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POSITIVE FIXED POINT THEOREMS ARISING FROM SEEKING STEADY STATES OF NEURAL NETWORKS

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Abstract. Biological systems are able to switch their neural systems into inhibitory states and it is therefore important to build mathematical models that can explain such phenomena. If we interpret such inhibitory modes as 'positive' or 'negative' steady states of neural networks, then we will need to find the corresponding fixed points. This paper shows positive fixed point theorems for a particular class of cellular neural networks whose neuron units are placed at the vertices of a regular polygon. The derivation is based on elementary analysis. However, it is hoped that our easy fixed point theorems have potential applications in exploring stationary states of similar biological network models.

Keywords: positive fixed point, neural network, periodic solution, difference equation, discrete boundary condition, critical point theory

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1. Introduction

In [1], several fixed point theorems for the problem

$$u = f(u), \ u \in \mathbb{R}^{\omega}, \tag{1}$$

arising from seeking steady states of a special class of cellular neural networks that admit potentials are established. These results deal with the existence of nontrivial fixed points. Since steady states of cellular neural networks may represent memory data, transient neural activities, etc., they are important in understanding the dynamics of such networks.

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There are now numerous studies related to nonlinear systems of the form (1) but when such a system admits a potential, a useful approach is by means of the critical point theory. For general information on critical point theory, we refer to [2] and [5]. For studies that approach (1) by the critical point theory, we refer to [1], [6]–[10]). For studies that approach (1) by other methods, we refer to [11]–[13] and the book by Cheng [13].

In applications, it is also of interest to seek fixed points with specific properties. In this paper, we continue our investigations in [1] and seek fixed points that are 'positive' and 'negative'.

For the sake of convenience, we quickly recall the neural networks mentioned in [1]. We consider ω neuron units $x_1, \ldots, x_\omega$ that are placed at the vertices of a regular ω-polygon. Let $x_n^{(t)}$ denote the state value of the n-th neuron unit during the time period $t \in \{0, 1, 2, \ldots\}$. Assume that each neuron unit is activated in the following manner:

$$x_n^{(t+1)} = f_n(x_1^{(t)}, \ldots, x_\omega^{(t)}), \quad n \in \{1, \ldots, \omega\},$$

where $f = (f_1, \ldots, f_\omega)^\dagger \in C(\mathbb{R}^\omega, \mathbb{R}^\omega)$ admits a 'primitive' or potential function $F \in C^1(\mathbb{R}^\omega, \mathbb{R})$ satisfying

$$\frac{\partial}{\partial u_k} F(u_1, u_2, \ldots, u_\omega) = f_k(u_1, u_2, \ldots, u_\omega), \quad k = 1, 2, \ldots, \omega.$$

If we seek steady state solutions $\{(x_1^{(t)}, \ldots, x_\omega^{(t)})^\dagger\}_{t=0}^\infty$ such that $x_n^{(t)} = x_n$ for $n \in \{1, \ldots, \omega\}$ and $t \geqslant 0$, then we are led to finding solutions of the steady state system of the form (1). Two specific examples of (1) are

$$x = \frac{8}{3}(5 - \sin x)x^{\frac{5}{3}} - x^{\frac{8}{3}} \cos x, \tag{2}$$

and

$$\begin{aligned} x_1 &= -x_3 - x_2 + 4x_1^3, \\ x_2 &= -x_1 - x_3 + 4x_2^3, \\ x_3 &= -x_2 - x_1 + 4x_3^3. \end{aligned} \tag{3}$$

Steady state solutions can be interpreted as transient neural activities [3], [4]. In applications, it is of interest to seek steady state solutions that are situated in specific regions of the state space $\mathbb{R}^\omega$. Here we consider the existence of a solution u of (1) such that all its components are positive (or all are negative)[1].

[1] One reason for doing so is the fact that some biological systems are able to switch their neural systems into inhibitory transient states (such as those found in animal hibernation).

To be more precise, a column vector $u = (u_1, u_2, \ldots, u_\omega)^\dagger \in \mathbb{R}^\omega$ is said to be a solution of (1) if substitution of u into (1) renders an identity. A vector $u = (u_1, u_2, \ldots, u_n)^\dagger$ is said to be positive if $u_k > 0$ for $k \in \{1, \ldots, n\}$, negative if $u_k < 0$ for $k \in \{1, \ldots, n\}$, and zero-free if $u_k \neq 0$ for all $k \in \{1, \ldots, n\}$. For the sake of convenience, we also let

$$\Omega_\omega^+ = \left\{u = (u_1, \ldots, u_\omega)^\dagger \in \mathbb{R}^\omega \colon\ u_k \geqslant 0 \text{ for } k \in \{1, \ldots, \omega\}\right\},$$
$$\Omega_\omega^- = \left\{u = (u_1, \ldots, u_\omega)^\dagger \in \mathbb{R}^\omega \colon\ -u_k \geqslant 0 \text{ for } k \in \{1, \ldots, \omega\}\right\}.$$

