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LI-YORKE PAIRS OF FULL HAUSDORFF DIMENSION FOR SOME CHAOTIC DYNAMICAL SYSTEMS

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Abstract. We show that for some simple classical chaotic dynamical systems the set of Li-Yorke pairs has full Hausdorff dimension on invariant sets.

Keywords: Li-Yorke chaos, Hausdorff dimension

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1. Introduction

The term "chaos" in mathematical theory of dynamical systems was used for the first time in the influential paper of Li and Yorke [6]. The approach of Li and Yorke is based on the existence of Li-Yorke pairs. These are pairs of points in the phase space that approaches each other for some sequence of moments in the time evolution and that remain separated for other sequences of moments. The characteristic property of dynamical systems that are chaotic in the sense of Li-Yorke is sensitivity to the initial condition. States that are physically indistinguishable result in physically distinguishable states for such systems.

Although there is an enormous literature on Li-Yorke chaos (see for instance [1] and [2] and the bibliography therein) there seem to be no results from a dimensional-theoretical point of view. In this paper we ask the natural question if for invariant sets like repellers, attractors or hyperbolic sets of chaotic dynamical systems, Li-Yorke pairs have full Hausdorff dimension, see section two for appropriate definitions. Our main result in Section 5 is that this is in fact the case, if the invariant set is self-similar or a product of self-similar sets and the dynamics of the system is homomorphic conjugated to a full shift, see Theorem 5.1 and 5.2 below. To prove this result we have a look at Li-Yorke pairs in symbolic dynamics in Section 3 and study Li-Yorke pairs in the context of iterated function systems in Section 4. Our general result can be applied to simple classical models of "chaotic" dynamics like the tent map, the Bakers transformation, Smales horseshoe, and solenoid-like systems, see Section 6. To prove that Li-Yorke pairs have full dimension for more general hyperbolic systems could be a task for further research.

2. Basic notation

Consider a dynamical system (X, T) on a complete separable metric space X. A pair of points $(x, y) \in X^2$ is called a Li-Yorke pair for T if

$$\liminf_{n\to\infty} d(T^n x, T^n y) = 0 \quad \text{and} \quad \limsup_{n\to\infty} d(T^n x, T^n y) > 0,$$

see [6]. Now given an invariant set $\Lambda \subseteq X$, i.e. $f(\Lambda) = \Lambda$, we define the set of Li-Yorke pairs in Λ for T by

$$LY_T(\Lambda) = \{(x, y) \in \Lambda^2;\ (x, y) \text{ is a Li-Yorke pair}\}.$$

We say that Li-Yorke pairs in Λ have full Hausdorff dimension for T if the Hausdorff dimension of $LY_T(\Lambda)$ coincides with the Hausdorff dimension of Λ^2, i.e.

$$\dim_H(LY_T(\Lambda)) = \dim_H(\Lambda^2).$$

Recall that the Hausdorff dimension of $A \subseteq X$ is given by

$$\dim_H(A) = \inf\{s;\ H^s(A) = 0\} = \sup\{s;\ H^s(A) = \infty\}$$

where $H^s(A)$ denotes the s-dimensional Hausdorff measure, i.e.

$$H^s(A) = \lim_{\varepsilon\to 0} \inf \Big\{ \sum |U_i|^s;\ A \subseteq \bigcup U_i, \quad |U_i| < \varepsilon \Big\},$$

and $|U_i|$ denotes the diameter of a covering element U_i. Moreover we note that the Hausdorff dimension of a Borel probability measure μ on X is defined by

$$\dim_H \mu = \inf\{\dim_H A;\ \mu(A) = 1\}.$$

Beside Hausdorff dimension we will use the Minkowski dimension in our proofs for some technical reasons. Let $N_\varepsilon(A)$ be the smallest number of balls needed to cover A. The Minkowski dimension of A is given by

$$\dim_M A = \lim_{\varepsilon\to 0} \frac{\log N_\varepsilon(A)}{-\log \varepsilon},$$

if the limit exists. We refer to the book of Falconer [3] for an introduction to the modern dimension theory. Given a dynamical system, invariant sets like repellers, attractors or hyperbolic sets often have a fractal geometry with non integer Hausdorff dimension. In this way the dimension theory comes into the study of dynamical systems, see the book of Pesin [11]. If Li-Yorke pairs have full dimension in an invariant set for a dynamical system, this means that the measure of chaos on this set is maximal from dimensional-theoretic point of view.

