \in J(\alpha)$ and there exists an index $i \in I^<(\alpha)$ such that $a_i(\tilde{x}) = \alpha < b_{ir} \wedge x_r \leq b_i(x)$ and according to the above considerations we have $a_{ir} = \alpha = a_{ir} \wedge \tilde{x}_r = a_{ir} \wedge x_r$. Since this equality holds for any index r with the property $\tilde{x}_r < x_r \leq \overline{x}_r$ and for the other indices $j \in J$ we have $x_j \leq \tilde{x}_j \leq \overline{x}_j$, we obtain that $a_i(x) = a_i(\tilde{x}) = \alpha < b_{ir} \wedge x_r \leq b_i(x)$, and therefore x does not satisfy the system (5), (6), which completes the proof.

(ii) Let us assume that $a_i(\tilde{x}) \neq b_i(\tilde{x})$ holds for some fixed $i \in I$. Then, in view of the assertion (i), we have $i \notin I^<(\alpha)$ and $i \notin I^=(\alpha)$. By definition of the set $I^<(\alpha)$ we get that either $i \notin I^<(\overline{x})$ (i.e. $a_i(\overline{x}) = b_i(\overline{x})$), or $\alpha < a_i(\overline{x}) < b_i(\overline{x})$. Further we get, by definition of the set $I^=(\alpha)$, that either $i \notin I^=(\overline{x})$ (i.e. $a_i(\overline{x}) < b_i(\overline{x})$), or $\alpha < a_i(\overline{x}) = b_i(\overline{x})$. Summarizing, the inequality (7) is fulfilled in the case when $a_i(\overline{x}) < b_i(\overline{x})$, as well as in the case $a_i(\overline{x}) = b_i(\overline{x})$.

For the proof of (8) we use first the fact that, in view of (7), there is $k \in J$ such that $a_{ik} \wedge \overline{x}_k > \alpha$, which implies $a_{ik} > \alpha$. If $k \in J(\alpha)$, then by definition of $\tilde{x}$ we have $\tilde{x}_k = \alpha$, hence $a_{ik} \wedge \tilde{x}_k = \alpha$. On the other hand, if $k \notin J(\alpha)$, then $\tilde{x}_k = \overline{x}_k$ and $a_{ik} \wedge \tilde{x}_k = a_{ik} \wedge \overline{x}_k > \alpha$. In both cases we get $a_i(\tilde{x}) = \max_{j \in J}(a_{ij} \wedge \tilde{x}_j) \geq a_{ik} \wedge \tilde{x}_k \geq \alpha$. The inequality $b_i(\tilde{x}) \geq \alpha$ is proved analogously. □

4. THE ALGORITHM

Summarizing the considerations of Section 3, we propose an algorithm to find the maximum element $x^{\max}$ of the set $M(\overline{x})$. In every iteration we find $\tilde{x}$ using formula (4). According to Theorem 3.1, $\tilde{x}$ is the maximum element of the set of all solutions of the system (5), (6). Therefore, if $\tilde{x} \in M(\overline{x})$, then $\tilde{x} = x^{\max}$ and we stop. Otherwise, we use $\tilde{x}$ as the new upper bound and repeat the procedure.

Let us assume that $\tilde{x} \notin M(\overline{x})$ and let us change the notation in such a way that $a_i(\tilde{x}) \leq b_i(\tilde{x})$ for all $i \in I$. To avoid confusion in further explanations, we return to the full notation $\alpha(\overline{x})$ for any upper bound $\overline{x}$. Then we have $\alpha(\tilde{x}) \equiv \min\{a_i(\tilde{x});\, i \in I^<(\tilde{x})\}$ and, according to the inequality (8) in Theorem 3.1 we get the inequality $\alpha(\tilde{x}) \geq \alpha = \alpha(\overline{x})$, which immediately implies the set inclusion $I^=(\alpha(\overline{x})) \subseteq I^=(\alpha(\tilde{x}))$. By this inclusion, the vector $\tilde{x}$ fulfills every equation with index $i \in I^=(\alpha(\overline{x}))$, which is fulfilled by the previous vector $\overline{x}$ and the common value $a_i(\overline{x}) = b_i(\overline{x})$ on both sides of the equation is less or equal than $\alpha = \alpha(\overline{x})$. However, if $a_i(\overline{x}) = b_i(\overline{x}) > \alpha(\overline{x})$, then the equation need not be satisfied by $\tilde{x}$.