As in [1], we will employ the concept of critical points for approaching our goal [2]. To this end, we let the functional $I\colon\ \mathbb{R}^\omega \to \mathbb{R}$ be defined by

$$I(u) = \frac{1}{2}\sum_{k=1}^{\omega} u_k^2 - F(u_1, u_2, \ldots, u_\omega),\ u \in \mathbb{R}^\omega. \tag{4}$$

Since

$$\frac{\partial I(u)}{\partial u_k} = u_k - f_k(u_1, u_2, \ldots, u_\omega),\ k \in \{1, \ldots, \omega\}, \tag{5}$$

we see that $I'(u) = 0$ if, and only if,

$$u_k - f_k(u_1, u_2, \ldots, u_\omega) = 0,\ k \in \{1, \ldots, \omega\}. \tag{6}$$

That is, $u \in \mathbb{R}^\omega$ is a critical point of I (i.e. $I'(u) = 0$) if, and only if, u is a solution of (1).

2. Main results

We first provide two results for the positive or negative fixed points.

Theorem 1. *Suppose that $F(0) = 0$. Suppose further that*

(G_1) *there are constants $\delta > 0$ and $\alpha \in \left(0, \frac{1}{2}\right)$ such that for any $u \in \Omega_\omega^+$ ($u \in \Omega_\omega^-$, or $u \in \Omega_\omega^+ \cup \Omega_\omega^-$) satisfying $|u| \leqslant \delta$, it follows that $F(u) \leqslant \alpha |u|^2$;*

(G_2) *there are constants $\varrho > 0$, $\gamma > 0$ and $\beta \in (\frac{1}{2}, \infty)$ such that for any $u \in \Omega_\omega^+$ ($u \in \Omega_\omega^-$, or $u \in \Omega_\omega^+ \cup \Omega_\omega^-$) satisfying $|u| \geqslant \varrho$, it follows that $F(u) \geqslant \beta |u|^2 - \gamma$;*

(G_3) *for any $u = (u_1, u_2, \ldots, u_\omega) \in \Omega_\omega^+$ ($u = (u_1, u_2, \ldots, u_\omega) \in \Omega_\omega^-$, or $u = (u_1, u_2, \ldots, u_\omega) \in \Omega_\omega^+ \cup \Omega_\omega^-$) and any $i \in \{1, 2, \ldots, \omega\}$,*

$$F(u_1, u_2, \ldots, u_\omega) \leqslant F(u_1, \ldots, u_{i-1}, 0, u_{i-1}, \ldots, u_\omega) + F(0, \ldots, 0, u_i, 0, \ldots, 0).$$

Then (1) possesses at least one positive solution (one negative solution, or one positive and one negative solution).

Theorem 2. *Suppose that $F(0)=0$. Suppose further that*

(G$_4$) *there are constants $\delta>0$ and $\alpha\in\left(\frac{1}{2},\infty\right)$ such that for any $u\in\Omega_\omega^+$ ($u\in\Omega_\omega^-$, or $u\in\Omega_\omega^+\cup\Omega_\omega^-$) satisfying $|u|\leqslant\delta$, we have $F(u)\geqslant\alpha|u|^2$;*

(G$_5$) *there are constants $\varrho>0$, $\gamma>0$ and $\beta\in(0,\frac{1}{2})$ such that for any $u\in\Omega_\omega^+$ ($u\in\Omega_\omega^-$, or $u\in\Omega_\omega^+\cup\Omega_\omega^-$) satisfying $|u|\geqslant\varrho$, we have $F(u)\leqslant\beta|u|^2+\gamma$;*

(G$_6$) *for any $u=(u_1,u_2,\dots,u_\omega)\in\Omega_\omega^+$ ($u=(u_1,u_2,\dots,u_\omega)\in\Omega_\omega^-$, or $u=(u_1,u_2,\dots,u_\omega)\in\Omega_\omega^+\cup\Omega_\omega^-$) and any $i\in\{1,2,\dots,\omega\}$,*

$$F(u_1,u_2,\dots,u_\omega)\geqslant F(u_1,\dots,u_{i-1},0,u_{i-1},\dots,u_\omega)+F(0,\dots,0,u_i,0,\dots,0).$$

Then (1) *possesses at least one positive solution (one negative, or one positive and one negative solution).*

Before proving our results, let us first show that they are not vacuous. Indeed, the system (3) is of the form (1) where $\omega=3$. Here

$$f_i(x)=-x_{i-1}-x_{i+1}+4x_i^3,\quad i=1,2,3,$$

by taking $x_0=x_3$ and $x_4=x_1$. It is easy to see that $F(x_1,x_2,x_3)=-x_1x_3-x_1x_2-x_2x_3+x_1^4+x_2^4+x_3^4$ satisfies $(\partial/\partial u_k)F(u_1,u,u_3)=f_k(u_1,u,u_3)$ for $k=1,2,3$ and $F(0,0,0)=0$. Note that for any $x=(x_1,x_2,x_3)\in\Omega_3^+$,

$$F(x_1,x_2,x_3)\leqslant -x_1x_3-x_1x_2-x_2x_3+x_1^4+x_2^4+x_3^4\leqslant x_1^4+x_2^4+x_3^4.$$