3. Li-Yorke pairs in symbolic dynamics

Consider the spaces of one and two sided sequences $\Sigma = \{1, 2, \ldots m\}^{\mathbb{N}}$, $\widetilde{\Sigma} = \{1, 2, \ldots m\}^{\mathbb{Z}}$ with the metric

$$\operatorname{dist}(\mathfrak{s}, \mathfrak{t}) = \sum m^{-|k|} |s_k - t_k|,$$

where $\mathfrak{s} = (s_k)$ and $\mathfrak{t} = (t_k)$. These are perfect, totally disconnected and compact metric spaces. The shift map σ on this spaces is given by

$$\sigma((s_k)) = (s_{k+1}).$$

For an introduction to symbolic dynamics consider for instance [5].

Two sequences $\mathfrak{s}$ and $\mathfrak{t}$ form a Li-Yorke pair in Σ (or $\widetilde{\Sigma}$) for σ with respect to the metric *dist*, if they coincide on a sequence of blocks with increasing length and do not coincide on one subsequence. We construct here Li-Yorke pairs in the following way: Fix $\mathfrak{s} \in \Sigma$ and an arbitrary sequence $\mathfrak{N} = (N_n)$ of natural numbers. Let the first digit of $\mathfrak{t} \in \Sigma$ be s_1 and let the second digit by $t_2 = s_2 + 1$ modulo m. Then we choose N_1 arbitrary digits. Next choose two digits of $\mathfrak{s}$ and one digit of $\mathfrak{s} + 1$ modulo m. Now we again choose N_2 arbitrary digits and three digits from $\mathfrak{s}$ and one of $\mathfrak{s} + 1$ modulo m and so on. Thus we consider subsets of Σ given by

$$\begin{aligned}\Sigma_{\mathfrak{N}}(\mathfrak{s}) = \{\mathfrak{t} \in \Sigma;\ & t_k = s_k \text{ for } k \in \{u_i, \ldots, u_i + i\},\\ & \text{and } t_{u_i+i+1} = s_{u_i+i+1} + 1 \bmod m \text{ for } i = 0, \ldots, \infty\}\end{aligned}$$

where $u_0 = 1$ and u_i is given by the recursion $u_{i+1} = u_i + N_i + i + 1$.

Proposition 3.1. *A pair $(\mathfrak{s}, \mathfrak{t}) \in \Sigma^2$ with $\mathfrak{t} \in \Sigma_{\mathfrak{N}}(\mathfrak{s})$ is a Li-Yorke pair for σ.*

Proof. Under the assumptions we have

$$\begin{aligned}&\lim_{i\to\infty} d(\sigma^{u_i}(\mathfrak{s}), \sigma^{u_i}(\mathfrak{t})) \leqslant \lim_{i\to\infty} 1/m^i = 0,\\ &\limsup_{i\to\infty} d(\sigma^{u_i+i+1}(\mathfrak{s}), \sigma^{u_i+i+1}(\mathfrak{t})) \geqslant 1/m\end{aligned}$$

giving the required asymptotic. □

By this proposition we obviously have

$$\Pi_{\mathfrak{N}} := \{(\mathfrak{s}, \mathfrak{t});\ \mathfrak{s} \in \Sigma \text{ and } \mathfrak{t} \in \Sigma_{\mathfrak{N}}(\mathfrak{s})\} \subseteq LY_\sigma(\Sigma),$$

where LY is the set of Li-Yorke pairs defined in the preceding section. Moreover, for the two-side sequence we have

$$\widetilde{\Pi}_{\mathfrak{N}} := \{(\tilde{\mathfrak{s}}, \tilde{\mathfrak{t}});\ \tilde{\mathfrak{s}} \in \Sigma \text{ and } \mathfrak{t} \in \Sigma_{\mathfrak{N}}(\mathfrak{s})\} \subseteq LY_\sigma(\widetilde{\Sigma}),$$

