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EXISTENCE RESULTS OF POSITIVE SOLUTIONS TO FRACTIONAL DIFFERENTIAL EQUATION WITH INTEGRAL BOUNDARY CONDITIONS

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Abstract. In this paper we consider the existence, multiplicity, and nonexistence of positive solutions to fractional differential equation with integral boundary conditions. Our analysis relies on the fixed point index.

Keywords: fractional differential equation, integral boundary conditions, positive solutions, fixed point index, cone

MSC 2010: 26Axx, 34B15

1. Introduction

We will consider the existence, multiplicity, and nonexistence of positive solutions for the fractional differential equation with integral boundary conditions

$$\begin{cases} D^{\alpha}u + \lambda p(t)f(u) = 0, \quad t \in (0,1), \\ u'(0) = \mu \int_0^1 p_0(s)g(u(s))\,\mathrm{d}s, \\ u(1) = \nu \int_0^1 p_1(s)h(u(s))\,\mathrm{d}s, \end{cases} \tag{1.1}$$

where $\lambda > 0$, $\mu \geqslant 0$, $\nu \geqslant 0$, $1 < \alpha \leqslant 2$, D^{α} is the Caputo fractional derivative of order $1 < \alpha \leqslant 2$ defined by (see [1])

$$D^{\alpha}u(t) = D^{\alpha}_{0+}u(t) - \frac{u(0)}{\Gamma(1-\alpha)}t^{-\alpha} - \frac{u'(0)}{\Gamma(2-\alpha)}t^{1-\alpha}, \quad 1 < \alpha \leqslant 2,$$

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where

$$D_{0+}^{\alpha}u(t) = \frac{\mathrm{d}^2}{\mathrm{d}t^2}I^{2-\alpha}u(t) = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\frac{1}{\Gamma(2-\alpha)}\int_0^t (t-s)^{1-\alpha}u(s)\,\mathrm{d}s$$

is the Riemann-Liouville fractional derivative of order α, and

$$I^{2-\alpha}u(t) = \frac{1}{\Gamma(2-\alpha)}\int_0^t (t-s)^{1-\alpha}u(s)\,\mathrm{d}s$$

is the Riemann-Liouville fractional integral of order $2-\alpha$, see [1].

Differential equations of fractional order occur more frequently in different research areas and engineering, such as physics, chemistry, control of dynamical systems, etc. Recently, many authors have paid attention to the existence of a solution to boundary value problems for fractional differential equations, such as [2]–[10], etc. However, as far as we know, there are few papers dealing with the existence, multiplicity, and nonexistence of positive solutions to integral boundary value problem for fractional differential equations, in which the derivative is the Caputo fractional derivative. It is well known that fractional derivatives are generalizations of the derivative of integer order, and there are several kinds of fractional derivatives, such as Riemann-Liouville fractional derivative, Marchaud fractional derivative, Caputo fractional derivative, etc. As cited in [11], a number of works have appeared, especially in the theory of viscoelasticity and in hereditary solid mechanics, where fractional derivatives are used to get a better description of material properties. Mathematical modelling based on enhanced rheological models naturally leads to differential equations of fractional order and the necessity of the formulation of initial conditions to such equations. Applied problems require definitions of fractional derivatives to illustrate the initial value problems in Physics containing the expressions of $f(a)$, $f'(a)$, etc. In fact, the same requirements apply for boundary conditions. Caputo fractional derivative exactly satisfies these demands. Herein, in this paper, we establish the existence, multiplicity, and nonexistence of positive solutions to integral boundary value problems for fractional differential equations, by means of the fixed point index.

Definition 1.1. We call a function $u(t)$ a positive solution of problem (1.1) if $u(t) \in C([0,1])$, $u(t) \geqslant 0$, $t \in (0,1)$ and it satisfies problem (1.1).

We assume:

(C1) $\lambda > 0$, $\mu \geqslant 0$, $\nu \geqslant 0$, $1 < \alpha \leqslant 2$;

(C2) $t^{r_i}p_i\colon [0,1] \to [0,+\infty)$ is continuous, $t^{r_i}p_i(t)$ is nonidenticaly vanishing, $0 \leqslant r_i < 1$, $i = 0,1$, and $\nu p_1 - \mu p_0 \geqslant 0$ for $t \in (0,1]$;

(C3) $f, g, h\colon [0,+\infty) \to [0,+\infty)$ are continuous, $h(u) \geqslant g(u)$ for all $u \in [0,+\infty)$, and $f(u) > 0$, $h(u) - g(u) > 0$, $h(u) > 0$ for $u \in [0,+\infty)$ and $\|u\| > 0$, where $\|u\| = \max_{t\in[0,1]} |u(t)|$ is the norm of the continuous functions space $C([0,1])$.

Let

$$f_0 = \lim_{|x|\to 0} \frac{f(x)}{|x|}, \quad f_\infty = \lim_{|x|\to\infty} \frac{f(x)}{|x|}, \quad (h-g)_0 = \lim_{|x|\to 0} \frac{h(x)-g(x)}{|x|},$$
$$(h-g)_\infty = \lim_{|x|\to\infty} \frac{h(x)-g(x)}{|x|}, \quad h_0 = \lim_{|x|\to 0} \frac{h(x)}{|x|}, \quad h_\infty = \lim_{|x|\to\infty} \frac{h(x)}{|x|}.$$

Our main results are:

Theorem 1.1. *Assume that conditions* (C1)–(C3) *hold.*

(a) *If $f_0 = 0 = h_0$ and $f_\infty = \infty$ (or $(h-g)_\infty = \infty$ or $h_\infty = \infty$), then for all $\lambda > 0$, $\mu \geqslant 0$, $\nu \geqslant 0$ problem* (1.1) *has a positive solution.*

(b) *If $f_0 = \infty$ and $f_\infty = 0 = h_\infty$, then for all $\lambda > 0$, $\mu \geqslant 0$, $\nu \geqslant 0$ problem* (1.1) *has a positive solution.*

(c) *If $(h-g)_0 = \infty$ and $f_\infty = 0 = h_\infty$, then for all $\lambda > 0$, $\mu \geqslant 0$, $\nu \geqslant 0$ problem* (1.1) *has a positive solution.*

(d) *If $h_0 = \infty$ and $f_\infty = 0 = h_\infty$, then for all $\lambda > 0$, $\mu \geqslant 0$, $\nu \geqslant 0$ problem* (1.1) *has a positive solution.*

Theorem 1.2. *Assume that conditions* (C1)–(C3) *hold.*

(a) *If $f_0 = 0 = h_0$ or $f_\infty = 0 = h_\infty = 0$, then there exist $\lambda_0 > 0$, $\mu_0 > 0$, $\nu_0 > 0$ such that for the three cases:* (1) *$\lambda > \lambda_0$, $\mu \geqslant 0$, $\nu \geqslant 0$;* (2) *$\lambda > 0$, $\mu > \mu_0$, $\nu \geqslant 0$;* (3) *$\lambda > 0$, $\mu \geqslant 0$, $\nu > \nu_0$, problem* (1.1) *has a positive solution.*

(b) *If $f_0 = \infty$ or $f_\infty = \infty$, then there exist $\lambda_0 > 0$, $\mu_0 > 0$, $\nu_0 > 0$ such that for all $0 < \lambda < \lambda_0$, $0 \leqslant \mu < \mu_0$, $0 \leqslant \nu < \nu_0$, problem* (1.1) *has a positive solution.*

