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Ádám Besenyei

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ON SOME NONLOCAL SYSTEMS CONTAINING A PARABOLIC PDE AND A FIRST ORDER ODE

Ádám Besenyei, Budapest

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Abstract. Two models of reaction-diffusion are presented: a non-Fickian diffusion model described by a system of a parabolic PDE and a first order ODE, further, porosity-mineralogy changes in porous medium which is modelled by a system consisting of an ODE, a parabolic and an elliptic equation. Existence of weak solutions is shown by the Schauder fixed point theorem combined with the theory of monotone type operators.

Keywords: Schauder fixed point theorem, system of parabolic and elliptic equations, monotone operator, reaction-diffusion

MSC 2010: 35K60, 35J60

1. Introduction

The present paper is concerned with two systems of differential equations containing a parabolic equation and a first order ordinary differential equation. Such a system is motivated by some models of reaction-diffusion: non-Fickian diffusion and flow in porous medium with variable porosity. The former consisting of two different types of differential equations mentioned above occurs, e.g., in diffusion processes in polymer media, see [8], [9], [12], which has applications, e.g., to adhesives and food packaging etc. The latter, which was set up by Logan et al. in [11], contains also an elliptic equation thus being made of three different types of equations. A similar model was considered in [7] by the method of Rothe. Both the above systems were studied from a numerical point of view, see [11], [12]. In [2], [3] similar degenerate systems of parabolic differential equations were considered without functional dependence.

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In the sequel, we give a brief summary of our work [4] regarding existence of solutions to systems consisting of different types of differential equations. We first present the two models mentioned before. We shall consider a general system of three different types of functional differential equations. Existence of weak solutions is shown by the Schauder fixed point theorem (see, e.g., [15]) combined with techniques of the theory of monotone type operators (see, e.g., [10]). At the end of the paper our theorem will be illustrated by some examples. We note that the system with two equations is a special case of the system with three equations, so the results also apply to the former system. A different approach, not involving fixed point theory, to the system with an ODE and a parabolic PDE can be found in [14].

2. Two models of reaction-diffusion

Non-Fickian diffusion in polymers. A polymer is a large molecule composed of repeating structural units connected by chemical bonds. The diffusion of certain polymers cannot be modelled by classical Fickian diffusion, it exhibits non-Fickian (or anomalous) behaviour. Due to the viscoelastic nature of the polymer, one must include also nonlocal effects. In [8] the following model was proposed to describe such diffusion:

$$D_t u = \nabla \cdot D \nabla u + \nabla \cdot K \nabla \sigma \quad \text{in } (0,T) \times \Omega, \tag{2.1}$$

with concentration u (and constants D, K) where the viscoelastic stress σ is given by the relaxation equation

$$D_t \sigma + \gamma(u)\sigma = \mu u, \tag{2.2}$$

with a given function γ and a constant μ. Notice that for fixed σ, (2.1) is a parabolic equation in u, and for fixed u, (2.2) is a first order ordinary differential equation in σ. We also note that from (2.2) it is easy to express σ in terms of u, namely

$$\sigma(t,x) = \int_0^t \exp\Big(- \int_s^t \gamma(u(\tau,x))\,\mathrm{d}\tau \Big) \mu u \,\mathrm{d}s.$$

By substituting the above expression in (2.1), it becomes a nonlocal parabolic equation. For a detailed description of the model we refer to [8], [9], and for a numerical consideration of the above system see, e.g., [12].