Then we have

$$\limsup_{x\in\Omega_3^+,x\to 0}\frac{F(x)}{|x|^2}\leqslant\lim_{x\to 0}\frac{x_1^4+x_2^4+x_3^4}{x_1^2+x_2^2+x_3^2}=0$$

and

$$\begin{aligned}F(x_1,x_2,x_3)&=-x_1x_3-x_1x_2-x_2x_3+x_1^4+x_2^4+x_3^4\\&\geqslant\left\{\max_{1\leqslant i\leqslant 3}x_i^2\right\}^2-3\max_{1\leqslant i\leqslant 3}x_i^2\\&\geqslant\{\tfrac{1}{3}\left(x_1^2+x_2^2+x_3^2\right)\}^2-3\left(x_1^2+x_2^2+x_3^2\right)\\&\geqslant\{\tfrac{1}{9}\left(x_1^2+x_2^2+x_3^2\right)-3\}\left(x_1^2+x_2^2+x_3^2\right)-1\\&\geqslant 6|x|^2-1,\end{aligned}$$

for $|x|\geqslant 9$. Thus the conditions (G$_1$) and (G$_2$) are satisfied. Further, it is also easy to check the condition (G$_3$).

P r o o f of Theorem 1. Let I be defined by (4). According to (G_2), if we let

$$\gamma_1 = \max\{|F(u) - \beta|u|^2 + \gamma|\colon\ |u| \leqslant \varrho, u \in \Omega_\omega^+\}$$

and $\gamma' = \gamma + \gamma_1$, then for any $u \in \Omega_\omega^+$ we have

$$F(u) \geqslant \beta|u|^2 - \gamma'. \tag{7}$$

By (4) and (7), we see that for any $u \in \Omega_\omega^+$,

$$\begin{aligned} I(u) &= \frac{1}{2}\sum_{k=1}^{\omega} u_k^2 - F(u_1, u_2, \ldots, u_\omega) \leqslant \frac{1}{2}|u|^2 - (\beta|u|^2 - \gamma') \\ &= \Bigl(\frac{1}{2} - \beta\Bigr)|u|^2 + \gamma'. \end{aligned}$$

Since $\beta > \frac{1}{2}$ by (G_2), for any $u \in \Omega_\omega^+$ we have $I(u) \leqslant \gamma'$.

Let $c_0 = \sup\limits_{u\in\Omega_\omega^+} I(u)$. We claim that $c_0 > 0$. Indeed, by the assumption (G_1), we see that there are constants $\delta > 0$ and $\alpha \in \left(0, \frac{1}{2}\right)$ such that for any $u \in \Omega_\omega^+$ satisfying $|u| \leqslant \delta$ we have

$$F(u) \leqslant \alpha|u|^2. \tag{8}$$

Then for any constant vector $u = (c, c, \ldots, c)^\dagger \in \Omega_\omega^+$ satisfying $|u| = \delta$,

$$\begin{aligned} I(u) &= \frac{1}{2}\sum_{k=1}^{\omega} u_k^2 - F(u_1, u_2, \ldots, u_\omega) \\ &\geqslant \frac{1}{2}|u|^2 - \alpha|u|^2 = \Bigl(\frac{1}{2} - \alpha\Bigr)|u|^2 = \sigma, \end{aligned}$$

where $\sigma = \left(\frac{1}{2} - \alpha\right)\delta^2 > 0$. Thus $c_0 = \sup\limits_{u\in\Omega_\omega^+} I(u) \geqslant \sigma > 0$. We know that there is a sequence $\{u^{(i)}\} \in \Omega_\omega^+$ such that $c_0 = \lim\limits_{i\to\infty} I(u^{(i)})$. It is easy to see that there exists a positive constant M such that

$$-M \leqslant I(u^{(i)}), \quad i = 1, 2, \ldots.$$

Then

$$-M \leqslant I(u^{(i)}) \leqslant \Bigl(\frac{1}{2} - \beta\Bigr)|u^{(i)}|^2 + \gamma',$$

which implies that

$$|u^{(i)}|^2 \leqslant \Bigl(\beta - \frac{1}{2}\Bigr)^{-1}(M + \gamma').$$

That is, $\{u^{(i)}\}$ is a bounded sequence in $\Omega_\omega^+ \subset \mathbb{R}^\omega$. Hence $\left\{u^{(i)}\right\}$ has a convergent subsequence $\left\{u^{(i_j)}\right\}$. Let $\left\{u^{(i_j)}\right\}$ tend to $u^{(0)} = (u_1^{(0)}, u_2^{(0)}, \ldots, u_n^{(0)})^\dagger$. Then $u^{(0)} \in \Omega_\omega^+$ and $I\left(u^{(0)}\right) = c_0$. Note that $I(0) = 0$ and $c_0 > 0$. Thus, there must exist $k_0 \in \{1, \ldots, n\}$ such that $u_{k_0}^{(0)} > 0$. Now we assert that $u^{(0)}$ is a positive critical point of the functional I. Assume that there exist $i_0 \in \{1, \ldots, n\}$ and $i_0 \neq k_0$ such that $u_{i_0}^{(0)} = 0$. Then

$$I(u^{(0)}) = \frac{1}{2}\sum_{k\neq i_0}^{\omega}(u_k^{(0)})^2 - F(u_1^{(0)}, \ldots, u_{i_0-1}^{(0)}, 0, u_{i_0+1}^{(0)}, \ldots, u_n^{(0)}). \tag{9}$$

