Gábor Czédli; Miklós Maróti

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ON THE HEIGHT OF ORDER IDEALS

Gábor Czédli, Miklós Maróti, Szeged

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Abstract. We maximize the total height of order ideals in direct products of finitely many finite chains. We also consider several order ideals simultaneously. As a corollary, a shifting property of some integer sequences, including digit sum sequences, is derived.

Keywords: order, product of chains, ideal of maximum height, digit sum sequence

MSC 2010: 06D99, 06A07

1. Introduction and the main results

Our notation and terminology are standard, see, for example, Davey and Priestley [3] or Grätzer [6]. In particular, let $B = (B; \leqslant)$ be a finite *order*; also called a partially ordered set or poset. A subset I of B is called an *order ideal*, in notation $I \lhd B$, if $x \leqslant y$ and $y \in I$ implies $x \in I$ for all $x, y \in B$. By definition, $\emptyset \lhd B$. For $b \in B$, let $\downarrow b$ denote the order ideal $\{x \in B\,;\ x \leqslant b\}$. Non-empty subsets of the form $\{x \in B\,;\ a \leqslant x \leqslant b\}$ are called *intervals* of B. By an n-element *chain* we mean $\{0 < 1 < \ldots < n-1\}$; the *length* of this chain is $n-1$.

For $b \in B$, the *height* of b is defined as the maximum of lengths of chains in $\downarrow b$. By the *(total) height* $h(X)$ of a subset X of B we mean $\sum\limits_{a \in X} h(a)$. We say that an order ideal I of B *maximizes the total height*, if for all $J \lhd B$ with $|J| = |I|$, we have $h(J) \leqslant h(I)$. For example, if B is the order (in fact, a distributive lattice) depicted in Figure 1, then $h(\downarrow a) = 6$ and $h(\downarrow b) = 8$, and both $\downarrow a$ and $\downarrow b$ maximize the total height.

We have two main motivations for studying the total height of order ideals.

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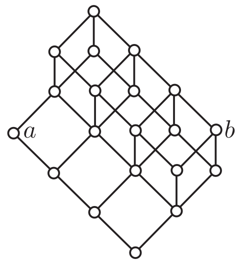

Figure 1. $\downarrow a$ and $\downarrow b$ maximize the total height.

First, total heights have a useful connection to digit sum sequences. This connection will be exploited in Corollary 4 and the remark following it.

Secondly, let $f_B\colon \{0,\ldots,|B|\} \to \mathbb{N}_0$, $n \mapsto \max\{h(I);\ I \lhd B \text{ and } |I| = n\}$. It seems to be an interesting problem to

(1) determine the function f_B.

The solution of (1) for finite direct powers of the two-element chain, that is for finite boolean algebras, has been heavily used in [2]. In the present paper, Theorem 5 will solve problem (1) for direct products of finitely many finite chains. Notice that while the usual greedy algorithm yields the solution of (1) in the boolean case, here the straightforward greedy algorithm can fail even in the direct square of a three-element chain. Therefore, the present approach to (1) is quite different from and more complex than that in [2].

Let $I_0 \subset I_1 \subset \ldots \subset I_{|B|}$ be order ideals of B such that, for all $j = 0, 1, \ldots, |B|$, $|I_j| = j$ and I_j maximizes the total height. Then we call $I_0 \subset I_1 \subset \ldots \subset I_{|B|}$ a *greedy chain of order ideals* of B. In connection with problem (1), one may ask

(2) which finite orders have a greedy chain of order ideals?

Although (2) is not targeted in the present paper, we mention two facts. The answer to (2) in case of Figure 1 is negative, since $\downarrow a$ and $\downarrow b$ are, respectively, the only four and five element order ideals maximizing the total height but $\downarrow a \not\subseteq \downarrow b$. Hence being a distributive lattice is not a sufficient condition to guarantee (2). On the other hand, it will be evident from Theorem 5 that the answer to (2) is affirmative for finite direct products of chains. This fact rather than the usual greedy algorithm motivates the term “greedy” in the following definition.

Definition 1. The following notation will be fixed throughout the paper. Let $t \in \mathbb{N}_0$. For $i = 1, \ldots, t$ let $k_i \in \mathbb{N} \setminus \{1\}$, and let $C_i = \{0 < 1 < \ldots < k_i - 1\}$ be a chain such that

$$k_1 \leqslant k_2 \leqslant \ldots \leqslant k_t, \text{ that is } |C_1| \leqslant |C_2| \leqslant \ldots \leqslant |C_t|.$$

Let $D = C_1 \times \ldots \times C_t$ be the direct product of these chains (which is the one element order if $t = 0$). By a *greedy order ideal* of D we mean an initial segment of the lexicographic order of D.