Porosity-mineralogy changes. A system consisting of three different types of differential equations may occur, e.g., as a model for flow of a fluid carrying a chemical solute in porous medium, i.e., a medium with lots of tiny holes, for example, limestone. The flow of a fluid through the medium is influenced by the large surface of the solid matrix and the closeness of the holes. If the fluid carries dissolved chemical species, chemical reactions can occur that can change the porosity. In [11] this process was modelled by J. Logan, M. R. Petersen, T. S. Shores by the following system of equations in one dimension:

$$\omega D_t u = D_x(\alpha|v|D_x u) + K(\omega)D_x p \cdot D_x u - kug(\omega), \tag{2.3}$$

$$D_t \omega = bug(\omega), \tag{2.4}$$

$$D_x(K(\omega)D_x p) = bug(\omega), \tag{2.5}$$

$$v = -K(\omega)D_x p \tag{2.6}$$

in $\mathbb{R}^+ \times (0,1)$ with initial conditions $u(0,x) = u_0(x)$, $\omega(0,x) = \omega_0(x)$ for $x \in (0,1)$, and boundary conditions $u(t,0) = u_1(t)$, $D_x u(t,1) = 0$, $p(t,0) = 1$, $p(t,1) = 0$ for $t > 0$, where ω is the porosity (i.e. the proportion of the holes), u is the concentration of the dissolved chemical solute carried by the fluid, p is the pressure, v is the velocity; further, α, k, b are given (positive) constants, K and g are given real functions. Clearly, equation (2.6) might be eliminated by substituting it into (2.3). Further, for fixed u, (2.4) is a first order ordinary differential equation with respect to ω; for fixed ω and p, (2.3) is a parabolic equation with respect to u; and for fixed ω and u, (2.5) is an elliptic equation with respect to p. So the above system consists of three different types of differential equations: a first order ODE, a parabolic and an elliptic PDE. We note that in [11] functions K, g were assumed to have the form

$$K(\omega) = \min\Big(\frac{\omega^3}{(1-\omega)^2}, 0.8216\Big), \quad g(\omega) = (\omega_f - \omega)\big(\omega(1-\omega)^{1/2} + (1-\omega)\omega^{1/2}\big), \tag{2.7}$$

where ω_f is the "final" porosity. From (2.4) we see that ω tends to ω_f as $t \to \infty$.

3. Formulation of the problem

Notation. Throughout the paper $\Omega \subset \mathbb{R}^n$ is a bounded domain with the cone property (see [1]), further, $0 < T < \infty$, $2 \leqslant p_1, p_2 < \infty$ are real numbers. We briefly write $Q_T := (0,T) \times \Omega$. Let V_i be a closed linear subspace of the usual Sobolev space $W^{1,p_i}(\Omega)$ $(i = 1,2)$ which contains $W_0^{1,p_i}(\Omega)$ (the closure of $C_0^\infty(\Omega)$). We denote by $X_i := L^{p_i}(0,T;V_i)$ the space of measurable functions $v\colon (0,T) \to V_i$ for which $\|v\|_{V_i}^{p_i}$ is integrable and the norm of X_i is given by

$$\|v\|_{X_i} := \Big(\int_0^T \|v(t)\|_{V_i}^{p_i}\,\mathrm{d}t\Big)^{1/p_i}.$$

It is well-known that $X_i^* = L^{q_i}(0,T;V_i^*)$ where $1/p_i + 1/q_i = 1$. The pairing between V_i^* and V_i, further, between X_i^* and X_i is denoted by $\langle \cdot, \cdot \rangle$ and $[\cdot, \cdot]$, respectively. Finally, $C(M,N)$ is the set of continuous functions $M \to N$.

A system with three different types of differential equations. We shall study a generalization of the above models where the equations may contain nonlocal (or functional) dependence on the unknowns. Functional dependence is motivated by the nonlocal nature of polymer diffusion demonstrated above. In addition, many naturally occuring diffusion processes (especially for population) show that it is reasonable to include nonlocal terms in the mathematical models. For example, in some cases the diffusion coefficient may depend on the unknowns in a nonlocal way, e.g., on the integral of certain quantities. In [5], [6] the following nonlocal quasilinear equation was studied:

$$D_t u - \sum_{i,j=1}^{n} D_j(a_{ij}(l(u))D_i u) + a_0(l(u))u = f \quad \text{in } (0,\infty) \times \Omega,$$