where $\mathfrak{s}$ is the part of $\tilde{\mathfrak{s}}$ with positive indices. In the next section the symbolic sets defined here will be used. In addition we will use the natural bijection Σ onto $\Sigma_{\mathfrak{N}}(\mathfrak{s})$ which we denote by $\mathrm{pr}_{\mathfrak{s},\mathfrak{N}}$. This bijection just fills arbitrary digits of sequences in $\Sigma_{\mathfrak{N}}(\mathfrak{s})$ successively with a given sequence of digits from Σ; compare with the construction of $\Sigma_{\mathfrak{N}}(\mathfrak{s})$ above. In our study of the dimension of Li-Yorke pairs of dynamical systems that are conjugated to shift systems, the symbolic approach will be useful in Section 5.

4. Li-Yorke pairs for iterated function systems

Consider a system of contracting similitudes, $S_i\colon\ \mathbb{R}^w \to \mathbb{R}^w$

$$|S_i x - S_i y| = c_i |x - y|$$

with $c_i \in (0,1)$ for $i = 1, \dots m$. It is well known [4] that there is a compact self-similar set Λ with

$$\Lambda = \bigcup_{i=1}^{m} S_i(\Lambda).$$

The set may be described using the projection $\pi\colon\ \Sigma \to \Lambda$ given by

$$\pi((s_k)) = \lim_{n\to\infty} S_{s_n} \circ \dots \circ S_{s_1}(K)$$

where K is a compact set with $S_i(K) \subseteq K$. In our study of Li-Yorke pairs for dynamical systems on X we are interested in the subset of Λ given by

$$\Lambda_{\mathfrak{N}}(\mathfrak{s}) = \pi(\Sigma_{\mathfrak{N}}(\mathfrak{s})),$$

where the set of symbols $\Sigma_{\mathfrak{N}}(\mathfrak{s})$ was introduced in the preceding section. We use an extension of the classical argument to proove the following result on the dimension of $\Lambda_{\mathfrak{N}}(\mathfrak{s})$:

Proposition 4.1. *Let $S_i\colon K \to K$ for $i = 1, \dots, m$ be contracting similitudes for some compact set $K \subseteq \mathbb{R}^w$ with $S_i(K) \cap S_j(K) = \emptyset$ for $i \neq j$.*

If $\mathfrak{N} = (N_n)$ is a sequence of natural numbers with

$$\lim_{M\to\infty} M^2 \Big/ \sum_{n=1}^{M} N_n = 0,$$

then for all $\mathfrak{s} \in \Sigma$

$$\dim_H \Lambda_{\mathfrak{N}}(\mathfrak{s}) = \dim_H \Lambda = D,$$

where D is the solution of

$$\sum_{i=1}^{m} c_i^D = 1$$

for the contraction constants c_i of the similitudes.

P r o o f. Fix $\mathfrak{s} = (s_k)$ and $\mathfrak{N} = (N_n)$ throughout the proof. Write $\bar{\Lambda}$ for $\Lambda_{\mathfrak{N}}(\mathfrak{s})$, $\bar{\Sigma}$ for $\Sigma_{\mathfrak{N}}(\mathfrak{s})$ and pr for the bijection from Σ onto $\bar{\Sigma}$ defined at the end of Section 3.

It is well known that $\dim_H \Lambda = D$, hence $\dim_H \bar{\Lambda} \leqslant D$, see [8]. For the opposite inequality we construct a Borel probability measure μ of dimension D on $\bar{\Lambda}$. To this end consider the probability vector $(c_1^D, \dots, c_m^D)$ on $\{1, \dots, n\}$ and the corresponding Bernoulli measure ν on Σ, which is the infinite product of this measure. Now we map this measure onto $\bar{\Sigma}$ using pr and further onto $\bar{\Lambda}$ using π, i.e.

$$\mu = \pi(\mathrm{pr}(\nu)) = \nu \circ \mathrm{pr}^{-1} \circ \pi^{-1}.$$

The local mass distribution princip states that

$$\liminf_{\varrho\to 0} \frac{\log \mu(B_\varrho(x))}{\log \varrho} \geqslant D$$

for all $x \in \bar{\Lambda}$ implies $\dim_H \mu \geqslant D$ and hence $\dim_H \bar{\Lambda} \geqslant D$. This is Proposition 2.1 of [13]. Hence if we prove this estimate on local dimension the proof is complete.