By definition of $\tilde{x}$ we have $b_{ij} \wedge \tilde{x}_j \leq \tilde{x}_j = \alpha(\overline{x})$ for all $i \in I$, $j \in J(\alpha(\overline{x}))$. On the other hand, the definition of $J(\alpha(\tilde{x}))$ says that for every $j \in J(\alpha(\tilde{x}))$ there is $i \in I^<(\alpha(\tilde{x}))$ such that $b_{ij} \wedge \tilde{x}_j > \alpha(\tilde{x}) \geq \alpha(\overline{x})$. These contradictory statements imply $J(\alpha(\tilde{x})) \cap J(\alpha(\overline{x})) = \emptyset$. Therefore, if we use $\tilde{x}$ as a new upper bound on the next iteration, we will decrease at least one new variable in (4). Therefore we will perform at most n iterations. Besides, since $\alpha(\tilde{x}) \geq \alpha(\overline{x})$, all the equations already satisfied with indexes $i \in I^<(\alpha(\overline{x})) \cup I^=(\alpha(\overline{x}))$ will remain satisfied, in accordance with Theorem 3.1. It follows that in the next iteration with the new upper bound $\tilde{x}$ after applying formula (4), the already satisfied equations remain satisfied and at least one new equation with index $i \in I^<(\alpha(\tilde{x}))$ will hold true. Therefore, the number of iterations does not exceed $\min(n, m)$. The corresponding algorithm is described explicitly below, step by step.

ALGORITHM $\mathcal{A}$

[1] Input m, n, $\overline{x}$;

[2] If $\overline{x} \in M(\overline{x})$, then $x^{\max} := \overline{x}$; STOP;

[3] Change notation so that $a_i(\overline{x}) \leq b_i(\overline{x})$ for all $i \in I$;

[4] Compute $\alpha(\overline{x})$, $I^<(\alpha(\overline{x}))$, $I^=(\alpha(\overline{x}))$;

[5] Set $\tilde{x}_j := \alpha(\overline{x})$ if $j \in J(\alpha(\overline{x}))$, $\tilde{x}_j := \overline{x}_j$ otherwise;

[6] If $\tilde{x} \in M(\overline{x})$, then $x^{\max} := \tilde{x}$; STOP;

$\boxed{7}$ Set $\overline{x} := \tilde{x}$; go to $\boxed{3}$;

One passing between steps $\boxed{3}$ and $\boxed{7}$ is called an iteration. The algorithm creates a sequence $\alpha_1, \alpha_2, \dots, \alpha_Q$ and the corresponding upper bounds for the solution set M. The key observation for obtaining the complexity bound is the inequality

$$\alpha_{q+1} \geq \alpha_q \quad \text{for each } q \ . \tag{9}$$

Let us call a variable x_j active in iteration q if $j \in J(\alpha_q)$. In other words, variable x_j is decreased to the value α_q in iteration Q. Thanks to inequality (9), a variable active in some iteration can never again become active. Then, since $J(\alpha_q)$ is never empty, we get that the number of iterations Q is not greater than n. Further, the set $I^<(\alpha_q)$ is nonempty in each iteration and if the index i of some equation belongs to $I^<(\alpha_q)$, then $i \in I^=(\alpha_{q'})$ for each $q' > q$. This proves $Q \leq m$.

Every iteration has the computational complexity $O(mn)$. Since the number of iterations does not exceed $\min(m, n)$, we obtain the total complexity $O(mn \cdot \min(m, n))$. If $m = n$, then the total complexity is $O(n^3)$.

Remark 4.1. Let us include additionally in our considerations some lower bound $\underline{x}$ and set $M(\underline{x}, \overline{x}) \equiv \{x \ ; \ x \geq \underline{x} \ \& \ x \in M(\overline{x})\}$. It follows immediately from our previous results that the set $M(\underline{x}, \overline{x})$ is nonempty if and only if $\underline{x} \leq x^{\max}$, where similarly as above $x^{\max}$ is the maximum element of $M(\overline{x})$.

