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# ON RELATIVELY ALMOST COUNTABLY COMPACT SUBSETS

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*Abstract.* A subset $Y$ of a space $X$ is almost countably compact in $X$ if for every countable cover $\mathscr{U}$ of $Y$ by open subsets of $X$, there exists a finite subfamily $\mathscr{V}$ of $\mathscr{U}$ such that $Y \subseteq \overline{\bigcup \mathscr{V}}$. In this paper we investigate the relationship between almost countably compact spaces and relatively almost countably compact subsets, and also study various properties of relatively almost countably compact subsets.

*Keywords*: countably compact space, almost countably compact space, relatively almost countably compact subset

*MSC 2010*: 54D15, 54D20

## 1. Introduction

By a space we mean a topological space. Let us recall that a space $X$ is *countably compact* if every countable open cover of $X$ has a finite subcover. As a generalization of countable compactness, Bonanzinga, Matveev and Pareek [1] defined a space $X$ to be *almost countably compact* if for every countable open cover $\mathscr{U}$ of $X$ there exists a finite subset $\mathscr{V}$ of $\mathscr{U}$ such that $\bigcup\{\overline{V}\colon V \in \mathscr{V}\} = X$. Clearly, every countably compact space is almost countably compact. Recall that a space $X$ is *Lindelöf* if every open cover of $X$ has a countable subcover. Sarsak [4] defined a subset $Y$ of a space $X$ to be *almost Lindelöf* in $X$ if for every cover $\mathscr{U}$ of $Y$ by open subsets of $X$ there exists a countable subfamily $\mathscr{V}$ of $\mathscr{U}$ such that $Y \subseteq \overline{\bigcup \mathscr{V}}$. This motivates the following definition.

**Definition 1.1.** A subset $Y$ of a space $X$ is *almost countably compact* in $X$ if for every countable cover $\mathscr{U}$ of $Y$ by open subsets of $X$ there exists a finite subset $\mathscr{V}$ of $\mathscr{U}$ such that $Y \subseteq \overline{\bigcup \mathscr{V}}$.

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The purpose of this paper is to investigate the relationship between almost countably compact spaces and relatively almost countably compact subsets. In particular, we study various properties of relatively almost countably compact subsets.

Throughout this paper, let $\mathfrak{c}$ denote the cardinality of the continuum, $\omega$ the first infinite cardinal and $\omega_1$ the first uncountable cardinal. As usual, a cardinal is the initial ordinal and an ordinal is the set of smaller ordinals. When viewed as a space, every cardinal has the usual order topology. Other terms and symbols that we do not define will be used as in [2].

## 2. On relatively countably compact subsets

First, we discuss the relationship between almost countably compact spaces and relatively almost countably compact subsets. We need the following lemma.

**Lemma 2.1** ([5]). *If $X$ has a dense countably compact subspace, then $X$ is almost countably compact.*

**Theorem 2.2.** *The following properties are equivalent for regular spaces $X$:*

(1) *$X$ is countably compact;*

(2) *all closed subspaces of $X$ are relatively almost countably compact in $X$.*

Proof. The implication (1) $\Rightarrow$ (2) is obvious. To show that (2) $\Rightarrow$ (1), suppose that $X$ is not countably compact. Then there exists a countably infinite subset $\{x_n\colon n\in\omega\}$ of $X$ that does not have an accumulation point. Hence $\{x_n\colon n\in\omega\}$ is a discrete closed subset of $X$. Since $X$ is regular, there exists a family $\{U_n\colon n\in\omega\}$ of open subsets of $X$ such that $U_n\cap U_m=\emptyset$ for $n,m\in\omega$ with $n\neq m$ and $\overline{U_n}\cap\{x_n\colon n\in\omega\}=x_n$ for each $n\in\omega$. This shows that $\{x_n\colon n\in\omega\}$ is not almost countably compact in $X$, which completes the proof. $\square$

**Corollary 2.3.** *There exists a Tychonoff almost countably compact space $X$ and a closed subset $Y$ of $X$ such that $Y$ is not almost countably compact in $X$.*

Proof. Let

$$X=((\omega_1+1)\times(\omega+1))\setminus\{\langle\omega_1,\omega\rangle\}$$

be the Tychonoff plank. Then $X$ is almost countably compact by Lemma 2.1. Since $X$ is not countably compact, there exists a closed subset $Y$ of $X$ such that $Y$ is not almost countably compact in $X$ by Theorem 2.2, which completes the proof. $\square$

The following two positive results can be easily proved.