For example, for $t = k_1 = k_2 = 2$, the greedy order ideals of D are exactly the following ones: $\emptyset$, $\{(0,0)\}$, $\{(0,0),(0,1)\}$, $\{(0,0),(0,1),(1,0)\}$, and D. It is straightforward to see that each greedy order ideal of D is an order ideal.

Definition 2. For $m \in \mathbb{N}$, let $\vec{X} = (X_1, \ldots, X_m)$ be a sequence of greedy order ideals of D. We say that $\vec{X}$ is *packed to the right*, if there is an $i \in \{1, \ldots, m\}$ such that $X_1 = \ldots = X_{i-1} = \emptyset$ and $X_{i+1} = \ldots = X_m = D$.

Theorem 3. *Let $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_m$ be greedy order ideals of D such that $\sum\limits_{i=1}^{m} |X_i| = \sum\limits_{i=1}^{m} |Y_i|$ and $(Y_1, \ldots, Y_m)$ is packed to the right. Then $\sum\limits_{i=1}^{m} h(X_i) \leqslant \sum\limits_{i=1}^{m} h(Y_i)$.*

Consider the sequence $\vec{\alpha}^{(D)} = (\alpha_0, \alpha_1, \ldots, \alpha_{|D|-1})$ where α_i is the height of the i-th member, with respect to the lexicographic enumeration, of D. That is, $\alpha_0 = h((0,\ldots,0,0)) = 0$, $\alpha_1 = h((0,\ldots,0,1)) = 1$, and so on. In some sense, Theorem 3 is a statement on this sequence. It has an interesting corollary, a relatively simple "shifting property", which seems to have been unnoticed so far:

Corollary 4. *Let $1 \leqslant n \leqslant |D| = k_1 k_2 \ldots k_t$, and let s be the sum of n consecutive members of $\vec{\alpha}^{(D)}$. Then*

$$\alpha_0 + \ldots + \alpha_{n-1} \leqslant s \leqslant \alpha_{|D|-n} + \ldots + \alpha_{|D|-1}.$$

Remark. For $2 \leqslant q \in \mathbb{N}$, let $\vec{\beta}^{(q)} = (\beta_0, \beta_1, \beta_2, \ldots)$ be the so-called *digit sum sequence* of base q, that is, β_i is the sum of digits of i written in base q. This sequence and the sum of its first n members have been studied for long, see Gel'fond [4] and Lindström [7] for example. A search for "title = sum of digits" returns more than seventy publications in MathSciNet. Notice that the first q^t elements of $\vec{\beta}^{(q)}$ coincide with $\vec{\alpha}^{(D)}$ where D is the t-th direct power of the q-element chain.

Our second result is the following one.

Theorem 5. *Let X and Y be order ideals of D such that $|X| = |Y|$ and Y is a greedy order ideal. Then $h(X) \leqslant h(Y)$.*

A slightly related problem has been studied by Bollobás and Leader [1]. That problem in [1] refers to the addition in $\mathbb{Z}_k^n$, whence it does not belong to the theory of lattices and orders. However, the compression method used in [1], which is an advanced tool of combinatorics, offers an alternative proof of Theorem 5. The present, more elementary, approach is justified by its connection to Theorem 3, digit sum sequences, and [2].

2. Proofs and auxiliary statements

If $t = 0$ or $t = 1$, then $|D| = 1$ or D is a chain. In both cases, Theorems 5 and 3 and Corollary 4 are trivial. Hence, from now on, we assume that $t \geqslant 2$. In addition to Definition 1, we also need the following one.

Definition 6.

- As usual, the covering relation in D is denoted by $\prec$. That is, $x \prec y$ means that $x < y$ but $x < z < y$ for no z.
- Let $E = C_1 \times C_2$, $F = C_3 \times \ldots \times C_t$, and $G = C_2 \times \ldots \times C_t$. For $i \in C_1 = \{0, 1, \ldots, k_1 - 1\}$, let $G_i = \{(i, a_2, \ldots, a_t);\ (a_2, \ldots, a_t) \in G\}$.
- Similarly, for $(i, j) \in E = C_1 \times C_2$, let $F_{ij} = \{(i, j, a_3, \ldots, a_t);\ (a_3, \ldots, a_t) \in F\}$.
- Let $d = |D| = k_1 \ldots k_t$, $g = |G| = k_2 \ldots k_t$ and $f = |F| = k_3 \ldots k_t$.
- For $i \in \{0, \ldots, d\}$, let $h'(i) = h'_D(i)$ denote $h(X)$ where X is the unique i-element greedy order ideal of D. The symbols h'_G and h'_E have analogous meaning.