We illustrate the work of algorithm $\mathcal{A}$ by the following numerical example.

Example 4.2. Let us consider system of equations (1) with the following matrices A, B:

$$A = \begin{pmatrix} 7 & 2 & 3 & 1 & 0 \\ 2 & 7 & 2 & 0 & -1 \\ 10 & 8 & 4 & 2 & 1 \\ 2 & 1 & 2 & 1 & 0 \\ 100 & 6 & 1 & 1 & 8 \end{pmatrix} \qquad B = \begin{pmatrix} 9 & 4 & 8 & 1 & 1 \\ 6 & 7 & 8 & -3 & 3 \\ 15 & 14 & 9 & -2 & 0 \\ 0 & -1 & 2 & 0 & 0 \\ 100 & 0 & -5 & 1 & 10 \end{pmatrix}$$

$\boxed{1}$ $m := 5,\ n := 5,\ I := \{1, 2, 3, 4, 5\},\ J := \{1, 2, 3, 4, 5\},\ k := 0,$
$\overline{x} := (100, 100, 100, \infty, 100) \in \overline{R}^5$;

$\boxed{2}$ $\overline{x} \notin M(\overline{x})$;

$\boxed{3}$ we have already $a_i(\overline{x}) \leq b_i(\overline{x})$ for all $i \in I$;

$\boxed{4}$ $\alpha(\overline{x}) = 7,\ I^<(\alpha(\overline{x})) = I^<(7) = \{1, 2\},\ I^=(\alpha(\overline{x})) = I^=(7) = \{4\},$
$J(\alpha(\overline{x})) = J(7) = \{1, 3\}$;

$\boxed{5}$ $\tilde{x} = (7, 100, 7, \infty, 100)$;

$\boxed{6}$ $I^<(7) \cup I^=((7) \neq I$;

================= 1st iteration =================

$\boxed{3}$ we have already $a_i(\overline{x}) \leq b_i(\overline{x})$ for all $i \in I$;

[4] $\alpha(\overline{x}) = 8,\ I^{<}(\alpha(\overline{x})) = I^{<}(8) = \{3,5\},\ I^{=}(\alpha(\overline{x})) = I^{=}(8) = \{1,2,4\}$,
$J(\alpha(\overline{x})) = J(8) = \{2,5\}$;

[5] $\tilde{x} = (7,8,7,\infty,8)$;

[6] $I^{<}(8) \cup I^{=}(8) = I,\ \tilde{x} = (7,8,7,\infty,8) = x^{\max}$, STOP .

================== 2nd iteration ==================

Below we present two examples, showing areas where the problems considered in this paper can be applied.

Example 4.3. Let us consider a situation, in which transportation means of different size are transporting goods from places $i \in I$ to one terminal T. The goods are unloaded in T and the transportation means (possibly with other goods uploaded in T) have to return to i.

We assume that the connection between i and T is only possible via one of the places (e.g. cities) $j \in J$, the roads between i and j are one-way roads, and the capacity of the road between $i \in I$ and $j \in J$ is equal to a_{ij}. We have to join places j with T by a two-way road with a capacity x_j in both directions. The total capacity of the connection between i and T is therefore equal to $\max\{a_{ij} \wedge x_j;\ j \in J\}$. The transport from T to i is carried out via other one-way roads between places $j \in J$ and $i \in I$ with (in general, different) capacities between j and i equal to b_{ij}. Since the roads between T and j are two-way roads, the total capacity of the connection between T and i is equal to $\max\{b_{ij} \wedge x_j;\ j \in J\}$ for all $i \in I$.

We assume that the transportation means can only pass through some roads with the capacity which is not smaller than the capacity of the transportation mean and our task is to choose appropriate capacities x_j, $j \in J$. In order that each of the transportation means may return to i, it is natural to require for each i that the maximal attainable capacity of connections between i and T via j is equal to maximal attainable capacity of connections between T and i on the way back. In other words, we have to choose x_j, $j \in J$ in such a way that

$$\max\{a_{ij} \wedge x_j;\ j \in J\} = \max\{b_{ij} \wedge x_j;\ j \in J\} \quad \text{for all} \ \ i \in I \ .$$

We see that the problem transforms to solving system (1) with some finite upper bounds on x, since in reality the chosen capacities cannot be unbounded.