where a_{ij} are real functions and the functional l is given by

$$l(u(t,x)) = \int_\Omega g(x)u(t,x)\,\mathrm{d}x. \tag{3.1}$$

We mention also the work [13] where one can find a detailed study of functional parabolic equations by means of monotone type operators. These models show that it is plausible to assume that the processes of Section 2 might be more complicated, involving functional dependence on its unknowns. So let us consider the following system consisting of three different types of nonlocal equations:

$$\begin{aligned} D_t u(t,x) - \sum_{i=1}^{n} D_i[a_i(t,x,\omega(t,x),u(t,x),Du(t,x),\mathsf{p}(t,x),D\mathsf{p}(t,x);\omega,u,\mathsf{p})] \\ + a_0(t,x,\omega(t,x),u(t,x),Du(t,x),\mathsf{p}(t,x),D\mathsf{p}(t,x);\omega,u,\mathsf{p}) \\ = g(t,x), \qquad u(0,x) = 0, \end{aligned} \tag{3.2}$$

$$D_t\omega(t,x) = f(t,x,\omega(t,x),u(t,x);u), \qquad \omega(0,x) = \omega_0(x), \tag{3.3}$$

$$\begin{aligned} \sum_{i=1}^{n} D_i[b_i(t,x,\omega(t,x),u(t,x),\mathsf{p}(t,x),D\mathsf{p}(t,x);\omega,u,\mathsf{p})] \\ + b_0(t,x,\omega(t,x),u(t,x),\mathsf{p}(t,x),D\mathsf{p}(t,x);\omega,u,\mathsf{p}) = h(t,x) \end{aligned} \tag{3.4}$$

with some boundary conditions (which may be assumed to be homogeneous) for $(t,x) \in (0,T) \times \Omega$ where the terms after ";" represent the nonlocal variables (variable p is written in sans-serif style in order to distinguish it from exponents p_1, p_2).

Assumptions. We shall seek weak solutions u and p in X_1, X_2, respectively, further, according to its original physical meaning, the space for ω will be $L^\infty(Q_T)$. In what follows, ξ, (ζ_0, ζ), (η_0, η) refer to the variables ω, (u, Du), $(\mathsf{p}, D\mathsf{p})$, respectively, while w, v_1, v_2 refer to the nonlocal variables (the subindices referring to the corresponding spaces X_1, X_2). Now we present the assumptions for the functions a_i.

(A1) Functions $a_i\colon Q_T\times\mathbb{R}\times\mathbb{R}^{n+1}\times\mathbb{R}^{n+1}\times L^2(Q_T)\times L^{p_1}(Q_T)\times X_2 \to \mathbb{R}$ $(i = 0,\dots,n)$ are of Carathéodory type for fixed $(w, v_1, v_2) \in L^2(Q_T) \times L^{p_1}(Q_T) \times X_2$, i.e., they are measurable in $(t, x) \in Q_T$ and continuous in $(\xi, \zeta_0, \zeta, \eta_0, \eta) \in \mathbb{R} \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$.

(A2) There exist bounded operators $\delta_1\colon L^2(Q_T) \times L^{p_1}(Q_T) \times X_2 \to \mathbb{R}^+$ and $k_1\colon L^2(Q_T) \times L^{p_1}(Q_T) \times X_2 \to L^{q_1}(Q_T)$ such that k_1 is continuous and

$$
\begin{aligned}
&|a_i(t, x, \xi, \zeta_0, \zeta, \eta_0, \eta; w, v_1, v_2)| \\
&\quad \leqslant \delta_1(w, v_1, v_2)(|\zeta_0|^{p_1-1} + |\zeta|^{p_1-1} + |\eta_0|^{p_2/q_1} + |\eta|^{p_2/q_1} + [k_1(w, v_1, v_2)](t, x))
\end{aligned}
$$

for a.e. $(t, x) \in Q_T$ and all $(\xi, \zeta_0, \zeta, \eta_0, \eta) \in \mathbb{R} \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$, $(w, v_1, v_2) \in L^2(Q_T) \times L^{p_1}(Q_T) \times X_2$ $(i = 0,\dots,n)$.