(c) *If $(h-g)_0 = \infty$ or $(h-g)_\infty = \infty$, then there exist $\lambda_0 > 0$, $\mu_0 > 0$, $\nu_0 > 0$ such that for all $0 < \lambda < \lambda_0$, $0 \leqslant \mu < \mu_0$, $0 \leqslant \nu < \nu_0$, problem* (1.1) *has a positive solution.*

(d) *If $h_0 = \infty$ or $h_\infty = \infty$, then there exist $\lambda_0 > 0$, $\mu_0 > 0$, $\nu_0 > 0$ such that for all $0 < \lambda < \lambda_0$, $0 \leqslant \mu < \mu_0$, $0 \leqslant \nu < \nu_0$, problem* (1.1) *has a positive solution.*

(e) *If $f_0 = 0 = h_0$, $f_\infty = 0 = h_\infty$, then there exists $\mu_0 > 0$ such that for the three cases:* (1) *$\lambda > \lambda_0$, $\mu \geqslant 0$, $\nu \geqslant 0$;* (2) *$\lambda > 0$, $\mu > \mu_0$, $\nu \geqslant 0$;* (3) *$\lambda > 0$, $\mu \geqslant 0$, $\nu > \nu_0$, problem* (1.1) *has two positive solutions;*

(f) *If $f_0 = f_\infty = \infty$ (or $(h-g)_0 = (h-g)_\infty = \infty$, or $h_0 = h_\infty = \infty$), then there exist $\lambda_0 > 0$, $\mu_0 > 0$, $\nu_0 > 0$ such that for all $0 < \lambda < \lambda_0$, $0 \leqslant \mu < \mu_0$, $0 \leqslant \nu < \nu_0$, problem* (1.1) *has two positive solutions.*

(g) *If $f_0 < \infty$, $h_0 < \infty$ and $f_\infty < \infty$, $h_\infty < \infty$, then there exist $\lambda_0 > 0$, $\mu_0 > 0$, $\nu_0 > 0$ such that for all $0 < \lambda < \lambda_0$, $0 \leqslant \mu < \mu_0$, $0 \leqslant \nu < \nu_0$, problem* (1.1) *has no positive solution.*

(h) *If $f_0 > 0$ and $f_\infty > 0$, then there exists $\lambda_0 > 0$ such that for all $\lambda > \lambda_0$, $\mu \geqslant 0$, $\nu \geqslant 0$, problem* (1.1) *has no positive solution.*

(j) *If $(h-g)_0 > 0$ and $(h-g)_\infty > 0$, then there exists $\mu_0 > 0$ such that for all $\lambda > 0$, $\mu > \mu_0$, $\nu \geqslant 0$, problem* (1.1) *has no positive solution.*

(k) *If $h_0 > 0$ and $h_\infty > 0$, then there exists $\nu_0 > 0$ such that for all $\lambda > 0$, $\mu \geqslant 0$, $\nu > \nu_0$, problem* (1.1) *has no positive solution.*

2. Preliminaries

We will have some results which are very important for us to prove Theorems 1.1 and 1.2.

Lemma 2.1. *Let $\eta \in C((0,1])$, $t^r\eta \in C([0,1])$, $0 \leqslant r < 1$ and $\sigma_0, \sigma_1 \in L_1(0,1)$. Then the unique solution of the boundary value problem*

$$\begin{cases} D^\alpha u + \eta(t) = 0, \quad t \in (0,1),\ 1 < \alpha \leqslant 2, \\ u'(0) = \mu \int_0^1 \sigma_0(s)\,\mathrm{d}s, \\ u(1) = \nu \int_0^1 \sigma_1(s)\,\mathrm{d}s \end{cases} \tag{2.1}$$

is

$$u(t) = \int_0^1 G(t,s)\eta(s)\,\mathrm{d}s - \mu(1-t)\int_0^1 \sigma_0(s)\,\mathrm{d}s + \nu\int_0^1 \sigma_1(s)\,\mathrm{d}s,$$

where

$$G(t,s) = \begin{cases} \dfrac{(1-s)^{\alpha-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leqslant s \leqslant t \leqslant 1, \\ \dfrac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leqslant t \leqslant s \leqslant 1. \end{cases} \tag{2.2}$$

Moreover, the function $G(t,s)$ satisfies the following conditions:

(1) *$G(t,s) \geqslant 0$ for all $t,s \in [0,1]$, $G(t,s) > 0$ for all $t,s \in (0,1)$, and*

$$\max_{0\leqslant t\leqslant 1} G(t,s) = G(s,s), \quad s \in [0,1].$$

(2) *$\min\limits_{\gamma\leqslant t\leqslant\delta} G(t,s) \geqslant (1-\delta^{\alpha-1})G(s,s) = (1-\delta^{\alpha-1})\max\limits_{0\leqslant t\leqslant 1} G(t,s)$, $0 \leqslant s \leqslant 1$, where $\frac{1}{2} < \gamma < \delta < 1$.*

P r o o f. Since $t^r\eta \in C[0,1]$, equation (2.1) means that the Caputo fractional derivative $D^\alpha u \in C_r[0,1]$ exists on $[0,1]$. According to the definition of the Caputo fractional derivative, we have that $u \in L_1(0,1)$ and $I^{2-\alpha}u(t) \in C_r^2[0,1]$. Thus, applying the operator I^α to both sides of equation (2.1) and using some facts from [1], we obtain

$$u(t) = c_1 + c_2 t - I^\alpha \eta(t)$$

for some constants c_i, $i=1,2$. Using the boundary conditions of problem (2.1), we can calculate that $c_3 = c_4 = 0, c_2 = \mu\int_0^1 \sigma_0(s)\,\mathrm{d}s,\ c_1 = \nu\int_0^1 \sigma_1(s)\,\mathrm{d}s + I^\alpha\eta(1) - \mu\int_0^1 \sigma_0(s)\,\mathrm{d}s$. Consequently, the solution of problem (2.1) is

$$u(t) = \nu\int_0^1 \sigma_1(s)\,\mathrm{d}s + I^\alpha\eta(1) - \mu\int_0^1 \sigma_0(s)\,\mathrm{d}s + \mu t\int_0^1 \sigma_0(s)\,\mathrm{d}s - I^\alpha\eta(t).$$

Thus, the unique solution $u(t)$ of problem (2.1) can be written as

$$u(t) = \int_0^1 G(t,s)\eta(s)\,\mathrm{d}s - \mu(1-t)\int_0^1 \sigma_0(s)\,\mathrm{d}s + \nu\int_0^1 \sigma_1(s)\,\mathrm{d}s$$

where $G(t,s)$ is defined in (2.2).