**Proposition 2.4.** *If $X$ is an almost countably compact space and $Y$ an open and closed subset of $X$, then $Y$ is almost countably compact in $X$.*

**Proposition 2.5.** *If $Y$ is almost countably compact in $X$ and $Y$ is a dense subset of $X$, then $X$ is almost countably compact.*

Next, we study various properties of relatively almost countably compact subsets. From the definition of a relatively almost countably compact subset, it is not difficult to see that if $A \subseteq B \subseteq X$ and $A$ is almost countably compact in $B$, then $A$ is almost countably compact in $X$. Recall from [3] that a subset $B$ of a space $X$ is called *preopen* if $B \subseteq \operatorname{Int} \overline{B}$. About the converse, we have the following result.

**Proposition 2.6.** *If $A \subseteq B \subseteq X$, $B$ is a preopen subset of $X$ and $A$ is almost countably compact in $X$, then $A$ is almost countably compact in $B$.*

P r o o f. Supposing that $A$ is almost countably compact in $X$, we shall show that $A$ is almost countably compact in $B$. Let $\{U_n \colon n \in \omega\}$ be a countable cover of $A$ by open subsets of $B$. Then there exists an open subset $V_n$ of $X$ such that $U_n = V_n \cap B$ for each $n \in \omega$, hence $\{V_n \colon n \in \omega\}$ is a countable cover of $A$ by open subsets of $X$. Since $B$ is preopen in $X$ and $A \subseteq B$, it follows that $\{V_n \cap \operatorname{Int} \overline{B} \colon n \in \omega\}$ is also a countable cover of $A$ by open subsets of $X$, hence there exists a finite subfamily $\{V_{n_i} \cap \operatorname{Int} \overline{B} \colon i = 1, 2, \ldots, m\}$ of $\{V_n \cap \operatorname{Int} \overline{B} \colon n \in \omega\}$ such that

$$A \subseteq \bigcup_{i \leqslant m} \overline{V_{n_i} \cap \operatorname{Int} \overline{B}} \subseteq \bigcup_{i \leqslant m} \overline{V_{n_i} \cap \overline{B}} = \bigcup_{i \leqslant m} \overline{V_{n_i} \cap B} = \bigcup_{i \leqslant m} \overline{U_{n_i}}.$$

Thus

$$A \subseteq B \cap \Bigl( \bigcup_{i \leqslant m} \overline{U_{n_i}} \Bigr) = \bigcup_{i \leqslant m} (B \cap \overline{U_{n_i}}) = \bigcup_{i \leqslant m} \overline{U_{n_i}}^{B},$$

which shows that $A$ is almost countably compact in $B$. □

Since an open set is preopen, we have the following corollary.

**Corollary 2.7.** *If $A \subseteq B \subseteq X$, $B$ is an open subset of $X$ and $A$ is almost countably compact in $X$, then $A$ is almost countably compact in $B$.*

**Proposition 2.8.** *Let $f \colon X \to Y$ be a continuous mapping from a space $X$ into a space $Y$. If $A$ is almost countably compact in $X$, then $f(A)$ is almost countably compact in $Y$.*

P r o o f. Suppose that $A$ is almost countably compact in $X$ and let $f \colon X \to Y$ be a continuous map. Let $\mathscr{U} = \{U_n \colon n \in \omega\}$ be a countable cover of $f(A)$ by open subsets of $Y$. Then $\mathscr{V} = \{f^{-1}(U_n)\colon n \in \omega\}$ is a countable cover of $A$ by open subsets of $X$. Since $A$ is almost countably compact in $X$, there exists a finite subset $\{n_i\colon i = 1, 2 \ldots m\}$ such that

$$A \subseteq \bigcup\{\overline{f^{-1}(U_{n_i})}\colon i = 1, 2 \ldots, m\},$$

hence

$$\begin{aligned} f(A) \subseteq f\Big(\bigcup\{\overline{f^{-1}(U_{n_i})}\colon i = 1, 2, \ldots, m\}\Big) &= \bigcup\{f(\overline{f^{-1}(U_{n_i})}\colon i = 1, 2, \ldots, m\} \\ \subseteq \bigcup\{\overline{f(f^{-1}(U_{n_i}))}\colon i = 1, 2, \ldots, m\} &\subseteq \bigcup\{\overline{U_{n_i}}\colon i = 1, 2, \ldots, m\}. \end{aligned}$$

This shows that $f(A)$ is almost countably compact in $Y$. □

Recall from [6] that a mapping $f$ from a space $X$ to a space $Y$ is called *almost open* if $f^{-1}(\overline{U}) \subseteq \overline{f^{-1}(U)}$ for each open subset $U$ of $Y$.