Note that the model is flexible enough to include different real situations. If, for instance, the road between i and j does not exist, we set simply $a_{ij} = 0$, if the road is a two way road with equal capacity in both directions, we set $b_{ij} = a_{ij}$, or, if we do not want to connect j with T, we set the lower and upper bound on x_j equal to 0. Also the connection of T with j does not need necessarily have the same capacity x_j in both directions, in such a case we can insert on the right-hand sides different variables y_j, and the transformation to system of the form (1) is then only a technical problem. Namely, in this case we have a vector of variables $(x,y) \in R^{2n}$ and we can introduce additional coefficients $a_{ij} = -\infty$ for $j = n+1,\ \ldots,\ 2n$ on the left hand side and $b_{ij} = -\infty$ for $j = 1,\ \ldots,\ n$ on the right hand side of every equation.

The next example shows an application within the fuzzy set theory.

Example 4.4. First let us recall that the height of a fuzzy set F is defined as the maximum of the values of its membership function μ_F on its support S. We will introduce the notation $\mathrm{Hght}(F) = \max\{\, \mu_F(y);\, y \in S\,\}$. Further let

$$\left\{\mu_i^{(1)} : J \to [0,1]\,;\, i \in I\right\}, \quad \left\{\mu_i^{(2)} : J \to [0,1]\,;\, i \in I\right\}$$

be membership functions of two groups of fuzzy sets

$$\left\{A_i^{(1)};\, i \in I\right\}, \quad \left\{A_i^{(2)};\, i \in I\right\}$$

with a finite support J. We want to find a fuzzy set X with membership function $\mu_X : J \to [0,1]$ (i.e. we want to find values $\mu_X(j)$ for $j \in J$) in such a way that

$$\mathrm{Hght}\left(A_i^{(1)} \cap^* X\right) = \mathrm{Hght}\left(A_i^{(2)}) \cap^* X\right) \quad \text{for all } \ i \in I \ ,$$

where the intersection $\cap^*$ means the intersection of two fuzzy sets. In other words, we have to solve the the following system of equations with respect to $\mu_X(j),\ j \in J$

$$\max_{j \in J}\left(\mu_i^{(1)}(j) \wedge \mu_X(j)\right) = \max_{j \in J}\left(\mu_i^{(2)}(j) \wedge \mu_X(j)\right) \quad \text{for all } \ i \in I$$

$$\mu_X(j) \in [0,1] \quad \text{for all } \ j \in J \ .$$

Let us set $a_{ij} \equiv \mu_i^{(1)}(j)$, $b_{ij} \equiv \mu_i^{(2)}(j)$, $x_j \equiv \mu_X(j)$, $\underline{x}_j = 0$, $\overline{x}_j = 1$ for all $i \in I$, $j \in J$. Then we see that we have to solve a system of the form (1). The maximum element $x^{\max}$ of $M(\underline{x}, \overline{x})$ can be found in this case by making use of the unchanged algorithm $\mathcal{A}$, because the lower bound $\underline{x} = 0 \in R^n$ satisfies the inequalities $\underline{x}_j \le a_{ij} \wedge b_{ij}$ for all $i \in I$, $j \in J$ (compare Remark 4.1). The maximum element $x^{\max}$ represents a fuzzy set X with the highest membership values to the set of feasible solutions of the problem. We could also require that $\underline{x} \neq 0$ and proceed in accordance with Remark 4.1.

The next remark shows one possible interpretation of the set X from the preceding example.

Remark 4.5. Let goals G_k, $k \in K \equiv \{1, \dots, p\}$ be given. The goals should be achieved by using treatments $T_j, j \in J$. The goals may be e.g. projects, plans or symptoms of diseases, the treatments may represent e.g. incentives, investments, medicaments etc. We assume that for each goal G_k a fuzzy set R_k of effective treatments with finite support J and membership function $\mu_{G_k} : J \to [0,1]$ is given (if $\mu_{G_k}(j) = 1$, then treatment T_j is strongly effective, if $\mu_{G_k}(j) = 0$, it is not effective). To simplify the notation, let us set in the sequel $\mu_{G_k}(j) = r_{kj}$.