(A3) There exists a constant $C > 0$ such that for a.e. $(t, x) \in Q_T$ and all $(\xi, \zeta_0, \zeta, \eta_0, \eta)$, $(\xi, \zeta_0, \tilde{\zeta}, \eta_0, \eta) \in \mathbb{R} \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$, $(w, v_1, v_2) \in L^2(Q_T) \times L^{p_1}(Q_T) \times X_2$,

$$
\begin{aligned}
\sum_{i=0}^{n}\bigl(a_i(t, x, \xi, \zeta_0, \zeta, \eta_0, \eta; w, v_1, v_2) - a_i(t, x, \xi, \tilde{\zeta}_0, \tilde{\zeta}, \eta_0, \eta; w, v_1, v_2)\bigr)(\zeta_i - \tilde{\zeta}_i)& \\
\geqslant C\cdot\bigl(|\zeta_0 - \tilde{\zeta}_0|^{p_1} + |\zeta - \tilde{\zeta}|^{p_1}\bigr).&
\end{aligned}
$$

(A4) There exist a constant $c_2 > 0$ and a bounded operator $k_2\colon L^{p_1}(Q_T) \to L^1(Q_T)$ such that

$$
\sum_{i=0}^{n} a_i(t, x, \xi, \zeta_0, \zeta, \eta_0, \eta; w, v_1, v_2)\zeta_i \geqslant c_2\,(|\zeta_0|^{p_1} + |\zeta|^{p_1}) - [k_2(v_1)](t, x)
$$

for a.e. $(t, x) \in Q_T$ and all $(\xi, \zeta_0, \zeta, \eta_0, \eta) \in \mathbb{R} \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$, $(w, v_1, v_2) \in L^2(Q_T) \times L^{p_1}(Q_T) \times X_2$. Further,

$$
\lim_{r\to+\infty}\ \sup_{\|v_1\|_{L^{p_1}(Q_T)}\leqslant r} \frac{\|k_2(v_1)\|_{L^1(Q_T)}}{r^{p_1}} = 0.
$$

(A5) If $\omega_k \to \omega$ in $L^2(Q_T)$, $u_k \to u$ in $L^{p_1}(Q_T)$, and $\mathsf{p}_k \to \mathsf{p}$ in X_2 then for all $(\zeta_0, \zeta) \in \mathbb{R}^{n+1}$ and a.e. $(t, x) \in Q_T$ we have

$$
a_i(t, x, \omega_k, \zeta_0, \zeta, \mathsf{p}_k, D\mathsf{p}_k; \omega_k, u_k, \mathsf{p}_k) \to a_i(t, x, \omega, \zeta_0, \zeta, \mathsf{p}, D\mathsf{p}; \omega, u, \mathsf{p})
$$

for a suitable subsequence.

R e m a r k 3.1. Notice that (A2) is the classical growth condition in the variable (ζ_0, ζ) if we fix the other variables. Moreover, (A3) represents uniform monotonicity in (ζ_0, ζ) and (A4) is a generalization of the classical coercivity condition. Finally, (A5) assumes a kind of continuity in the functional variables, which will be crucial for applying continuity and compactness arguments.

Now we sketch the conditions imposed on the functions b_i and f; in the case of analogous assumptions we just refer to those for a_i (for details, see [4]).