By bijectivity of the coding map for all points $x \in \bar{\Lambda}$ there is a unique sequence $\mathfrak{a} = (a_k)$ such that $\pi(\mathrm{pr}(\mathfrak{a})) = x$, where pr is defined at the end of Section 3. We have

$$\{x\} = \bigcap_{k=1}^{\infty} \pi(\mathrm{pr}[a_1, \dots, a_k]),$$

where pr acts pointwise on cylinder sets

$$[a_1, \dots, a_k] := \{\mathfrak{t} = (t_i) \in \Sigma;\ t_i = a_i \quad \text{for} \quad i = 1, \dots k\}$$

in Σ. For further use note that $\mathrm{pr}[a_1,\dots,a_k]$ is itself a cylinder set in $\bar{\Sigma}$, the length of this cylinder set is k plus the digits coming from the fixed sequence $\mathfrak{s}$.

Given an arbitrary real $\varrho > 0$ we choose $k = k(\varrho)$ such that

$$d \cdot c(\mathrm{pr}[a_1,\dots,a_k]) \leqslant \varrho < d \cdot c(\mathrm{pr}[a_1,\dots,a_{k-1}]),$$

where $c([a_1,\dots,a_p]) := c_{a_1} \cdot \dots \cdot c_{a_p}$ for all cylinder sets and d is the minimal distance of two sets in the construction of $\bar{\Lambda}$, i.e.

$$d = \min_{i \neq j} d(S_i(K), S_k(K)).$$

Given another finite sequence $(\bar{a}_1,\dots,\bar{a}_k) \neq (a_1,\dots,a_k)$ the contraction property of the maps S_i implies

$$\mathrm{dist}(\pi(\mathrm{pr}[a_1,\dots,a_k]), \pi(\mathrm{pr}[\bar{a}_1,\dots,\bar{a}_k])) \geqslant d \cdot c(\mathrm{pr}[a_1,\dots,a_k]) > \varrho,$$

here again $[a_1,\dots,a_k]$ is a cylinder set in Σ and $\mathrm{pr}[a_1,\dots,a_k]$ is the corresponding cylinder set in $\bar{\Sigma}$. Hence we have

$$\bar{\Lambda} \cap B_\varrho(x) \subseteq \pi(\mathrm{pr}[a_1,\dots,a_k])$$

and

$$\begin{aligned}\mu(B_\varrho(x)) &\leqslant \mu(\pi(\mathrm{pr}[a_1,\dots,a_k])) = (c([a_1,\dots,a_k]))^D \\ &= \frac{(c([a_1,\dots,a_k]))^D}{(c([\mathrm{pr}(a_1,\dots,a_k]))^D}(c(\mathrm{pr}[a_1,\dots,a_k]))^D \leqslant \frac{(c([a_1,\dots,a_k]))^D}{(c(\mathrm{pr}[a_1,\dots,a_k]))^D} d^{-D} \varrho^D\end{aligned}$$

by the construction of the measure μ and the choice of k. Now taking logarithm this yields

$$\log \mu(B_\varrho(x)) \leqslant D\Big(\log \varrho - \log d - \log \frac{c(\mathrm{pr}[a_1,\dots,a_k])}{c([a_1,\dots,a_k])}\Big) \leqslant D(\log \varrho - \log d - \sharp(k) \log \underline{c}),$$

where $\underline{c} = \min_{1 \leqslant i \leqslant m} c_i$ and $\sharp(k)$ is the length of the cylinder set $\mathrm{pr}[a_1,\dots,a_k]$ minus k, the length of the cylinder set $[a_1,\dots,a_k]$. Now dividing by $\log \varrho$ and using the definition of k we obtain,