Choose $0 < \overline{u}_{i_0} \leqslant \delta$ and let

$$\begin{aligned} I_0\left(\overline{u}_{i_0}\right) &= \frac{1}{2}\left(\overline{u}_{i_0}\right)^2 - F\left(0, \ldots, 0, \overline{u}_{i_0}, 0, \ldots, 0\right), \\ \overline{u} &= (u_1^{(0)}, \ldots, u_{i_0-1}^{(0)}, \overline{u}_{i_0}, u_{i_0-1}^{(0)}, \ldots, u_n^{(0)})^\dagger. \end{aligned} \tag{10}$$

By (8) and (10), we have

$$\begin{aligned} I_0\left(\overline{u}_{i_0}\right) &= \frac{1}{2}\left(\overline{u}_{i_0}\right)^2 - F\left(0, \ldots, 0, \overline{u}_{i_0}, 0, \ldots, 0\right) \\ &\geqslant \frac{1}{2}\left(\overline{u}_{i_0}\right)^2 - \alpha\left(\overline{u}_{i_0}\right)^2 = \left(\frac{1}{2} - \alpha\right)\left(\overline{u}_{i_0}\right)^2 > 0. \end{aligned} \tag{11}$$

Thus, from (G_3), (9), (10) and (11) we have

$$\begin{aligned} I(\overline{u}) &= \frac{1}{2}\sum_{k\neq i_0}^{\omega}(u_k^{(0)})^2 + \frac{1}{2}(\overline{u}_{i_0})^2 - F(u_1^{(0)}, \ldots, u_{i_0-1}^{(0)}, \overline{u}_{i_0}, u_{i_0+1}^{(0)}, \ldots, u_n^{(0)}) \\ &\geqslant \frac{1}{2}\sum_{k\neq i_0}^{\omega}(u_k^{(0)})^2 + \frac{1}{2}(\overline{u}_{i_0})^2 - F(u_1^{(0)}, \ldots, u_{i_0-1}^{(0)}, 0, u_{i_0+1}^{(0)}, \ldots, u_n^{(0)}) \\ &\quad - F(0, \ldots, 0, \overline{u}_{i_0}, 0, \ldots, 0) \\ &= I(u^{(0)}) + I_0(\overline{u}_{i_0}) > I(u^{(0)}) = c_0, \end{aligned}$$

which is contrary to the definition of c_0. Thus $u^{(0)}$ belongs to the interior of Ω_ω^+ and $I\left(u^{(0)}\right) = \sup\limits_{u\in\Omega_\omega^+} I(u)$. It follows that $u^{(0)}$ is a critical point of I, that is, $u^{(0)} = (u_1^{(0)}, u_2^{(0)}, \ldots, u_n^{(0)})^\dagger$ is a positive solution of (1).

In case the condition $u \in \Omega_\omega^+$ is replaced by $u \in \Omega_\omega^-$ in (G_1), (G_2) and (G_3), then to find a negative solution of (1), we let the functional $I\colon \Omega_\omega^- \to \mathbb{R}$ be defined by

$$I(v) = \frac{1}{2}\sum_{k=1}^{\omega} v_k^2 - F(v_1, v_2, \ldots, v_\omega), \; v \in \Omega_\omega^-,$$

and let $v = -u$. Then it suffices to find positive critical points of the functional

$$J(u) = \frac{1}{2}\sum_{k=1}^{\omega} u_k^2 - F(-u_1, -u_2, \ldots, -u_\omega), \quad u \in \Omega_\omega^+.$$

The rest of the reasoning is similar to that described above and may be skipped. Finally, if the condition $u \in \Omega_\omega^+$ is replaced by $u \in \Omega_\omega^- \cup \Omega_\omega^+$ in (G_1), (G_2) and (G_3), then we may infer from the facts obtained that there are two zero free solutions of (1), one is positive and the other is negative. The proof is complete. □

P r o o f of Theorem 2. According to (G_5), if we let

$$\gamma_2 = \max\{|F(u) - \beta|u|^2 - \gamma| : |u| \leqslant \varrho, u \in \Omega_\omega^+\}$$

and $\gamma'' = \gamma + \gamma_2$, then for any $u \in \Omega_\omega^+$ we have

$$F(u) \leqslant \beta|u|^2 + \gamma''. \tag{12}$$

By (4) and (12), we see that for any $u \in \Omega_\omega^+$

$$\begin{aligned} I(u) &= \frac{1}{2}\sum_{k=1}^{\omega} u_k^2 - F(u_1, u_2, \ldots, u_\omega) \\ &\geqslant \frac{1}{2}|u|^2 - (\beta|u|^2 - \gamma'') = \left(\frac{1}{2} - \beta\right)|u|^2 - \gamma''. \end{aligned}$$

Since $\beta < \frac{1}{2}$ by (G_4), for any $u \in \Omega_\omega^+$ we have $I(u) \geqslant -\gamma''$.