As for the properties of (1), (2) of the function $G(t,s)$, we can easily obtain them in the same way as in Lemma 2.8 [10]. In view of the expression for the function $G(t,s)$, we easily find that $G(t,s) \geqslant 0$, $t,s\in[0,1]$, and $G(t,s)>0$, $t,s\in(0,1)$, $\max\limits_{0\leqslant t\leqslant 1} G(t,s) = G(s,s) = (1-s)^{\alpha-1}/\Gamma(\alpha)$. On the other hand, we have

$$\min_{\gamma\leqslant t\leqslant\delta} G(t,s) = \begin{cases} \dfrac{(1-s)^{\alpha-1}-(\delta-s)^{\alpha-1}}{\Gamma(\alpha)}, & s\in[0,\gamma],\\ \min\Big\{\dfrac{(1-s)^{\alpha-1}-(\delta-s)^{\alpha-1}}{\Gamma(\alpha)}, \dfrac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\Big\} & \\ \qquad = \dfrac{(1-s)^{\alpha-1}-(\delta-s)^{\alpha-1}}{\Gamma(\alpha)}, & s\in[\gamma,\delta],\\ \dfrac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}, & s\in[\delta,1], \end{cases}$$
$$= \begin{cases} \dfrac{(1-s)^{\alpha-1}-(\delta-s)^{\alpha-1}}{\Gamma(\alpha)}, & s\in[0,\delta],\\ \dfrac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}, & s\in[\delta,1], \end{cases}$$

and

$$\begin{aligned}(1-s)^{\alpha-1}-(\delta-s)^{\alpha-1} &= (1-s)^{\alpha-1} - \delta^{\alpha-1}\Big(1-\frac{s}{\delta}\Big)^{\alpha-1}\\ &\geqslant (1-\delta^{\alpha-1})(1-s)^{\alpha-1},\ s\in[0,\delta],\\ (1-s)^{\alpha-1} &\geqslant (1-\delta^{\alpha-1})(1-s)^{\alpha-1},\ s\in[\delta,1].\end{aligned}$$

Thus, we have

$$\min_{\gamma\leqslant t\leqslant\delta} G(t,s) \geqslant (1-\delta^{\alpha-1})\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} = (1-\delta^{\alpha-1})G(s,s), \quad 0\leqslant s\leqslant 1.$$

The proof is complete. $\square$

Hence, we know that the solution to problem (1.1) is

$$\begin{aligned} u(t) &= \lambda\int_0^1 G(t,s)p(s)f(u)\,\mathrm{d}s - \mu(1-t)\int_0^1 p_0(s)g(u)\,\mathrm{d}s + \nu\int_0^1 p_1(s)h(u)\,\mathrm{d}s \\ &= \lambda\int_0^1 G(t,s)p(s)f(u)\,\mathrm{d}s + \mu(1-t)\int_0^1 p_0(s)(h(u)-g(u))\,\mathrm{d}s \\ &\quad + \int_0^1 (\nu p_1(s)-\mu(1-t)p_0(s))h(u)\,\mathrm{d}s. \end{aligned}$$

As a result, the solution u of problem (1.1) can be written in the form

$$\begin{aligned} u(t) = \lambda\int_0^1 G(t,s)p(s)f(u(s))\,\mathrm{d}s &+ \mu(1-t)\int_0^1 p_0(s)(h(u)-g(u))\,\mathrm{d}s \\ &+ \int_0^1 (\nu p_1(s)-\mu(1-t)p_0(s))h(u)\,\mathrm{d}s. \end{aligned} \tag{2.3}$$

Lemma 2.2. *Assume that conditions* (C1)–(C3) *hold. Then $u\in C[0,1]$ is a solution of problem* (1.1) *if and only if $u\in C[0,1]$ is a solution of the integral equation* (2.3).

P r o o f. First we prove the necessity. Let $u\in C[0,1]$ be a solution of problem (1.1). Applying the method used for proving Lemma 2.1, we can obtain that $u\in C[0,1]$ is a solution of the integral equation (2.3), and hence the necessity is proved.

Now, we prove the sufficiency. Let $u\in C[0,1]$ be a solution of the integral equation (2.3). Applying the operator D^α to both sides of (2.3), we have $D^\alpha u(t) = -\lambda p(t)f(u(t))$; moreover, we can also verify that $u(t)$ satisfies the boundary conditions $u'(0)=\mu\int_0^1 p_0(s)g(u(s))\,\mathrm{d}s, u(1)=\nu\int_0^1 p_1(s)h(u(s))\,\mathrm{d}s$. Thus, we verify that $u\in C[0,1]$ is a solution of problem (1.1). Thus, the sufficiency is proved.

The following is a well-known result concerning the fixed point index. $\square$

Lemma 2.3 ([12]). *Let E be a Banach space and K a cone in E. For $r > 0$, define $K_r = \{u \in K;\ \|u\| < r\}$. Assume that $T\colon \overline{K}_r \to K$ is completely continuous such that $Tx \neq 0$ for $x \in \partial K_r = \{u \in K;\ \|u\| = r\}$.*

(i) *If $\|Tu\| \geqslant \|u\|$ for $u \in \partial K_r$, then*

$$i(T, K_r, K) = 0.$$

(ii) *If $\|Tu\| \leqslant \|u\|$ for $u \in \partial K_r$, then*

$$i(T, K_r, K) = 1.$$

Let $\frac{1}{2} < \gamma < \delta < 1$, and define K to be a cone in $C[0,1]$ by

$$K = \{u \in C[0,1];\ u(t) \geqslant 0, t \in [0,1] \text{ and } \min_{\gamma \leqslant t \leqslant \delta} u(t) \geqslant (1 - \delta^{\alpha-1})\|u\|\}.$$

Define $T_\lambda\colon K \to C[0,1]$ by

$$Tu(t) = \lambda \int_0^1 G(t,s)p(s)f(u)\,\mathrm{d}s + \mu(1-t)\int_0^1 p_0(s)(h(u) - g(u))\,\mathrm{d}s + \int_0^1 (\nu p_1(s) - \mu(1-t)p_0(s))h(u)\,\mathrm{d}s. \tag{2.4}$$

Lemma 2.4. *Let assumptions* (C1)–(C3) *hold, then $T\colon K \to K$ is completely continuous.*

P r o o f. By the assumptions (C_1)–(C_3) and since $G(t,s) \geqslant 0$, $t, s \in [0,1]$, we have $Tu(t) \geqslant 0$, $Tu(t) \in C[0,1]$ for $u \in K$. Moreover, for $u \in K$, from assumptions (C1)–(C3), Lemma 2.1 and $1 - \gamma > 1 - \delta > 1 - \delta^{\alpha-1}$, $\delta^{\alpha-1} - \gamma > \delta - \gamma > 0$, $1 = \frac{1}{2} + \frac{1}{2} < \gamma + \delta^{\alpha-1} < 1 + 1 = 2$, we have

$$\begin{aligned}
Tu(t) &= \lambda \int_0^1 G(t,s)p(s)f(u)\,\mathrm{d}s + \mu(1-t)\int_0^1 p_0(s)(h(u) - g(u))\,\mathrm{d}s \\
&\quad + \int_0^1 (\nu p_1(s) - \mu(1-t)p_0(s))h(u)\,\mathrm{d}s \\
&\leqslant \lambda \int_0^1 G(s,s)p(s)f(u)\,\mathrm{d}s \\
&\quad + \frac{(1-\gamma)\mu}{1-\delta^{\alpha-1}} \int_0^1 p_0(s)(h(u) - g(u))\,\mathrm{d}s + \int_0^1 \nu p_1(s)h(u)\,\mathrm{d}s \\
&\leqslant \lambda \int_0^1 G(s,s)p(s)f(u)\,\mathrm{d}s - \frac{(1-\gamma)\mu}{1-\delta^{\alpha-1}} \int_0^1 p_0(s)g(u)\,\mathrm{d}s \\
&\quad + \frac{(2-\gamma-\delta^{\alpha-1})\nu}{1-\delta^{\alpha-1}} \int_0^1 p_1(s)h(u)\,\mathrm{d}s,
\end{aligned}$$