$$\begin{aligned}\frac{\log \mu(B_\varrho(x))}{\log \varrho} &\geqslant D + D\Big(-\frac{\log d}{\log \varrho} - \frac{\sharp(k) \log \underline{c}}{\log d + \log(c(\mathrm{pr}[a_1,\dots,a_{k-1}]))}\Big) \\ &\geqslant D + D\Big(-\frac{\log d}{\log \varrho} - \frac{\sharp(k) \log \underline{c}}{\log d + ((k-1) + \sharp(k-1)) \log \overline{c}}\Big),\end{aligned}$$

where $\overline{c} = \max\limits_{1 \leqslant i \leqslant m} c_i$. We have $\lim\limits_{\varrho \to 0} k(\varrho) = \infty$, hence it remains to show that

$$\lim_{k \to \infty} \frac{\sharp(k)}{k} = 0.$$

Given k choose $M(k)$ such that

$$\sum_{n=1}^{M(k)-1} N_n < k \leqslant \sum_{n=1}^{M(k)} N_n.$$

By the definition of $\bar{\Sigma}$ and the map pr we have

$$\sharp(k) \leqslant \sum_{v=1}^{M(k)} (v+1) < M(k)^2,$$

hence

$$\frac{\sharp(k)}{k} \leqslant M(k)^2 \Big/ \sum_{n=1}^{M(k)-1} N_n.$$

By the assumption on (N_n) the righthand side goes to zero with $k \to \infty$. This completes the proof. $\square$

Using general results in dimensions theory we may go one step further and show that the projection of the set $\Pi_{\mathfrak{N}}$ has full dimension in Λ^2. This is the image of Li-Yorke pairs for the shift map in the symbolic space, compare with Section 3. We will use the following fact:

Theorem 4.1. *Let F be a subset of $\mathbb{R}^n$ and E be a subset of $\mathbb{R}^k$ with $k < n$. Let L_x be the $n - k$ dimensional affine linear subspace of $\mathbb{R}^n$ given by the translation $x \in E$. If $\dim_H(F \cap L_x) \geqslant t$ for all $x \in E$ than $\dim_H F \geqslant t + \dim_H E$.*

This is Corollary 7.1 of Falconer [3] based on the work of Marstrand [7]. With the help of this theorem we get:

Proposition 4.2. *Under the assumptions of Proposition* 4.1 *we have*

$$\dim_H S = \dim_H \Lambda^2 = 2 \dim_H \Lambda$$

where $S = \pi(\Pi_{\mathfrak{N}}) = \{(x, y);\ x \in \Lambda, y \in \Lambda_{\mathfrak{N}}(\pi^{-1}(x))\}$.

P r o o f. Since Hausdorff and Minkowski dimension of Λ coincide we have $\dim_H \Lambda^2 = 2 \dim_H \Lambda$, see Corollary 7.4 of [3]. Obviously $S \subseteq \Lambda^2$, hence $\dim_H S \leqslant 2 \dim_H \Lambda$. On the other hand Proposition 4.1 implies $\dim_H(S \cap \Lambda_{\mathfrak{N}}(\pi^{-1}(x))) = \dim_H \Lambda$ for all $x \in \Lambda$. By Theorem 4.1 we get $\dim_H S \geqslant 2 \dim_H \Lambda$, which completes the proof. $\square$

We remark at the end of this section that the results proved here for $\mathbb{R}^w$ remain true on complete separable metric spaces of finite multiplicity which have the Besicovitsh property, compare Appendix I of [11]. The techniques we have used apply in the general setting.

5. Li-Yorke pairs of full dimension for systems conjugated to a shift

In this section we state and prove our main results on the Hausdorff dimension of Li-Yorke pairs of dynamical systems conjugated to a shift and having a self-similar invariant set. The results are consequences of Proposition 3.1, Proposition 4.1 and Proposition 4.2 below.