Let $c_1 = \inf_{u \in \Omega_\omega^+} I(u)$. We claim that $c_1 < 0$. Indeed, by the assumption (G_4), there are constants $\delta > 0$ and $\alpha \in \left(\frac{1}{2}, \infty\right)$ such that for any $u \in \Omega_\omega^+$ satisfying $|u| \leqslant \delta$, we have

$$F(u) \geqslant \alpha|u|^2. \tag{13}$$

Then for any constant vector $u = (c, c, \ldots, c)^\dagger \in \Omega_\omega^+$ satisfying $|u| = \delta$,

$$\begin{aligned} I(u) &= \frac{1}{2}\sum_{k=1}^{\omega} u_k^2 - F(u_1, u_2, \ldots, u_\omega) \\ &\leqslant \frac{1}{2}|u|^2 - \alpha|u|^2 = \left(\frac{1}{2} - \alpha\right)|u|^2 = \lambda, \end{aligned}$$

where $\lambda = \left(\frac{1}{2} - \alpha\right)\delta^2 < 0$. Thus $c_1 = \inf_{u \in \Omega_\omega^+} I(u) \leqslant \lambda < 0$. We know that there is a sequence $\left\{u^{(i)}\right\} \in \Omega_\omega^+$ such that $c_1 = \lim_{i \to \infty} I\left(u^{(i)}\right)$. Hence there exists a positive constant M such that

$$I(u^{(i)}) \leqslant M, \quad i = 1, 2, \ldots.$$

Thus

$$\Big(\frac{1}{2}-\beta\Big)|u^{(i)}|^2-\gamma''\leqslant I(u^{(i)})\leqslant M,$$

which implies that

$$|u^{(i)}|^2\leqslant\Big(\frac{1}{2}-\beta\Big)^{-1}(M+\gamma^{''}).$$

That is, $\{u^{(i)}\}$ is a bounded sequence in $\Omega_\omega^+\subset\mathbb{R}^\omega$. Hence $\{u^{(i)}\}$ has a convergent subsequence $\{u^{(i_j)}\}$. Let $\{u^{(i_j)}\}$ tend to $u^{(0)}=(u_1^{(0)},u_2^{(0)},\dots,u_n^{(0)})^\dagger$. Then $u^{(0)}\in\Omega_\omega^+$ and $I\left(u^{(0)}\right)=c_1$. Note that $I(0)=0$ and $c_1<0$. Clearly, there exists $k_0\in\{1,\dots,n\}$ such that $u_{k_0}^{(0)}>0$. Now we assert that $u^{(0)}$ is a positive critical point of the functional I . Assume that there exist $i_0\in\{1,\dots,n\}$ and $i_0\neq k_0$ such that $u_{i_0}^{(0)}=0$. Then

$$I(u^{(0)})=\frac{1}{2}\sum_{k\neq i_0}^{\omega}(u_k^{(0)})^2-F(u_1^{(0)},\dots,u_{i_0-1}^{(0)},0,u_{i_0+1}^{(0)},\dots,u_n^{(0)}). \tag{14}$$

Choose $0<\overline{u}_{i_0}\leqslant\delta$ and let

$$\begin{aligned} I_0\left(\overline{u}_{i_0}\right)&=\frac{1}{2}\left(\overline{u}_{i_0}\right)^2-F(0,\dots,0,\overline{u}_{i_0},0,\dots,0),\\ \overline{u}&=(u_1^{(0)},\dots,u_{i_0-1}^{(0)},\overline{u}_{i_0},u_{i_0-1}^{(0)},\dots,u_n^{(0)})^\dagger. \end{aligned}\tag{15}$$

By (13) and (15) we have

$$\begin{aligned} I_0\left(\overline{u}_{i_0}\right)&=\frac{1}{2}\left(\overline{u}_{i_0}\right)^2-F\left(0,\dots,0,\overline{u}_{i_0},0,\dots,0\right)\\ &\leqslant\frac{1}{2}(\overline{u}_{i_0})^2-\alpha\left(\overline{u}_{i_0}\right)^2=\Big(\frac{1}{2}-\alpha\Big)\left(\overline{u}_{i_0}\right)^2<0. \end{aligned}\tag{16}$$

Thus, from (G_6), (14), (15) and (16), we have

$$\begin{aligned} I(\overline{u})&=\frac{1}{2}\sum_{k\neq i_0}^{\omega}(u_k^{(0)})^2+\frac{1}{2}(\overline{u}_{i_0})^2-F(u_1^{(0)},\dots,u_{i_0-1}^{(0)},\overline{u}_{i_0},u_{i_0+1}^{(0)},\dots,u_n^{(0)})\\ &\leqslant\frac{1}{2}\sum_{k\neq i_0}^{\omega}(u_k^{(0)})^2+\frac{1}{2}(\overline{u}_{i_0})^2-F(u_1^{(0)},\dots,u_{i_0-1}^{(0)},0,u_{i_0+1}^{(0)},\dots,u_n^{(0)})\\ &\quad-F(0,\dots,0,\overline{u}_{i_0},0,\dots,0)\\ &=I(u^{(0)})+I_0(\overline{u}_{i_0})<I(u^{(0)})=c_1, \end{aligned}$$

which is contrary to the definition of c_1. Thus $u^{(0)}$ belongs to the interior of Ω_ω^+ and $I\left(u^{(0)}\right)=\inf\limits_{u\in\Omega_\omega^+}I\left(u\right)$. It follows that $u^{(0)}$ is a critical point of I, that is, $u^{(0)}=(u_1^{(0)},u_2^{(0)},\dots,u_n^{(0)})^\dagger$ is a positive solution of (1). The rest of the proof is similar to that of Theorem 1 and is skipped. The proof is complete. □