$$
\begin{aligned}
\min_{\gamma\leqslant t\leqslant\delta} Tu(t) = \min_{\gamma\leqslant t\leqslant\delta}\Big(\lambda\int_0^1 G(t,s)p(s)f(u)\,\mathrm{d}s - \mu(1-t)\int_0^1 p_0(s)g(u)\,\mathrm{d}s \\
+ \nu\int_0^1 p_1(s)h(u)\,\mathrm{d}s\Big) \\
\geqslant \lambda(1-\delta^{\alpha-1})\int_0^1 G(s,s)p(s)f(u)\,\mathrm{d}s - \mu(1-\gamma)\int_0^1 p_0(s)g(u)\,\mathrm{d}s \\
+ \nu\int_0^1 p_1(s)h(u)\,\mathrm{d}s \\
= (1-\delta^{\alpha-1})\Big(\lambda\int_0^1 G(s,s)p(s)f(u)\,\mathrm{d}s - \frac{(1-\gamma)\mu}{1-\delta^{\alpha-1}}\int_0^1 p_0(s)g(u)\,\mathrm{d}s \\
+ \frac{\nu}{1-\delta^{\alpha-1}}\int_0^1 p_1(s)h(u)\,\mathrm{d}s\Big) \\
\geqslant (1-\delta^{\alpha-1})\Big(\lambda\int_0^1 G(s,s)p(s)f(u)\,\mathrm{d}s - \frac{(1-\gamma)\mu}{1-\delta^{\alpha-1}}\int_0^1 p_0(s)g(u)\,\mathrm{d}s \\
+ \frac{\nu(2-\gamma-\delta^{\alpha-1})}{1-\delta^{\alpha-1}}\int_0^1 p_1(s)h(u)\,\mathrm{d}s\Big) \\
\geqslant (1-\delta^{\alpha-1})\|Tu\|.
\end{aligned}
$$

Hence, the operator $T\colon K\to K$ is well defined. $\square$

We can complete the remaining part of the proof, by standard arguments; here we omit it.

Lemma 2.5. *Assume that* (C1)–(C3) *hold. Let $u\in K$ and $r>0$. Then there exists $\lambda_0>0$ such that*

$$\|Tu\| > \|u\| \quad \text{for } u\in\partial\Omega_r,\ \lambda>\lambda_0,\ \mu\geqslant 0,\ \nu\geqslant 0,$$

where $\Omega_r=\{u\in K;\ \|u\|<r\}$, $\partial\Omega_r=\{u\in K;\ \|u\|=r\}$.

Proof. For fixed $r>0$, let $\lambda_0 = r\big((1-\delta^{\alpha-1})m_r\int_\gamma^\delta G(s,s)p(s)\,\mathrm{d}s\big)^{-1}$, where $m_r = \min\limits_{r(1-\delta^{\alpha-1})\leqslant\|u\|\leqslant r} f(u)$ (by (C3), $m_r>0$). Then, for $u\in\partial\Omega_r$, $\lambda>\lambda_0,\mu\geqslant 0$, $\nu\geqslant 0$, by Lemma 2.1, we have

$$
\begin{aligned}
\min_{\gamma\leqslant t\leqslant\delta} Tu(t) = \min_{\gamma\leqslant t\leqslant\delta}\Big(\lambda\int_0^1 G(t,s)p(s)f(u(s))\,\mathrm{d}s + \mu(1-t)\int_0^1 p_0(s)(h(u)-g(u))\,\mathrm{d}s \\
+ \int_0^1 (\nu p_1(s) - \mu(1-t)p_0(s))h(u)\,\mathrm{d}s\Big) \\
\geqslant \lambda(1-\delta^{\alpha-1})\int_\gamma^\delta G(s,s)p(s)f(u(s))\,\mathrm{d}s
\end{aligned}
$$

$$\geqslant \lambda m_r(1-\delta^{\alpha-1})\int_\gamma^\delta G(s,s)p(s)\,\mathrm{d}s$$
$$> \lambda_0 m_r(1-\delta^{\alpha-1})\int_\gamma^\delta G(s,s)p(s)\,\mathrm{d}s = r = \|u\|,$$

which implies that $\|Tu\| > \|u\|$ for $u\in\partial\Omega_r$, $\lambda>\lambda_0$, $\mu\geqslant 0$, $\nu\geqslant 0$. $\square$

By similar arguments, we can also obtain the following results:

Lemma 2.6. *Assume that* (C1)–(C3) *hold. Let* $u\in K$ *and* $r>0$. *Then there exists* $\mu_0>0$ *such that*

$$\|Tu\| > \|u\| \quad \textit{for } u\in\partial\Omega_r,\ \lambda>0,\ \mu>\mu_0,\ \nu\geqslant 0,$$

where $\Omega_r=\{u\in K;\ \|u\|<r\}$, $\partial\Omega_r=\{u\in K;\ \|u\|=r\}$.

Lemma 2.7. *Assume that* (C1)–(C3) *hold. Let* $u\in K$ *and* $r>0$. *Then there exists* $\nu_0>0$ *such that*

$$\|Tu\| > \|u\| \quad \textit{for } u\in\partial\Omega_r,\ \lambda>0,\ \mu\geqslant 0,\ \nu>\nu_0,$$

where $\Omega_r=\{u\in K;\ \|u\|<r\}$, $\partial\Omega_r=\{u\in K;\ \|u\|=r\}$.

We also have the following result:

Lemma 2.8. *Assume that* (C1)–(C3) *hold. Let* $u\in K$ *and* $r>0$. *Then there exist positive constants* $\lambda_0,\mu_0,\nu_0>0$ *such that*

$$\|Tu\| < \|u\| \quad \textit{for } u\in\partial\Omega_r,\ 0<\lambda<\lambda_0,\ 0\leqslant\mu<\mu_0,\ 0\leqslant\nu<\nu_0,$$

where $\Omega_r=\{u\in K;\ \|u\|<r\}$, $\partial\Omega_r=\{u\in K;\ \|u\|=r\}$.

P r o o f. For fixed $r>0$, let

$$\lambda_0=\frac{r}{3M_{fr}\int_0^1 G(s,s)p(s)\,\mathrm{d}s},\qquad \mu_0=\frac{r}{3M_{hr}\int_0^1 p_0(s)\,\mathrm{d}s},$$
$$\nu_0=\frac{r}{3M_{hr}\int_0^1 p_1(s)\,\mathrm{d}s},$$

where $M_{fr} = \max\limits_{\|u\|\leqslant r} f(u) + 1$, $M_{hr} = \max\limits_{\|u\|\leqslant r} h(u) + 1$. Then, for $u \in \partial\Omega_r$, $0 < \lambda < \lambda_0$, $0 \leqslant \mu < \mu_0$, $0 \leqslant \nu < \nu_0$, we have

$$
\begin{aligned}
|Tu(t)| &= \lambda \int_0^1 G(t,s)p(s)f(u(s))\,\mathrm{d}s + \mu(1-t)\int_0^1 p_0(s)(h(u)-g(u))\,\mathrm{d}s \\
&\quad + \int_0^1 (\nu p_1(s) - \mu(1-t)p_0(s))h(u)\,\mathrm{d}s \\
&\leqslant \lambda \int_0^1 G(t,s)p(s)f(u(s))\,\mathrm{d}s + \mu(1-t)\int_0^1 p_0(s)h(u)\,\mathrm{d}s + \nu\int_0^1 p_1(s)h(u)\,\mathrm{d}s \\
&\leqslant \lambda M_{fr}\int_0^1 G(s,s)p(s)\,\mathrm{d}s + \mu M_{hr}\int_0^1 p_0(s)\,\mathrm{d}s + \nu M_{hr}\int_0^1 p_1(s)\,\mathrm{d}s \\
&< \lambda_0 M_{fr}\int_0^1 G(s,s)p(s)\,\mathrm{d}s + \mu_0 M_{hr}\int_0^1 p_0(s)\,\mathrm{d}s + \nu_0 M_{hr}\int_0^1 p_1(s)\,\mathrm{d}s \\
&= \frac{r}{3} + \frac{r}{3} + \frac{r}{3} = \|u\|,
\end{aligned}
$$

which implies that $\|Tu\| < \|u\|$ for $u \in \partial\Omega_r$, $0 < \lambda < \lambda_0$, $0 \leqslant \mu < \mu_0$, $0 \leqslant \nu < \nu_0$.