Theorem 5.1. *Let $f\colon\ \mathbb{R}^w \to \mathbb{R}^w$ be a dynamical system with a compact invariant set Λ. If (Λ, f) is homomorphic conjugated to a one-sided full shift (Σ, σ) and Λ is self-similar then the Li-Yorke pairs in Λ have full Hausdorff dimension for f.*

Proof. By the assumption of the theorem there is a homeomorphism $\pi\colon\ \Sigma \to \Lambda$ with

$$f \circ \pi = \pi \circ f. \tag{$\star$}$$

If $(s,t) \in \Sigma^2$ is a Li-Yorke pair for σ then by definition

$$\liminf_{n\to\infty} d(\sigma^n s, \sigma^n t) = 0 \quad \text{and} \quad \limsup_{n\to\infty} d(\sigma s, \sigma t) > 0.$$

By continuity this implies

$$\liminf_{n\to\infty} d(\pi(\sigma^n s), \pi(\sigma^n t)) = 0 \quad \text{and} \quad \limsup_{n\to\infty} d(\pi(\sigma^n s), \pi(\sigma^n t)) > 0$$

and using $(\star)$

$$\liminf_{n\to\infty} d(f^n(\pi(s)), f^n\pi(t)) = 0 \quad \text{and} \quad \limsup_{n\to\infty} d(f^n(\pi(s)), f^n(\pi(t))) > 0,$$

which means $(\pi(s), \pi(t)) \in \Sigma^2$ is a Li-Yorke pair of f. Since $\Pi_{\mathfrak{N}} \subseteq LY_\sigma(\Sigma)$, see Section 3, we get

$$S = \pi(\Pi_{\mathfrak{N}}) \subseteq LY_f(\Lambda).$$

Furthermore, since Λ is self-similar and fulfils the condition of Proposition 4.1, by the properties of the coding π we have by Proposition 4.2 $\dim_H S = \dim_H \Lambda^2$ and hence $\dim_H LY_f(\Lambda) = \dim_H \Lambda^2$ concluding the proof. $\square$

For dynamical systems which have a product structure we obtain the following result:

Theorem 5.2. *Let $f\colon \mathbb{R}^w \to \mathbb{R}^w$ be a dynamical system with a compact invariant set Λ. If (Λ, f) is homeormorphic conjugated to a two-side full shift $(\widetilde{\Sigma}, \sigma)$ and the coding map is a product with two self-similar images, $\Lambda = \Lambda_1 \times \Lambda_2$, then the Li-Yorke pairs in Λ have full dimension for f.*

Proof. By the same argument as in the proof of the Theorem 5.1 we have $S = \pi(\widetilde{\Pi}_{\mathfrak{N}}) \subseteq LY_f(\Lambda)$. Using $\pi = (\pi_1, \pi_2)$ we have

$$\begin{aligned} S = \{&(\pi_1(s^+), \pi_2(s^-), \pi_1(t^+), \pi_2(t^-);\ \tilde{\mathfrak{s}} = (s^+, s^-) \in \widetilde{\Sigma}, \\ &\tilde{\mathfrak{t}} = (t^+, t^-) \in \widetilde{\Sigma} \quad \text{with } t^+ \in \Sigma_{\mathfrak{N}}(\mathfrak{s}^+)\}. \end{aligned}$$

Since Λ is self-similar and fulfils the condition of Proposition 4.1, by the properties of the coding π_1 we have

$$\dim_H \{\pi(t^+);\ t^+ \in \Sigma_{\mathfrak{N}}(\mathfrak{s}^+)\} = \dim_H \Lambda_1.$$

By the argument used in the proof of Proposition 4.2 this implies

$$\dim_H S \geqslant 2 \dim_H \Lambda_1 + 2 \dim_H \Lambda_2,$$

see again Corollary 7.12 of [3] and [7]. On the other hand, since Minkowski and Hausdorff dimensions of self-similar sets Λ_1, Λ_2 coincides, we have

$$\dim_H S \leqslant \dim_H \Lambda = 2 \dim_H \Lambda_1 + 2 \dim_H \Lambda_2$$

by Corollary 7.4 of [3]. This concludes the proof. □

The last section of this paper is devoted to examples.

6. Examples

In this section we consider four classical examples of chaotic dynamical systems, namely the tent map in dimension one, the skinny Backers transformation and a linear horseshoe in dimension two and linear solenoid-like systems in dimension three. All these systems have self-similar invariant sets with dynamics conjugated to a full shift on two symbols.