**Proposition 2.9.** *If $f\colon X \to Y$ is an almost open and closed continuous mapping, $f^{-1}(y)$ is compact for each $y \in A$ and $A$ is almost countably compact in $Y$, then $f^{-1}(A)$ is almost countably compact in $X$.*

Proof. Let $\mathscr{U}$ be a countable cover of $f^{-1}(A)$ by open subsets of $X$ and let

$$\mathscr{V} = \Big\{V\colon \text{ there exists a finite subfamily } \mathscr{F} \text{ of } \mathscr{U} \text{ such that } V = \bigcup \mathscr{F}\Big\}.$$

Then $\mathscr{V}$ is countable, since $\mathscr{U}$ is countable. Hence we can enumerate $\mathscr{V}$ as $\{V_n\colon n \in \omega\}$. For each $n \in \omega$, let

$$W_n = Y \setminus f(X \setminus V_n).$$

Then $W_n$ is an open subset of $Y$, since $f$ is closed. Let $\mathscr{W} = \{W_n\colon n \in \omega\}$, then $\mathscr{W}$ is a countable cover of $A$ by open subsets of $Y$. In fact, for every $y \in A$ there exists a $V_n \in \mathscr{V}$ such that $f^{-1}(y) \subseteq V_n$, and since $f^{-1}(y)$ is compact, hence $W_n = Y \setminus f(X \setminus V_n)$ is an open neighborhood of $y$. Since $A$ is almost countably compact in $Y$, there exists a finite subfamily $\{W_{n_i}\colon i = 1, 2 \ldots m\}$ of $\mathscr{W}$ such that

$$A \subseteq \bigcup_{i \leqslant m} \overline{W_{n_i}}.$$

Since $f$ is almost open, we have

$$\begin{aligned} f^{-1}(A) = f^{-1}\Big(\bigcup_{i \leqslant m} \overline{W_{n_i}}\Big) &= \bigcup_{i \leqslant m} f^{-1}(\overline{W_{n_i}}) \\ &\subseteq \bigcup_{i \leqslant m} \overline{f^{-1}(W_{n_i})} \subseteq \bigcup_{i \leqslant m} \overline{V_{n_i}}, \end{aligned}$$

and since every element of $\mathscr{V}$ is the union of a finite subfamily of $\mathscr{U}$. This shows that $f^{-1}(A)$ is almost countably compact in $X$, which completes the proof. □

We have the following corollary by Proposition 2.9.

**Corollary 2.10.** *If $A$ is almost countably compact in $X$ and $Y$ is compact, then $A \times Y$ is almost countably compact in $X \times Y$.*

It is well known that a subset $A$ of a space $X$ is compact in $X$ if and only if $A$ is countably compact in $X$ and $A$ is Lindelöf in $X$. About the class of almost countably compact subsets, we have the following result.

**Proposition 2.11.** *Let $X$ be a regular space and let $A$ be almost countably compact in $X$ and Lindelöf in $X$. Then $A$ is compact in $X$*

P r o o f. Let $X$ be a regular space. Let $\mathscr{U}$ be any cover of $A$ by open subsets of $X$. For each $x \in A$ there exists a $U_x \in \mathscr{U}$ such that $x \in U_x$, hence there exists an open neighbourhood $V_x$ of $x$ such that $x \in V_x \subseteq \overline{V_x} \subseteq U_x$. Let $\mathscr{V} = \{V_x \colon x \in A\}$. Then $\mathscr{V}$ is an open cover of $A$. Hence $\mathscr{V}$ has a countable subcover, since $A$ is Lindelöf in $X$, say $\{V_{x_n} \colon n \in \omega\}$. Thus $\{V_{x_n} \colon n \in \omega\}$ has a finite subset $\{V_{x_{n_j}} \colon j = 1, 2, \ldots, m\}$ such that $A \subseteq \bigcup\{\overline{V_{x_{n_j}}} \colon j = 1, 2, \ldots, m\}$, since $A$ is almost countably compact in $X$. Clearly, $\{U_{x_{n_j}} \colon j = 1, 2, \ldots, m\}$ is a finite subfamily of $\mathscr{U}$ and

$$A \subseteq \bigcup\{U_{x_{n_j}} \colon j = 1, 2, \ldots, m\},$$

which completes the proof. $\square$

R e m a r k 1. The first author constructed an example showing that there exists a Hausdorff almost countably compact and Lindelöf space which is not compact (see Example 2.2, [5]). In fact, the example also shows that the condition of regularity in Proposition 2.12 is necessary.

**Proposition 2.12.** *Let $X$ be a regular space. If $A$ is almost countably compact in $X$ and is Lindelöf in $X$, and $Y$ is a compact space, then $A \times Y$ is almost countably compact in $X \times Y$.*

P r o o f. Since $X$ is regular and $A$ is an almost countable compact subset in $X$ and Lindeöf in $X$, hence $A$ is compact by 2.11. Clearly $A \times Y$ is compact in $X \times Y$, hence $A \times Y$ is almost countably compact in $X \times Y$. $\square$

In Proposition 2.12, if $X$ is Hausdorff, then $A$ need not be compact in $X$ by Remark 1. But we still have the following proposition.