3. Proof of theorem 1.1

In view of Lemma 2.2, it is sufficient to find a fixed point $u \in K$ of the operator $T\colon\ K \to K$ defined by (2.4). It follows from Lemma 2.4 that the operator $T\colon\ K \to K$ is completely continuous. Moreover, from Lemma 2.4 and assumptions (C1)–(C3), if we fix $r > 0$, and $K_r = \{u \in K;\ \|u\| < r\}$, and $\partial K_r = \{u \in K;\ \|u\| = r\}$, we can obtain that $T\colon\ K_r \to K$ is completely continuous, $Tu \neq 0$ for $u \in \partial K_r$.

(a) By virtue of $f_0 = 0 = h_0$, for $0 < \varepsilon < \big(\lambda \int_0^1 G(s,s)p(s)\,\mathrm{d}s + \mu\int_0^1 p_0(s)\,\mathrm{d}s + \nu\int_0^1 p_1(s)\,\mathrm{d}s\big)^{-1}$, there exists a positive $r_1 > 0$ such that

$$
f(u) < \varepsilon|u|,\ h(u) < \varepsilon|u| \quad \text{for } 0 < |u| < r_1.
$$

Let $\Omega_{r_1} = \{u \in K\colon\ \|u\| < r_1\}$. For $u \in \partial\Omega_{r_1}$ we have

$$
\begin{aligned}
|Tu(t)| &= \lambda \int_0^1 G(t,s)p(s)f(u)\,\mathrm{d}s + \mu(1-t)\int_0^1 p_0(s)(h(u)-g(u))\,\mathrm{d}s \\
&\qquad\qquad + \int_0^1 (\nu p_1(s) - \mu(1-t)p_0(s)h(u)\,\mathrm{d}s \\
&\leqslant \lambda \int_0^1 G(s,s)p(s)f(u)\,\mathrm{d}s + \mu(1-t)\int_0^1 p_0(s)h(u)\,\mathrm{d}s + \nu\int_0^1 p_1(s)h(u)\,\mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant \lambda\varepsilon \int_0^1 G(s,s)p(s)|u|\,\mathrm{d}s + \mu\varepsilon \int_0^1 p_0(s)|u|\,\mathrm{d}s + \nu\varepsilon \int_0^1 p_1(s)|u|\,\mathrm{d}s \\
&\leqslant \varepsilon r_1 \biggl(\lambda \int_0^1 G(s,s)p(s)\,\mathrm{d}s + \mu \int_0^1 p_0(s)\,\mathrm{d}s + \nu \int_0^1 p_1(s)\,\mathrm{d}s \biggr) < r_1,
\end{aligned}
$$

which implies that $\|Tu\| \leqslant r_1 = \|u\|$ for $u \in \partial\Omega_{r_1}$.

If $f_\infty = \infty$, then, for $M > \bigl(\lambda(1-\delta^{\alpha-1})^2 \int_\gamma^\delta G(s,s)p(s)\,\mathrm{d}s\bigr)^{-1}$, there exists a positive $N > 0$ such that

$$
f(u) > M|u| \quad \text{for } |u| > N.
$$

Take $r_2 > \max\{r_1, N/(1-\delta^{\alpha-1})\}$, and let $\Omega_{r_2} = \{u \in K;\ \|u\| < r_2\}$. Then, for $u \in \partial\Omega_{r_2}$, $u(t) \geqslant (1-\delta^{\alpha-1})\|u\| = (1-\delta^{\alpha-1})r_2 > N$ for $\gamma \leqslant t \leqslant \delta$, hence we have

$$
\begin{aligned}
Tu(t) &= \lambda \int_0^1 G(t,s)p(s)f(u(s))\,\mathrm{d}s + \mu(1-t)\int_0^1 p_0(s)(h(u)-g(u))\,\mathrm{d}s \\
&\qquad + \int_0^1 (\nu p_1(s) - \mu(1-t)p_0(s))h(u)\,\mathrm{d}s \\
&\geqslant \lambda \int_0^1 G(t,s)p(s)f(u(s))\,\mathrm{d}s \\
&\geqslant \lambda \int_\gamma^\delta G(t,s)p(s)f(u(s))\,\mathrm{d}s \\
&\geqslant M\lambda(1-\delta^{\alpha-1}) \int_\gamma^\delta G(s,s)p(s)|u|\,\mathrm{d}s \\
&= M\lambda(1-\delta^{\alpha-1}) \int_\gamma^\delta G(s,s)p(s)u(s)\,\mathrm{d}s \\
&\geqslant M\lambda(1-\delta^{\alpha-1})^2 \int_\gamma^\delta G(s,s)p(s)\|u\|\,\mathrm{d}s \\
&> \|u\|,
\end{aligned}
$$

which implies that $\|Tu\| \geqslant \|u\|$ for $u \in \partial\Omega_{r_2}$.

It follows from Lemma 2.3 that

$$
i(T, \Omega_{r_1}, K) = 1 \quad \text{and} \quad i(T, \Omega_{r_2}, K) = 0.
$$

By the additivity of the fixed point index we have

$$
i(T, \Omega_{r_2} \setminus \overline{\Omega}_{r_1}, K) = -1,
$$

which implies that T has a fixed point $u \in \Omega_{r_2} \setminus \overline{\Omega}_{r_1}$, by virtue of the existence property of the fixed point index. From Lemma 2.2, the fixed point $u \in \Omega_{r_2} \setminus \overline{\Omega}_{r_1}$ is a positive solution of problem (1.1).

By the same reasoning, we can obtain that problem (1.1) has a positive solution, under $f_0 = 0 = h_0$ and $(h-g)_\infty = \infty$ or $h_\infty = \infty$.

(b) By $f_0 = \infty$, for $M > \bigl(\lambda(1-\delta^{\alpha-1})^2 \int_\gamma^\delta G(s,s)p(s)\,\mathrm{d}s\bigr)^{-1}$ there exists a positive $r_1 > 0$ such that

$$f(u) > M|u| \quad \text{for } 0 < |u| < r_1.$$

Let $\Omega_{r_1} = \{u \in K;\ \|u\| < r_1\}$. For $u \in \partial\Omega_{r_1}$ we have

$$\begin{aligned}
Tu(t) &\geqslant \lambda \int_0^1 G(t,s)p(s)f(u(s))\,\mathrm{d}s \\
&\geqslant \lambda \int_\gamma^\delta G(t,s)p(s)f(u(s))\,\mathrm{d}s \\
&\geqslant M\lambda(1-\delta^{\alpha-1})^2 \int_\gamma^\delta G(s,s)p(s)\|u\|\,\mathrm{d}s \\
&> \|u\|,
\end{aligned}$$

which implies that $\|Tu\| \geqslant \|u\|$ for $u \in \partial\Omega_{r_1}$.