First consider the expansive tent map, see [5], $t\colon \mathbb{R} \to \mathbb{R}$ given by

$$t(x) = a - 2a|x - 1/2|,$$

where $a > 1$. The map has an invariant repeller Λ which is given by the iterated function system

$$T_1 x = \frac{1}{2a}\,x, \qquad T_2 x = 1 - \frac{1}{2a}\,x.$$

The dynamical system (Λ, t) is conjugated to a one-sided shift on two symbols via the coding homomorphism induced by the iterated function system:

$$\pi(\mathfrak{s}) = \lim_{n\to\infty} T_{s_1} \circ T_{s_2} \circ \ldots \circ T_{s_n}([0,1]).$$

By Theorem 4.1 the Li-Yorke pairs in Λ have full Hausdorff dimension for t.

Now consider the skinny Backers transformation, see [9], $b\colon\ [0,1]^2 \to [0,1]^2$ given by

$$b(x,y) = \begin{cases} (\beta_1 x, 2y) & \text{if } y \leqslant 1/2, \\ (1-\beta_2+\beta_2 x, 1-2y) & \text{if } y > 1/2 \end{cases}$$

for $\beta_1, \beta_2 \in (0,1)$ with $\beta_1 + \beta_2 < 1$. The map has an attractor given by $\Lambda \times [0,1]$, where Λ is given by the iterated function system

$$T_1 x = \beta_1 x, \quad T_2 x = 1 - \beta_2 + \beta_2 x.$$

The system $(\Lambda \times [0,1], b)$ is homomorphic conjugated to a two-sided full shift on two symbols via $\pi = (\pi_1, \pi_2)$ where π_1 is given by the iterated function system and π_2 is just the map coming from dyadic expansion. The assumptions of Theorem 4.2 are fulfilled and we again have Li-Yorke pairs of full Hausdorff dimension.

Next consider Smale's horseshoe $h\colon\ [0,1]^2 \to \mathbb{R}^2$, see [12], fulfilling

$$h(x,y) = \begin{cases} (\beta x, \tau y) & \text{if } y \leqslant 1/\tau, \\ (-\beta x + 1, -\tau x + \tau) & \text{if } y > 1 - 1/\tau \end{cases}$$

on horizontal strips. Here we assume $\beta \in (0, 1/2)$ and $\tau > 2$. The map may be extended to a diffeomorphism of $\mathbb{R}^2$ using stretching and folding of the middle strip $1/\tau > y > 1 - 1/\tau$. The hyperbolic invariant set

$$\Lambda = \bigcap_{n=-\infty}^{\infty} b([0,1]^2)$$

is given by $\Lambda = \Lambda_1 \times \Lambda_2$ where Λ_1 is given by the iterated function system

$$T_1 x = \beta x, \qquad T_2 x = -\beta x + 1$$

and Λ_2 is given by the iterated function system

$$G_1 y = \frac{1}{\tau} y, \qquad G_2 y = -\frac{1}{\tau} y + 1.$$

To this iterated function system there corresponds a homomorphic coding $\pi = (\pi_1, \pi_2)$ of $(\Lambda_1 \times \Lambda_2, b)$. Again the assumptions of Theorem 4.2 are fulfilled and we have Li-Yorke pairs of full Hausdorff dimension.

Our last example is a solenoid like system, see [10], given by $s\colon [0,1]^3 \to [-1,1]^3$ given by $b\colon [0,1]^2 \to [0,1]^2$ given by

$$s(x,y,z) = \begin{cases} (\beta_1 x, \beta_1 y, 2z) & \text{if } z \leqslant 1/2, \\ (1-\beta_2+\beta_2 x, 1-\beta_2+\beta_2 y, 1-2z) & \text{if } z > 1/2 \end{cases}$$

for $\beta_1, \beta_2 \in (0,1)$ with $\beta_1 + \beta_2 < 1$. By exactly the same argument we used in the case of the skinny Bakers transformation we see that the Li-Yorke pairs have full Hausdorff dimension on the attractor Λ for the map s.

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