We remark that the conditions (G_1) and (G_2) in Theorem 1 of [1] are slightly modified and an additional condition (G_3) is added so as to guarantee the existence of positive solutions of (1). The proof of Theorem 1 in [1] relies on the mountain pass theorem, however, here we use a direct proof.

3. Examples and remarks

We remark that if F is even in the sense that

$$F(-u_1,\ldots,-u_\omega)=F(u_1,\ldots,u_\omega),\ \ (u_1,\ldots,u_\omega)^\dagger\in\Omega_\omega^+,$$

then clearly either Theorem 1 or Theorem 2 will also guarantee the existence of positive as well as negative solutions of (1). Yet such an assumption is not necessary. We will illustrate our results by several examples.

E x a m p l e 1. Consider the system

$$\begin{aligned} f_1\left(x_1,x_2,x_3,x_4\right) &= 20x_1^3-\frac{1}{3}x_1\left(x_4^2+x_2^2\right), \\ f_2(x_1,x_2,x_3,x_4) &= 24x_2^3-\frac{1}{3}x_2\left(x_1^2+x_3^2\right), \\ f_3(x_1,x_2,x_3,x_4) &= 28x_3^3-\frac{1}{3}x_3\left(x_2^2+x_4^2\right), \\ f_4(x_1,x_2,x_3,x_4) &= 20x_4^3-\frac{1}{3}x_4\left(x_3^2+x_1^2\right). \end{aligned} \tag{17}$$

Let

$$F\left(x_1,x_2,x_{3,}x_4\right)=5x_1^4+6x_2^4+7x_3^4+5x_4^4-\tfrac{1}{6}(x_1^2x_2^2+x_2^2x_3^2+x_3^2x_4^2+x_4^2x_1^2).$$

Then $F\left(0\right)=0$ and

$$\frac{\partial}{\partial x_k}F(x_1,x_2,x_{3,}x_4)=f_k(x_1,x_2,x_{3,}x_4).$$

Furthermore, it is easy to see that $\lim\limits_{x\to 0}\left|F(x)/|x|^2\right|=0$ and

$$\begin{aligned} F\left(x_1,x_2,x_{3,}x_4\right) &= 5x_1^4+6x_2^4+7x_3^4+5x_4^4-\frac{1}{6}(x_1^2x_2^2+x_2^2x_3^2+x_3^2x_4^2+x_4^2x_1^2) \\ &\geqslant 5\Big\{\max_{1\leqslant i\leqslant 4}x_i^2\Big\}^2-\frac{2}{3}\Big\{\max_{1\leqslant i\leqslant 4}x_i^2\Big\}^2 \\ &\geqslant 4\Big\{\max_{1\leqslant i\leqslant 4}x_i^2\Big\}^2 \\ &= \frac{1}{4}\Big\{4\max_{1\leqslant i\leqslant 4}x_i^2\Big\}^2\geqslant\frac{1}{4}\left(x_1^2+x_2^2+x_3^2+x_4^2\right)^2-1 \\ &\geqslant 4\left|x\right|^2-1 \end{aligned}$$

for $|x| \geqslant 4$, which implies that (G_1) and (G_2) are satisfied. It is also easy to check that (G_3) is satisfied. Thus by Theorem 1, (1) possesses at least 2 zero-free solutions, one of them is positive and the other is negative.

E x a m p l e 2. The equation

$$x = \begin{cases} x^3 & x \geqslant 0, \\ x^5 & x < 0 \end{cases} \tag{18}$$

is of the form (1). Here the function f is defined by

$$f(x) = \begin{cases} x^3 & x \geqslant 0, \\ x^5 & x < 0. \end{cases}$$

The function

$$F(x) = \begin{cases} \frac{1}{4}x^4 & x \geqslant 0, \\ \frac{1}{6}x^6 & x < 0 \end{cases}$$

is not even but satisfies $F(0) = 0$, $F'(x) = f(x)$ for any $x \in \mathbb{R}$, $F(x)/x^2 \to 0$ as $x \to 0$, and

$$F(x) \geqslant x^2 - 1, \quad |x| \geqslant 2.$$

By Theorem 1, we see that (18) possesses at least 2 zero-free solutions, one of them is positive and the other is negative. Indeed, the solutions are given precisely by $x = 0, \pm 1$. Thus Theorem 1 is 'sharp' when $\omega = 1$.

We remark that 'sharpness' holds in the sense that all the positive and negative solutions are found.