By $f_\infty = 0 = h_\infty$, for $0 < \varepsilon < \frac{1}{2}(\lambda\int_0^1 G(s,s)p(s)\,\mathrm{d}s + \mu\int_0^1 p_0(s)\,\mathrm{d}s + \nu\int_0^1 p_1(s)\,\mathrm{d}s)^{-1}$ there exists a positive $N > 0$ such that

$$f(u) < \varepsilon|u|, \quad h(u) < \varepsilon|u| \qquad \text{for } |u| > N.$$

Therefore,

$$f(u) \leqslant \varepsilon|u| + C_1, \quad h(u) \leqslant \varepsilon|u| + C_2$$

for $u \in [0,+\infty)$, where $C_1 = \max\limits_{0\leqslant u\leqslant N} f(u) + 1$, $C_2 = \max\limits_{0\leqslant u\leqslant N} h(u) + 1$. Let $\Omega_{r_2} = \{u \in K;\ \|u\| < r_2\}$, where

$$r_2 > \max\Biggl\{r_1, 2\biggl(\lambda C_1 \int_0^1 G(s,s)p(s)\,\mathrm{d}s + \mu C_2 \int_0^1 p_0(s)\,\mathrm{d}s + \nu C_2 \int_0^1 p_1(s)\,\mathrm{d}s\biggr)\Biggr\}.$$

Then, for $u \in \partial\Omega_{r_2}$ we have

$$
\begin{aligned}
|Tu(t)| &= \lambda \int_0^1 G(t,s)p(s)f(u)\,\mathrm{d}s + \mu(1-t)\int_0^1 p_0(s)(h(u)-g(u))\,\mathrm{d}s \\
&\quad + \int_0^1 (\nu p_1(s) - \mu(1-t)p_0(s))h(u)\,\mathrm{d}s \\
&\leqslant \lambda \int_0^1 G(t,s)p(s)f(u)\,\mathrm{d}s + \mu(1-t)\int_0^1 p_0(s)h(u)\,\mathrm{d}s + \nu\int_0^1 p_1(s)h(u)\,\mathrm{d}s \\
&\leqslant \varepsilon\Big(\lambda\int_0^1 G(s,s)p(s)|u|\,\mathrm{d}s + \mu\int_0^1 p_0(s)|u|\,\mathrm{d}s + \nu\int_0^1 p_1(s)|u|\,\mathrm{d}s\Big) \\
&\quad + \lambda C_1\int_0^1 G(s,s)p(s)\,\mathrm{d}s + \mu C_2\int_0^1 p_0(s)\,\mathrm{d}s + \nu C_2\int_0^1 p_1(s)\,\mathrm{d}s \\
&\leqslant \varepsilon r_2\Big(\lambda\int_0^1 G(s,s)p(s)\,\mathrm{d}s + \mu\int_0^1 p_0(s)\,\mathrm{d}s + \nu\int_0^1 p_1(s)\,\mathrm{d}s\Big) \\
&\quad + \lambda C_1\int_0^1 G(s,s)p(s)\,\mathrm{d}s + \mu C_2\int_0^1 p_0(s)\,\mathrm{d}s + \nu C_2\int_0^1 p_1(s)\,\mathrm{d}s \\
&\leqslant \frac{r_2}{2} + \frac{r_2}{2} = r_2,
\end{aligned}
$$

which implies that $\|Tu\| \leqslant r_2 = \|u\|$ for $u \in \partial\Omega_{r_2}$.

It follows from Lemma 2.3 that

$$i(T,\Omega_{r_1},K) = 0, \quad i(T,\Omega_{r_2},K) = 1.$$

By the additivity of the fixed point index we have

$$i(T,\Omega_{r_2}\setminus\overline{\Omega}_{r_1},K) = 1,$$

which implies that T has a fixed point $u \in \Omega_{r_2}\setminus\overline{\Omega}_{r_1}$, by virtue of the existence property of the fixed point index. From Lemma 2.2, the fixed point $u \in \Omega_{r_2}\setminus\overline{\Omega}_{r_1}$ is a positive solution of problem (1.1).

In the same way we can complete the proofs of (c) and (d). Here, we omit them. □

4. Proof of theorem 1.2

In view of Lemma 2.2, it is sufficient to find a fixed point $u \in K$ of the operator $T\colon K \to K$ defined by (2.4). It follows from Lemma 2.4 that operator $T\colon K \to K$ is completely continuous. Moreover, from Lemma 2.4 and assumptions (C1)–(C3), if we fix $r > 0$, put $K_r = \{u \in K;\ \|u\| < r\}$, and $\partial K_r = \{u \in K;\ \|u\| = r\}$, we can obtain that $T\colon K_r \to K$ is completely continuous, $Tu \neq 0$ for $u \in \partial K_r$.

(a) For fixed $r_1 > 0$, by Lemma 2.5 there exists $\lambda_0 > 0$ such that

$$\|Tu\| > \|u\| \quad \text{for } u \in \partial\Omega_{r_1},\ \lambda > \lambda_0,\ \mu \geqslant 0,\ \nu \geqslant 0.$$

If $f_0 = 0 = h_0$, then, according to Theorem 1.1(a), there exists a positive constant $r_2 < r_1$ such that

$$\|Tu\| \leqslant r_2 = \|u\| \quad \text{for} \quad u \in \partial\Omega_{r_2}.$$

If $f_\infty = 0 = h_\infty$, then, according to Theorem 1.1(b), there exists a positive constant $r_3 > \max\{r_1, r_2\}$ such that

$$\|Tu\| \leqslant r_3 = \|u\| \quad \text{for} \quad u \in \partial\Omega_{r_3}.$$

It follows from Lemma 2.3 that

$$i(T, \Omega_{r_1}, K) = 0 \quad \text{and} \quad i(T, \Omega_{r_2}, K) = 1 \quad \text{and} \quad i(T, \Omega_{r_3}, K) = 1.$$

By the additivity of the fixed point index we have

$$i(T, \Omega_{r_1} \setminus \overline{\Omega}_{r_2}, K) = -1, \quad i(T, \Omega_{r_3} \setminus \overline{\Omega}_{r_1}, K) = 1,$$

which implies that T has a fixed point $u \in \Omega_{r_1} \setminus \overline{\Omega}_{r_2}$ or $u \in \Omega_{r_3} \setminus \overline{\Omega}_{r_1}$ provided $f_0 = 0 = h_0$ or $f_\infty = 0 = h_\infty$, respectively. Consequently, problem (1.1) has a positive solution for $\lambda > \lambda_0$, $\mu \geqslant 0$, $\nu \geqslant 0$.

According to Lemmas 2.6, 2.7 and Theorem 1.2(a), we can complete the proof.