**Proposition 2.13.** *Let $X$ be a Hausdorff space. If $A$ is almost countably compact in $X$ and Lindelöf in $X$ and $Y$ is a compact space, then $A \times Y$ is almost countably compact in $X \times Y$.*

Proof. Let $\mathscr{U}$ be a countable cover of $A \times Y$ by open subsets of $X \times Y$. Without loss of generality, we can assume that $\mathscr{U}$ consists of basic open sets of $X \times Y$. Since $\{x\} \times Y$ is a compact subset of $X \times Y$ for each $x \in A$, there exists a finite subfamily $\{U_{x,i} \times V_{x,i} \colon\ i = 1, 2, \ldots, n_x\}$ of $\mathscr{U}$ such that

$$\{x\} \times Y \subseteq \bigcup\{U_{x,i} \times V_{x,i} \colon\ 1 \leqslant i \leqslant n_x\}.$$

Let $W_x = \bigcap\{U_{x,i} \colon\ 1 \leqslant i \leqslant n_x\}$. Then

$$\{x\} \times Y \subseteq \bigcup\{W_x \times V_{x,i} \colon\ 1 \leqslant i \leqslant n_x\}.$$

Let $\mathscr{W} = \{W_x \colon\ x \in A\}$. Then $\mathscr{W}$ is an open cover of $A$. Hence there is a countable subfamily $\{W_{x,j} \colon\ j \in \omega\}$ of $\mathscr{W}$ such that

$$A \subseteq \bigcup_{j \in \omega} W_{x,j},$$

since $A$ is Lindelöf in $X$. Thus there exists a finite subfamily

$$\{W_{x,j_k} \colon\ k = 1, 2, \ldots, n\}$$

such that

$$A \subseteq \bigcup\{\overline{W_{x,j_k}} \colon\ k = 1, 2, \ldots, n\},$$

since $A$ is almost countably compact in $X$.

Let

$$\mathscr{V} = \{U_{x,i_{j_k}} \times V_{x,i_{j_k}} \colon\ i \leqslant n_x, k \leqslant n\}.$$

Then $\mathscr{V}$ is a finite subfamily of $\mathscr{U}$. To show that $A \times Y \subseteq \overline{\bigcup \mathscr{V}}$, let $\langle s, t\rangle \in A \times Y$ be fixed. Let $U_s \times V_t$ be any open neighborhood of $\langle s, t\rangle$ in $X \times Y$ where $U_s$ and $V_t$ are open neighborhoods of $x$ and $y$ in $X$ and $Y$, respectively. Since $A \subseteq \overline{\bigcup\limits_{k \leqslant n} W_{x,j_k}}$, there exists a $k \leqslant n$ such that

$$U_s \cap W_{x,j_k} \neq \emptyset.$$

Thus,

$$(U_s \times V_t) \cap \Big(\bigcup\{W_{x,j_k} \times V_{x,i_{j_k}} \colon\ k \leqslant n\}\Big) \neq \emptyset.$$

Therefore,

$$(U_s \times V_t) \cap \Big(\bigcup\{U_{x,i_{j_k}} \times V_{x,i_{j_k}} \colon\, i \leqslant n_x, k \leqslant n\}\Big) \neq \emptyset.$$

We have

$$(U_s \times V_t) \cap \Big(\bigcup \mathscr{V}\Big) \neq \emptyset.$$

This implies $\langle s,t\rangle \in \overline{\bigcup \mathscr{V}}$. Hence, $A \times Y \subseteq \overline{\bigcup \mathscr{V}}$, which shows that $A \times Y$ is almost countably compact in $X \times Y$. □

It is well-known that the product of two countably compact spaces is not necessarily countably compact. In the following, we show that the product of two relatively countably compact subsets is not necessarily relatively almost countably compact by using an example. Here, we give the proof mainly for the sake of completeness (see [2] Example 3.10.19). For a Tychonoff space $X$, the symbol $\beta X$ denotes the Čech-Stone compactification of a space $X$.

Example 2.14. There exist two Tychonoff countably compact spaces $X$ and $Y$ such that $X \times Y$ is not almost countably compact.

Proof. Let $\omega$ be endowed with the discrete topology. We can define two countably compact spaces $X$ and $Y$ such that $\omega \subseteq X, Y \subset \beta\omega$, and $X \cap Y = \omega$ (see [2], Example 3.10.19). However $X \times Y$ is not almost countably compact, since the diagonal $\{\langle n,n\rangle \colon\, n \in \omega\}$ is a discrete open and closed subset of $X \times Y$ with cardinality $\omega$, and almost countable compactness is preserved by open and closed subsets. □

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