E x a m p l e 3. Consider the system

$$\begin{cases} u_1 = \begin{cases} \dfrac{1}{2\omega} u_1^{1/3}, & u_1 \geqslant 0, \\ \dfrac{5}{9\omega} u_1^{1/3}, & u_1 < 0, \end{cases} \\ u_k = \frac{1}{2} u_k^{1/3}, \quad k = 2, \dots, \omega, \end{cases} \tag{19}$$

where $\omega \geqslant 2$. Here the function f is defined by

$$f_1(u_1, u_2, \dots, u_\omega) = \begin{cases} \dfrac{1}{2\omega} u_1^{1/3}, & u_1 \geqslant 0, \\ \dfrac{5}{9\omega} u_1^{1/3}, & u_1 < 0, \end{cases}$$
$$f_k(u_1, u_2, \dots, u_\omega) = \frac{1}{2} u_k^{1/3}, \quad k = 2, \dots, \omega.$$

The function

$$F(u_1, u_{2,} \dots, u_\omega) = \begin{cases} \dfrac{3}{8\omega}(u_1^{4/3} + u_2^{4/3} + \dots + u_\omega^{4/3}), & u_1 \geqslant 0, \\ \dfrac{5}{12\omega} u_1^{4/3} + \frac{3}{8}(u_2^{4/3} + \dots + u_\omega^{4/3}), & u_1 < 0, \end{cases}$$

is not even but satisfies $F(0) = 0$, $(\partial/\partial u_k)F(u_1, u_2, \dots, u_\omega) = f_k(u_1, u_2, \dots, u_\omega)$ for $k = 1, 2, \dots, \omega$. Furthermore, it is easy to see that

$$\lim_{u \to 0} F(x) / |u|^2 = \infty,$$

which implies (G_4). Note that, for $|u| \geqslant \omega$, $\max\limits_{1 \leqslant i \leqslant \omega} |u_i| \geqslant 1$, thus we have

$$\begin{aligned} F(u_1, u_{2,} \dots, u_\omega) &\leqslant \frac{5}{12\omega}(u_1^{4/3} + u_2^{4/3} + \dots + u_\omega^{4/3}) \\ &\leqslant \frac{5}{12}\Big\{\max_{1 \leqslant i \leqslant \omega} |u_i|\Big\}^{4/3} + 1 \\ &\leqslant \frac{5}{12}\Big\{\max_{1 \leqslant i \leqslant \omega} |u_i|\Big\}^{2} + 1 \\ &\leqslant \frac{5}{12}|u|^2 + 1 \end{aligned}$$

for $|u| \geqslant \omega$, that is, (G_5) is satisfied. Further, it is also easy to check the condition (G_6). Thus by Theorem 2, (19) possesses at least 2 zero-free solutions, one of them is positive and the other is negative. Indeed, the solutions are given precisely by the form $u = (u_1, u_2, \dots, u_\omega)$, where $u_1 = 0$ or $(1/2)^{3/2}$ or $-(1/2)^{5/4}$, and $u_k = 0$ or $(1/2)^{3/2}$ or $-(1/2)^{3/2}$, $k = 2, \dots, \omega$. Hence $\big((1/2)^{3/2}, \dots, (1/2)^{3/2}\big)$ and $\big(-(1/2)^{5/4}, -(1/2)^{3/2}, \dots, -(1/2)^{3/2}\big)$ are the only positive and negative solutions. Thus Theorem 6 is sharp when $\omega \geqslant 2$.

We remark that although our fixed point theorems are motivated by considering steady states of digital neural networks, we may also use them in a number of practical problems including systems of nonlinear equations, boundary value problems involving difference equations, etc.

Example 4. Consider a second order difference equation of the form

$$\Delta^2 u_{k-1} + g_k(u_{k-1}, u_k, u_{k+1}) = 0, \quad k \in \{1, 2\}, \tag{20}$$

under the boundary conditions

$$u_0 = 0 = u_3, \tag{21}$$

where

$$\begin{aligned} g_1\left(x_0,x_1,x_2\right) &= 40x_1^3-\tfrac{2}{3}x_1x_2^2+\tfrac{1}{3}x_1^2x_2-24x_2^3, \\ g_2\left(x_1,x_2,x_3\right) &= -20x_1^3+\tfrac{1}{3}x_1x_2^2-\tfrac{2}{3}x_1^2x_2+48x_2^3. \end{aligned}$$

The above boundary value problem (20)–(21) is equivalent to

$$\begin{aligned} 2u_1-u_2 &= 40u_1^3-\tfrac{2}{3}u_1u_2^2+\tfrac{1}{3}u_1^2u_2-24u_2^3, \\ -u_1+2u_2 &= -20u_1^3+\tfrac{1}{3}u_1u_2^2-\tfrac{2}{3}u_1^2u_2+48u_2^3, \end{aligned}$$

and since

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}^{-1} = \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix},$$

it is also equivalent to

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 20u_1^3-\tfrac{1}{3}u_1u_2^2 \\ 24u_2^3-\tfrac{1}{3}u_1^2u_2 \end{pmatrix}.$$