- Functions $b_i\colon Q_T \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n+1} \times L^\infty(Q_T) \times X_1 \to \mathbb{R}$ are of Carathéodory type and admit growth, uniform monotonicity, coercivity and some continuity properties, analogous to (A1)–(A5).
- The function $f\colon Q_T \times \mathbb{R}^2 \times L^{p_1}(Q_T) \to \mathbb{R}$ is of Carathéodory type and has some continuity properties in the nonlocal variable, analogous to (A5).
- In addition, f is locally Lipschitz continuous in the variable ξ for fixed other variables with continuously depending Lipschitz constant, namely

$$|f(t, x, \xi, \zeta_0; v_1) - f(t, x, \tilde{\xi}, \zeta_0; v_1)| \leqslant \mathcal{K}(v_1) K(\zeta_0) L_I \cdot |\xi - \tilde{\xi}|$$

  for $\xi, \tilde{\xi} \in I \subset \mathbb{R}$ where L_I is the local Lipschitz constant corresponding to the bounded interval I. Further, $\mathcal{K}\colon L^{p_1}(Q_T) \to \mathbb{R}^+$ is bounded and $K \in C(\mathbb{R}, \mathbb{R}^+)$ satisfies $|K(\zeta_0)| \leqslant d(|\zeta_0|^{\varrho_1} + 1)$ for some $0 \leqslant \varrho_1 < p_1$ and $d > 0$.
- Finally, f satisfies the following "sign" condition for some $\omega^* \in L^\infty(\Omega)$:

$$(\xi - \omega^*(x)) \cdot f(t, x, \xi, \zeta_0; v_1) \leqslant 0.$$

R e m a r k 3.2. The Lipschitz continuity will guarantee the existence of a unique weak solution ω to the ODE (3.3) for a fixed u. The sign condition (which ensures the existence of a stable equilibrium point of f) is established according to (2.7). It will imply the boundedness of ω. We note that this condition is not necessary, one may seek solutions $\omega \in L^2(Q_T)$, see [14].

Weak formulation. If the above assumptions are satisfied, we may define operators $A\colon L^\infty(Q_T) \times X_1 \times X_2 \to X_1^*$, $B\colon L^\infty(Q_T) \times X_1 \times X_2 \to X_2^*$ for $v_i \in X_i$ $(i = 1, 2)$:

$$\begin{aligned}
&[A(\omega, u, \mathsf{p}), v_1] \\
&:= \int_{Q_T} \sum_{i=1}^{n} a_i(t, x, \omega(t, x), u(t, x), Du(t, x), \mathsf{p}(t, x), D\mathsf{p}(t, x); \omega, u, \mathsf{p}) D_i v_1(t, x) \,\mathrm{d}t \,\mathrm{d}x \\
&\quad + \int_{Q_T} a_0(t, x, \omega(t, x), u(t, x), Du(t, x), \mathsf{p}(t, x), D\mathsf{p}(t, x); \omega, u, \mathsf{p}) v_1(t, x) \,\mathrm{d}t \,\mathrm{d}x,
\end{aligned}$$

$$
\begin{aligned}
&[B(\omega,u,\mathsf{p}),v_2]\\
&:=\int_{Q_T}\sum_{i=1}^{n} b_i(t,x,\omega(t,x),u(t,x),\mathsf{p}(t,x),D\mathsf{p}(t,x);\omega,u)D_iv_2(t,x)\,\mathrm{d}t\,\mathrm{d}x\\
&\quad+\int_{Q_T} b_0(t,x,\omega(t,x),u(t,x),\mathsf{p}(t,x),D\mathsf{p}(t,x);\omega,u)v_2(t,x)\,\mathrm{d}t\,\mathrm{d}x.
\end{aligned}
$$

In addition, let us introduce the operator of differentiation $L\colon D(L)\to X_1^*$ by

$$
D(L)=\{u\in X_1\colon\ D_tu\in X_1^*,\ u(0)=0\},\qquad Lu=D_tu.
$$

Finally, supposing $g\in L^{q_1}(Q_T)$ and $h\in L^{q_2}(Q_T)$ we define $G\in X_1^*, H\in X_2^*$ by

$$
[G,v_1]:=\int_{Q_T} g(t,x)v_1(t,x)\,\mathrm{d}t\,\mathrm{d}x,\quad [H,v_2]:=\int_{Q_T} h(t,x)v_2(t,x)\,\mathrm{d}t\,\mathrm{d}x.
$$