We want to find values $\mu_X(j),\ j \in J$ of fuzzy set X, of intensively applied treatments, the values of the membership function μ_X may be interpreted as intensity of application of T_j (i.e. if $\mu_X(j) = 1$, then T_j is strongly applied, if $\mu_X(j) = 0$, then T_j

is not applied at all). Value $\mathrm{Hght}(R_k \cap^* X) = \max_{j\in J}(r_{kj} \wedge x_j)$ can be interpreted as a level, with which goal G_k is achieved ('hit') if $x = (x_1, \dots, x_n)$ is chosen and the effectiveness of the treatments with respect to goal G_k is taken into account.

In other words, we can introduce fuzzy set G of 'achieved' goals with membership function $\mu_G : K \to [0,1]$, $\mu_G(k, X) \equiv \mathrm{Hght}(R_k \cap^* X) = \max_{j\in J}(r_{kj} \wedge x_j)$ for every $k \in K$ (the value $\mu_G(k) = 1$ means that goal G_k is achieved, and $\mu_G(k) = 0$ means that G_k is not achieved, if the application intensity X with $\mu_X(j) = x_j$, $j \in J$ is chosen). To simplify the notation, we set $r_k(x) \equiv \max_{j\in J}(r_{kj} \wedge x_j)$. Because of some technological, ecological, medical or other reasons, we may require that some relations between values $r_k(x)$, $k \in K$ must hold e.g. $r_s(x) = r_t(x)$ for some s, $t \in K$, $s \neq t$ (it can be interpreted that some goals should be achieved on equal level). The maximum element $x^{\max}$ ensures the maximal level of achievement of the goals under the given conditions.

For similar reasons it can be required that some upper and lower bounds $\underline{x}$, $\overline{x} \in [0,1]^n$ for intensities x are given. By appropriate change of notation, such system of relations can be transformed to a system of (max, min)-linear equations with lower and upper bounds imposed on x of a similar form like the one considered in Example 4.4.

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REFERENCES

[1] F. L. Baccelli, G. Cohen, G. J. Olsder, and J. P. Quadrat: Synchronization and Linearity. An Algebra for Discrete Event Systems. Wiley, Chichester, 1992.

[2] P. Butkovič and G. Hegedüs: An elimination method for finding all solutions of the system of linear equations over an extremal algebra. Ekonomicko–matematický obzor *20* (1984), 203–215.

[3] P. Butkovič and K. Zimmermann: A strongly polynomial algorithm for solving two-sided linear systems in max-algebra. Discrete Applied Mathematics *154* (2006), 437–446.

[4] K. Cechlárová: Efficient computation of the greatest eigenvector in fuzzy algebra. Tatra Mt. Math. Publications *12* (1997), 73–79.

[5] K. Cechlárová: Eigenvectors of interval matrices over max-plus algebra. Discrete Applied Mathematics *150* (2005), Nos. 1–3, 2–15.

[6] R. A. Cuninghame-Green: Minimax Algebra. (Lecture Notes in Economics and Mathematical Systems *166*.) Springer–Verlag, Berlin 1979.

[7] R. A. Cuninghame-Green and P. Butkovič: The equation $A \otimes x = B \otimes y$ over (max,+). Theoretical Computer Science *293* (2003), 3–12.

[8] R. A. Cuninghame-Green and K. Zimmermann: Equation with residual functions. Comment. Math. Univ. Carolinae *42* (2001), 729–740.

[9] E. Sanchez: Resolution of eigen fuzzy sets equations. Fuzzy Sets and Systems *1* (1978), 69–74.

[10] E. Sanchez: Inverses of fuzzy relations. Applications to possibility distributions and medical diagnosis. Fuzzy Sets and Systems *1* (1978), 75–86.

[11] N. N. Vorobjov: Extremal algebra of positive matrices (in Russian). Datenverarbeitung und Kybernetik *3* (1967), 39–71.

Martin Gavalec, University of Hradec Králové, Faculty of Informatics and Management, Rokitanského 62, 500 03 Hradec Králové. Czech Republic.
e-mail: martin.gavalec@uhk.cz

Karel Zimmermann, Charles University Prague, Faculty of Mathematics and Physics, Malostranské nám. 25, 118 00 Praha 1. Czech Republic.
e-mail: karel.zimmermann@mff.cuni.cz