Let

(22) $$F\left(u_1,u_2\right)=5u_1^4+6u_2^4-\tfrac{1}{6}u_1^2u_2^2.$$

Then $F\left(0\right)=0$ and

$$\begin{aligned} \frac{\partial}{\partial u_1}F(u_1,u_2) &= 20u_1^3-\tfrac{1}{3}u_1u_2^2, \\ \frac{\partial}{\partial u_2}F(u_1,u_2) &= 24u_2^3-\tfrac{1}{3}u_1^2u_2. \end{aligned}$$

Furthermore, it is easy to see that

$$\lim_{x\to 0} F\left(x\right)/\left|x\right|^2=0,$$

which implies (G_1). Since

$$\begin{aligned} F\left(u_1,u_2\right) &= 5u_1^4+6u_2^4-\frac{1}{6}u_1^2u_2^2 \\ &\geqslant 5u_1^4+5u_2^4-\frac{1}{6}u_1^2u_2^2 \\ &\geqslant 5\left\{\max_{1\leqslant i\leqslant 2} u_i^2\right\}^2-\frac{1}{6}\left\{\max_{1\leqslant i\leqslant 2} u_i^2\right\}^2 \\ &\geqslant \frac{5}{4}\left\{2\max_{1\leqslant i\leqslant 2} u_i^2\right\}^2-1 \\ &\geqslant \frac{5}{4}\left|u\right|^4-1 \\ &\geqslant \frac{5}{4}\left|u\right|^2-1 \end{aligned}$$

for $|u| \geqslant 1$, hence (G_2) is satisfied. Further, it is also easy to check the condition (G_3). Thus by Theorem 3, (20)–(21) possesses at least 2 zero-free solutions, one of them is positive and the other is negative.

Our results are also good for looking for zero-free solutions of (1) in other quadrants of $\mathbb{R}^n$. The idea is quite simple. For instance, if we need a solution of (1) in

$$\Omega = \left\{(x_1, \ldots, x_n)^\dagger \colon\ x_1 > 0, x_2 < 0, \ldots, x_n < 0\right\},$$

we may consider the system

$$\begin{aligned} u_1 &= f_1(u_1, -u_2, \ldots, -u_n), \\ u_2 &= -f_2(u_1, -u_2, \ldots, -u_n), \\ &\vdots \\ u_n &= -f_n(u_1, -u_2, \ldots, -u_n). \end{aligned}$$

If $(u_1, \ldots, u_n)^\dagger$ is a positive solution, then $(u_1, -u_2, -u_3, \ldots, -u_n)^\dagger$ is a solution of (1) in Ω.

As our final remark, in Example 4 we have encountered a fixed problem of the form

$$Gu = f(u),$$

where G is an invertible matrix so that $u = G^{-1}f(u)$, and our fixed point theorems are applicable. In other situations, G may not be invertible and therefore the corresponding fixed point theorems may be of interest.

References

[1] *G. Q. Wang, S. S. Cheng*: Fixed point theorems arising from seeking steady states of neural networks. Appl. Math. Modelling *33* (2009), 499–506.

[2] *J. Mawhin, M. Willem*: Critical Point Theory and Hamiltonian Systems. Springer, New York, 1989.

[3] *J. S. Roberts*: Artificial Neural Networks. McGraw-Hill, Singapore, 1997.

[4] *S. Haykin*: Neural Networks: a Comprehensive Foundation. Englewood Cliffs, Macmillan Company, NJ, 1994.

[5] *P. H. Rabinowitz*: Minimax Methods in Critical Point Theory with Applications to Differential Equations. CBMS, AMS, number 65, 1986.

[6] *Z. Zhou, J. S. Yu, Z. M. Guo*: Periodic solutions of higher-dimensional discrete systems. Proc. Royal Soc. Edinburgh *134A* (2004), 1013–1022.

[7] *G. Q. Wang, S. S. Cheng*: Notes on periodic solutions of discrete steady state systems. Portugaliae Math. *64* (2007), 3–10.

[8] *Z. M. Guo, J. S. Yu*: Existence of periodic and subharmonic solutions for second-order superlinear difference equations. Science in China (Series A) *46* (2003), 506–515.

[9] *Z. M. Guo, J. S. Yu*: The existence of periodic and subharmonic solutions to subquadratic second-order difference equations. J. London Math. Soc. *68* (2003), 419–430.
[10] *Z. Zhou*: Periodic orbits on discrete dynamical systems. Comput. Math. Appl. *45* (2003), 1155–1161.
[11] *G. Q. Wang, S. S. Cheng*: Positive periodic solutions for nonlinear difference equations via a continuation theorem. Advance in Difference Equations *4* (2004), 311–320.
[12] *S. S. Cheng, S. S. Lin*: Existence and uniqueness theorems for nonlinear difference boundary value problems. Utilitas Math. *39* (1991), 167–186.
[13] *S. S. Cheng, H. T. Yen*: On a nonlinear discrete boundary value problem. Linear Alg. Appl. *312* (2000), 193–201.
[14] *S. S. Cheng*: Partial Difference Equations. Taylor and Francis, 2003.

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