(In fact one may consider arbitrary $G\in X_1^*, H\in X_2^*$ not necessarily having the above special form.) Then the weak form of the system (3.2)–(3.4) is defined as

$$
Lu+A(\omega,u,\mathsf{p})=G, \tag{3.5}
$$

$$
\omega(t,x)=\omega_0(x)+\int_0^t f(s,x,\omega(s,x),u(s,x);u)\,\mathrm{d}s\quad\text{a.e. in } Q_T, \tag{3.6}
$$

$$
B(\omega,u,\mathsf{p})=H. \tag{3.7}
$$

4. Existence of solutions

The main result of [4] is

Theorem 4.1. *Suppose that conditions* (A1)–(A5), (B1)–(B5), (F1)–(F3) *are fulfilled. Then for every $\omega_0\in L^\infty(\Omega)$, $G\in X_1^*$, $H\in X_2^*$ there exist solutions $\omega\in L^\infty(Q_T)$, $u\in D(L)$, $\mathsf{p}\in L^{p_2}(0,T;V_2)$ of the problem* (3.5)–(3.7).

Idea of the proof. We define an operator $\tilde{A}\colon X_1\times L^2(Q_T)\times L^{p_1}(Q_T)\times X_2\to X_1^*$ by

$$
\begin{aligned}
&[\tilde{A}(\tilde{u},\omega,u,\mathsf{p}),v_1]\\
&:=\int_{Q_T}\sum_{i=1}^{n} a_i(t,x,\omega(t,x),\tilde{u}(t,x),D\tilde{u}(t,x),\mathsf{p}(t,x),D\mathsf{p}(t,x);\omega,u,\mathsf{p})D_iv_1(t,x)\,\mathrm{d}t\,\mathrm{d}x\\
&\quad+\int_{Q_T} a_0(t,x,\omega(t,x),\tilde{u}(t,x),D\tilde{u}(t,x),\mathsf{p}(t,x),D\mathsf{p}(t,x);\omega,u,\mathsf{p})v_1(t,x)\,\mathrm{d}t\,\mathrm{d}x
\end{aligned}
$$

where $v_1 \in X_1$. Now for fixed $\omega \in L^2(Q_T)$, $u \in L^{p_1}(Q_T)$, $\mathsf{p} \in X_2$ we consider the (modified) system

$$L\tilde{u} + \tilde{A}(\tilde{u}, \omega, u, \mathsf{p}) = G,$$

$$\tilde{\omega}(t,x) = \omega_0(x) + \int_0^t f(s, x, \tilde{\omega}(s,x), \tilde{u}(s,x); \tilde{u})\, \mathrm{d}s \ \text{ a.e. in } Q_T,$$

$$B(\tilde{\omega}, \tilde{u}, \tilde{\mathsf{p}}) = H.$$

Conditions (A1)–(A4) imply that for a fixed (ω, u, p), the operator $\tilde{A}(\cdot, \omega, u, \mathsf{p})\colon X_1 \to X_1^*$ is bounded, demicontinuous, uniformly monotone and coercive, thus by the classical theory of monotone operators (see [13]) there exists a unique solution $\tilde{u} \in D(L)$ of equation (4.1). Now substituting this $\tilde{u}$ into (4.2), the Lipschitz continuity of f yields a unique absolutely continuous solution ω of that equation, which is in fact in $L^\infty(Q_T)$ by the sign condition. Finally, the assumptions made on b_i imply the boundedness, demicontinuity, uniform monotonicity and coercivity of the operator $B(\tilde{\omega}, \tilde{u}, \cdot)$, therefore one obtains a unique solution $\tilde{\mathsf{p}} \in X_2$ of the equation (4.3) for the given $\tilde{\omega}$ and $\tilde{u}$. So for every fixed $(\omega, u, \mathsf{p}) \in L^2(Q_T) \times L^{p_1}(Q_T) \times X_2$, there exists a unique solution $(\tilde{\omega}, \tilde{u}, \tilde{\mathsf{p}}) \in L^\infty(Q_T) \times D(L) \times X_2$ of the system (4.1)–(4.3).