(b) Fix a number $r_1 > 0$. Lemma 2.8 implies that there exist $\lambda_0 > 0$, $\mu_0 > 0$, $\nu_0 > 0$ such that

$$\|Tu\| < \|u\| \quad \text{for } u \in \partial\Omega_{r_1},\ 0 < \lambda < \lambda_0,\ 0 \leqslant \mu < \mu_0,\ 0 \leqslant \nu < \nu_0.$$

If $f_0 = \infty$, then, for $M > 1/\lambda(1-\delta^{\alpha-1})^2 \int_\gamma^\delta G(s,s)p(s)\,\mathrm{d}s$, there exists a positive $\overline{r} > 0$ such that

$$f(u) > M|u| \quad \text{for } 0 < |u| < \overline{r}.$$

Let $\Omega_{r_2} = \{u \in K\colon \|u\| < r_2\}$, $0 < r_2 < \{r_1, \overline{r}\}$. For $u \in \partial\Omega_{r_2}$ we have

$$\begin{aligned} Tu(t) &\geqslant \lambda M(1-\delta^{\alpha-1}) \int_\gamma^\delta G(s,s)p(s)|u|\,\mathrm{d}s \\ &\geqslant \lambda M(1-\delta^{\alpha-1})^2 \int_\gamma^\delta G(s,s)p(s)\|u\|\,\mathrm{d}s \\ &> \|u\|, \end{aligned}$$

which implies that $\|Tu\| \geqslant \|u\|$ for $u \in \partial\Omega_{r_2}$.

If $f_\infty = \infty$, then, for $M > \big(\lambda(1-\delta^{\alpha-1})^2 \int_\gamma^\delta G(s,s)p(s)\,\mathrm{d}s\big)^{-1}$ there exists a positive $N > 0$ such that

$$f(u) > M|u| \quad \text{for } |u| > N.$$

Let $r_3 > \max\{r_1, N/(1-\delta^{\alpha-1})\}$ and let $\Omega_{r_3} = \{u \in K\colon \|u\| < r_3\}$. Then for $u \in \partial\Omega_{r_3}$ we have $u(t) \geqslant (1-\delta^{\alpha-1})\|u\| = (1-\delta^{\alpha-1})r_3 > N$ for $\gamma \leqslant t \leqslant \delta$, hence we have

$$\begin{aligned} Tu(t) &\geqslant \lambda \int_\gamma^\delta G(t,s)p(s)f(u(s))\,\mathrm{d}s \\ &\geqslant \lambda M(1-\delta^{\alpha-1})^2 \int_\gamma^\delta G(s,s)p(s)\|u\|\,\mathrm{d}s \\ &> \|u\|, \end{aligned}$$

which implies that $\|Tu\| \geqslant \|u\|$ for $u \in \partial\Omega_{r_3}$.

It follows from Lemma 2.3 that

$$i(T, \Omega_{r_1}, K) = 1 \quad \text{and} \quad i(T, \Omega_{r_2}, K) = 0 \quad \text{and} \quad i(T, \Omega_{r_3}, K) = 0.$$

By the additivity of the fixed point index we have

$$i(T, \Omega_{r_1} \setminus \overline{\Omega}_{r_2}, K) = 1, \quad i(T, \Omega_{r_3} \setminus \overline{\Omega}_{r_1}, K) = -1,$$

which implies that T has a fixed point $u \in \Omega_{r_1} \setminus \overline{\Omega}_{r_2}$ or $u \in \Omega_{r_3} \setminus \overline{\Omega}_{r_1}$ provided $f_0 = \infty$ or $f_\infty = \infty$, respectively. Consequently, problem (1.1) has a positive solution for $0 < \lambda < \lambda_0$, $0 \leqslant \mu < \mu_0$, $0 \leqslant \nu < \nu_0$.

By the same method, we can verify the results of (c) and (d).

(e) Fix two numbers $0 < r_3 < r_4$. Lemma 2.5 implies that there exists $\lambda_0 > 0$ such that

$$\|Tu\| > \|u\| \quad \text{for } u \in \partial\Omega_{r_i}\ (i = 3, 4),\ \lambda > \lambda_0,\ \mu \geqslant 0,\ \nu \geqslant 0.$$

Since $f_0 = 0 = h_0$, $f_\infty = 0 = h_\infty$, according to the proof of Theorem 1.2(a), $0 < r_1 < r_3$ and $r_2 > r_4$ such that

$$\|Tu\| < \|u\| \quad \text{for } u \in \partial\Omega_{r_i}(i = 1, 2),$$

and it follows from Lemma 2.3 that

$$i(T, \Omega_{r_1}, K) = 1, \quad i(T, \Omega_{r_2}, K) = 1,$$

and

$$i(T, \Omega_{r_3}, K) = 0, \quad i(T, \Omega_{r_4}, K) = 0.$$

Hence, $i(T, \Omega_{r_3} \setminus \overline{\Omega}_{r_1}, K) = -1$, $i(T, \Omega_{r_2} \setminus \overline{\Omega}_{r_4}, K) = 1$. Thus, T has two fixed points $u_1 \in \Omega_{r_3} \setminus \overline{\Omega}_{r_1}$, $u_2 \in \Omega_{r_2} \setminus \overline{\Omega}_{r_4}$, which are distinct positive solutions of problem (1.1) for all $\lambda > \lambda_0$, $\mu \geqslant 0$, $\nu \geqslant 0$.

By very similar arguments, we can complete the proof.

(f) Fix two numbers $0 < r_3 < r_4$. Lemma 2.8 implies that there exist $\lambda_0 > 0$, $\mu_0 > 0$, $\nu_0 > 0$ such that

$$\|Tu\| < \|u\| \quad \text{for } u \in \partial\Omega_{r_i}(i = 3, 4),\ 0 < \lambda < \lambda_0,\ 0 \leqslant \mu < \mu_0,\ 0 \leqslant \nu < \nu_0.$$

Since $f_0 = f_\infty = \infty$, it follows from the proof of Theorem 1.2(b) that we can choose $0 < r_1 < r_3$ and $r_2 > r_4$ such that

$$\|Tu\| > \|u\| \quad \text{for } u \in \partial\Omega_{r_i}(i = 1, 2),$$

and it follows from Lemma 2.3 that

$$i(T, \Omega_{r_1}, K) = 0, \quad i(T, \Omega_{r_2}, K) = 0,$$

and

$$i(T, \Omega_{r_3}, K) = 1, \quad i(T, \Omega_{r_4}, K) = 1.$$

Hence, $i(T, \Omega_{r_3} \setminus \overline{\Omega}_{r_1}, K) = 1$, $i(T, \Omega_{r_2} \setminus \overline{\Omega}_{r_4}, K) = -1$. Thus, T has two fixed points $u_1 \in \Omega_{r_3} \setminus \overline{\Omega}_{r_1}$, $u_2 \in \Omega_{r_2} \setminus \overline{\Omega}_{r_4}$, which are distinct positive solutions of problem (1.1) for all $0 < \lambda < \lambda_0, 0 \leqslant \mu < \mu_0, 0 \leqslant \nu < \nu_0$.

According to Lemma 2.8, using similar arguments we can complete the proof, under the assumption $(h - g)_0 = (h - g)_\infty = \infty$ or $h_0 = h_\infty = \infty$.