Now we may uniquely define an operator $\Phi\colon\ L^2(Q_T) \times L^{p_1}(Q_T) \times X_2 \to L^2(Q_T) \times L^{p_1}(Q_T) \times X_2$ by $\Phi(\omega, u, \mathsf{p}) = (\tilde{\omega}, \tilde{u}, \tilde{\mathsf{p}})$. It is not so difficult to show that operator Φ is continuous, compact and there is a (bounded, closed, convex) ball $B(0,r)$ with radius r large enough such that $\Phi(B(0,r)) \subset B(0,r)$. Then by the Schauder fixed point theorem, Φ has a fixed point (in $B(0,r)$), i.e. $(\tilde{\omega}, \tilde{u}, \tilde{\mathsf{p}}) = (\omega, u, \mathsf{p})$. A fixed point of Φ is a solution to the system (3.5)–(3.7) and also $\omega \in L^\infty(Q_T)$, $u \in D(L)$, $\mathsf{p} \in X_2$ hold since Φ in fact maps into $L^\infty(Q_T) \times D(L) \times X_2$.

Continuity and compactness of Φ follows from the uniform monotonicity and continuity assumptions made on a_i, b_i and f. These conditions imply continuous dependence of solutions of the individual equations (3.5), (3.6), (3.7). By applying the coercivity assumptions made on a_i, b_i and the sign condition for f one may argue by contradiction to show that $\Phi(B(0,r)) \subset B(0,r)$ for some r, see [4]. $\square$

R e m a r k 4.1. If the functions a_i do not depend on p, one may formulate assumptions on a_i and f which imply (by Theorem 4.1) existence of solutions to systems consisting of a parabolic PDE and a first order ODE.

E x a m p l e. We now show a simple example which satisfies the assumptions of Section 3. Consider the following system containing a generalization of the p-Laplacian:

$$D_t u - \operatorname{div}(\varphi(\mathsf{p})|Du|^{p_1-2}Du) + \varphi(\mathsf{p})u|u|^{p_1-2} = g(t,x),$$

$$D_t\omega + u(\omega - \omega^*)\psi(u) = 0,$$

$$-\operatorname{div}(\pi(\omega)|D\mathsf{p}|^{p_2-2}D\mathsf{p}) + \pi(\omega)\mathsf{p}|\mathsf{p}|^{p_2-2} = h(t,x),$$

where the operator $\varphi\colon L^2(Q_T) \to L^\infty(Q_T)$ may have the form

$$[\varphi(u)](t,x) = \Phi\biggl(\int_H |u|^\beta\biggr), \quad \text{or} \quad [\varphi(u)](t,x) = \Phi\biggl(\int_H |Du|^\beta\biggr) \tag{4.1}$$

with $H = (0,t)$ or $H = \Omega$ or $H = Q_t$. Further, $1 \leqslant \beta \leqslant 2$ and $\Phi \in C(\mathbb{R}, \mathbb{R}^+) \cap L^\infty(\mathbb{R})$. One may have similar examples for π and ψ by suitably modifying the exponents, see [4]. Note that the functionals (4.4) match with the functional (3.1) considered before.

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Author's address: *Ádám Besenyei*, Eötvös Loránd University, H-1117 Budapest, Pázmány P. s. 1/C, Hungary, e-mail: badam@cs.elte.hu.