(g) Since $f_0 < \infty$, $h_0 < \infty$ and $f_\infty < \infty$, $h_\infty < \infty$, then there exist positive number ε_i, $i = 1, 2, 3, 4$ and $0 < r_1 < r_2$ such that,

$$f(u) \leqslant \varepsilon_1 |u|, \quad h(u) \leqslant \varepsilon_2 |u| \quad \text{for } |u| \leqslant r_1,$$

and

$$f(u) \leqslant \varepsilon_3|u|, \quad h(u) \leqslant \varepsilon_4|u| \quad \text{for } |u| \geqslant r_2.$$

Let

$$\varepsilon = \max\Big\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \max\Big\{\frac{f}{|u|}\colon\ r_1 \leqslant |u| \leqslant r_2\Big\},\ \max\Big\{\frac{h}{|u|}\colon\ r_1 \leqslant |u| \leqslant r_2\Big\}\Big\}.$$

We obtain that

$$f(u) \leqslant \varepsilon|u|, \quad h(u) \leqslant \varepsilon|u| \quad \text{for } u \in \mathbb{R}_+.$$

Assume that $v(t)$ is a positive solution of problem (1.1). Then, for $0 < \lambda < \lambda_0$, $0 \leqslant \mu < \mu_0$, $0 \leqslant \nu < \nu_0$, where $\lambda_0 = \big(6\varepsilon \int_0^1 G(s,s)p(s)\,\mathrm{d}s\big)^{-1}$, $\mu_0 = \big(6\varepsilon \int_0^1 p_0(s)\,\mathrm{d}s\big)^{-1}$, $\nu_0 = \big(6\varepsilon \int_0^1 p_1(s)\,\mathrm{d}s\big)^{-1}$, we have

$$\begin{aligned}
|Tv(t)| = Tv(t) &= \lambda \int_0^1 G(s,s)p(s)f(v)\,\mathrm{d}s + \mu(1-t)\int_0^1 p_0(s)(h(v) - g(v))\,\mathrm{d}s \\
&\quad + \int_0^1 (\nu p_1(s) - \mu(1-t)p_0(s))h(v)\,\mathrm{d}s \\
&\leqslant \lambda \int_0^1 G(s,s)p(s)f(u)\,\mathrm{d}s + \mu(1-t)\int_0^1 p_0(s)h(v)\,\mathrm{d}s + \nu\int_0^1 p_1(s)h(v)\,\mathrm{d}s \\
&\leqslant \varepsilon\|v\|\Big(\lambda\int_0^1 G(s,s)p(s)\,\mathrm{d}s + \mu\int_0^1 p_0(s)\,\mathrm{d}s + \nu\int_0^1 p_1(s)\,\mathrm{d}s\Big) \\
&< \varepsilon\|v\|\Big(\lambda_0\int_0^1 G(s,s)p(s)\,\mathrm{d}s + \mu_0\int_0^1 p_0(s)\,\mathrm{d}s + \nu_0\int_0^1 p_1(s)\,\mathrm{d}s\Big) \\
&= \frac{\|v\|}{2},
\end{aligned}$$

that is, $\|v\| = \|Tv\| \leqslant \frac{1}{2}\|v\| < \|v\|$, which is a contradiction.

(h) Since $f_0 > 0$ and $f_\infty > 0$, there exist positive numbers η_1, η_2 and $0 < r_1 < r_2$ such that,

$$f(u) \geqslant \eta_1|u| \quad \text{for } |u| \leqslant r_1,$$

and

$$f(u) \geqslant \eta_2|u| \quad \text{for } |u| \geqslant r_2.$$

Let

$$\eta_3 = \min\Big\{\eta_1, \eta_2, \min\Big\{\frac{f}{|u|}\colon\ (1-\delta^{\alpha-1})r_1 \leqslant |u| \leqslant r_2\Big\}\Big\}.$$

We can obtain that

$$f(u) \geqslant \eta_3|u| \quad \text{for } u \in \mathbb{R}_+,\ |u| \leqslant r_1, \tag{4.1}$$

$$f(u) \geqslant \eta_3|u| \quad \text{for } u \in \mathbb{R}_+,\ |u| \geqslant (1-\delta^{\alpha-1})r_1. \tag{4.2}$$

Let

$$\lambda_0 = \frac{1}{(1-\delta^{\alpha-1})^2 \eta_3 \int_\gamma^\delta G(s,s)p(s)\,\mathrm{d}s} + 1.$$

Assume that $v(t) \in K$ is a positive solution of problem (1.1). If $\|v\| \leqslant r_1$, then, by (4.1) , we have

$$f(v(t)) \geqslant \eta_3 |v(t)|, \quad t \in [0,1].$$

On the other hand, if $\|v\| > r_1$, then

$$\min_{\gamma \leqslant t \leqslant \delta} v(t) \geqslant (1-\delta^{\alpha-1})\|v\| > (1-\delta^{\alpha-1}) r_1,$$

which, together with (4.2), implies that

$$f(v(t)) \geqslant \eta_3 |v(t)|, \quad t \in [\gamma, \delta].$$

Hence, for $\lambda > \lambda_0, \mu \geqslant 0, \nu \geqslant 0$, one has

$$\begin{aligned}
v(t) = Tv(t) &\geqslant \lambda \int_0^1 G(t,s)p(s)f(v(s))\,\mathrm{d}s \\
&\geqslant \lambda \int_\gamma^\delta G(t,s)p(s)f(v(s))\,\mathrm{d}s \\
&\geqslant (1-\delta^{\alpha-1})\eta_3 \lambda \int_\gamma^\delta G(s,s)p(s)|v|\,\mathrm{d}s \\
&\geqslant (1-\delta)^{\alpha-1})^2 \eta_3 \lambda \int_\gamma^\delta G(s,s)p(s)\|v\|\,\mathrm{d}s \\
&> (1-\delta^{\alpha-1})^2 \eta_3 \lambda_0 \int_\gamma^\delta G(s,s)p(s)\|v\|\,\mathrm{d}s \\
&> \|v\|,
\end{aligned}$$

which produces a contradiction $\|v\| > \|v\|$.

By the same arguments as in Theorem 1.2 (h), we can complete the proofs of Theorem 1.2 (j), (k).

References

[1] *A. A. Kilbas, H. M. Srivastava, J. J. Trujillo*: Theory and Applications of Fractional Differential Equations. Elsevier, Amsterdam, 2006.

[2] *Y.-K. Chang, J. J. Nieto*: Some new existence results for fractional differential inclusions with boundary conditions. Mathematical and Computer Modelling *49* (2009), 605–609.

[3] *R. P. Agarwal, M. Benchohra, S. Hamani*: Boundary value problems for differential inclusions with fractional order. Adv. Stu. Contemp. Math. *16* (2008), 181–196.

[4] *Varsha Daftardar-Gejji, Sachin Bhalekar*: Boundary value problems for multi-term fractional differential equations. J. Math. Anal. Appl. *345* (2008), 754–765.

[5] *Sihua Liang, Jihui Zhang*: Positive solutions for boundary value problems of nonlinear fractional differential equation. Nonlinear Anal., Theory Methods Appl. *71* (2009), 5545–5550.

[6] *M. Benchohra, S. Hamani, S. K. Ntouyas*: Boundary value problems for differential equations with fractional order and nonlocal conditions. Nonlinear Anal., Theory Methods Appl. *71* (2009), 2391–2396.

[7] *M. Benchohra, J. R. Graef, S. Hamani*: Existence results for boundary value problems with nonlinear fractional differential equations. Appl. Anal. *87* (2008), 851–863.

[8] *M. Benchohra, S. Hamani*: Boundary value problems for differential equations with fractional order. Surv. Math. Appl. *3* (2008), 1–12.

[9] *Xiaojie Xu, Daqing Jiang, Chengjun Yuan*: Multiple positive solutions for the boundary value problem of a nonlinear fractional differential equation. Nonlinear Anal., Theory Methods Appl. *71* (2009), 4676–4688.

[10] *Shuqin Zhang*: Existence results of positive solutions to boundary value problem for fractional differential equation. Positivity *13* (2009), 585–599.

[11] *I. Podlubny*: Fractional Differential Equations. Mathematics in Science and Engineering, Academic Press, New York, 1999.

[12] *M. Krasnoselskii*: Positive Solutions of Operator Equations. Noordhoff, Groningen, 1964.

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