Notice that the F_{ij} and G_i are intervals of D. For $t = 3$, $k_1 = 3$, $k_2 = 4$ and $k_3 = 5$, the situation is depicted in Figure 2. The black-filled elements indicate how E is embedded in D.

Lemma 7. *For $i \in C_1$, let X_i be a greedy order ideal of G_i such that $i \prec j$ implies $|X_i| \geqslant |X_j|$. Then $\bigcup_{i \in C_1} X_i$ is an order ideal of D. Similarly, if Y_{ij} is a greedy order ideal of F_{ij} for each $(i, j) \in E$ such that $(i, j) \prec (k, l)$ implies $|Y_{ij}| \geqslant |Y_{kl}|$, then $\bigcup_{(i,j) \in E} Y_{ij}$ is an order ideal of D.*

P r o o f. The proof is almost trivial. If (k, l) covers (i, j) in E, then F_{ij} and F_{kl} are transposed intervals in the distributive lattice D, whence they are isomorphic. Let $\varphi\colon F_{kl} \to F_{ij}$ denote this isomorphism. Clearly, each greedy order ideal of F is determined by its size. Since $|Y_{ij}| \geqslant |Y_{kl}|$, $\varphi(Y_{kl})$ is an initial segment of Y_{ij}. Hence the second part of the lemma follows. The first part is even simpler. □

Definition 8. Let $q \in \mathbb{N}$. We say that $(y_1, \ldots, y_n) \in \{0, \ldots, q\}^n$ is *packed to the right* in $\{0, \ldots, q\}$, if, for $i \in \{1, \ldots, n-1\}$, $y_i \neq 0$ implies $y_{i+1} = q$. Similarly, we say that $(y_1, \ldots, y_n)$ is *packed to the left* in $\{0, \ldots, q\}$, if $(y_n, \ldots, y_1)$ is *packed to the right* in $\{0, \ldots, q\}$.

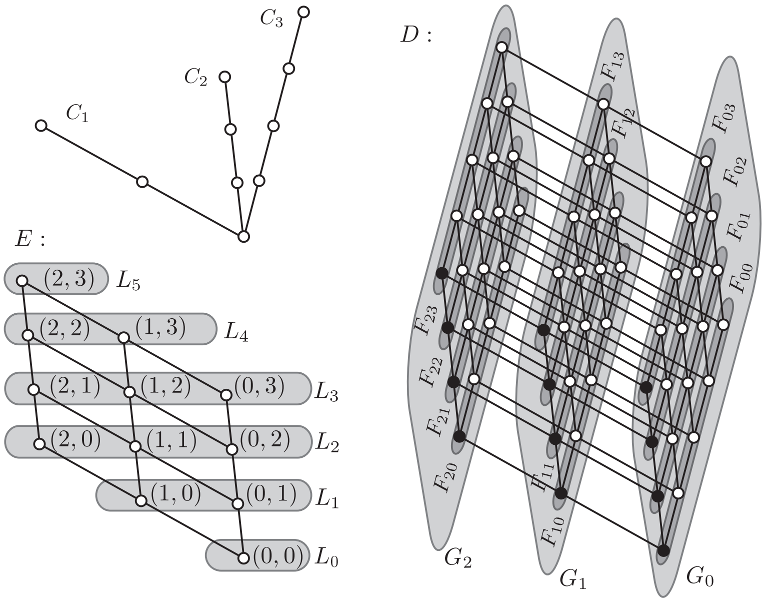

Figure 2. The structure of D for $(k_1, k_2, k_3) = (3, 4, 5)$

The following lemma is a trivial counterpart of Theorem 3; its evident proof will be omitted.

Lemma 9. *Let $n, q \in \mathbb{N}$, and consider integers $x_1, \ldots, x_n, y_1, \ldots, y_n \in \{0, \ldots, q\}$ and $a_1 \leqslant \ldots \leqslant a_n \in \mathbb{N}_0$ such that $x_1 + \ldots + x_n = y_1 + \ldots + y_n$ and $(y_1, \ldots, y_n)$ is packed to the right in $\{0, \ldots, q\}$. Then $a_1x_1 + \ldots + a_nx_n \leqslant a_1y_1 + \ldots + a_ny_n$.*

P r o o f of Theorem 3. We prove the theorem by induction on t. We assume that $t \geqslant 2$ and the statements hold for direct products of less than t finite chains. Since each greedy order ideal X of D is determined by its size $|X|$, the statement is evident for $m = 1$.

Next, we deal with the case $m = 2$. Assume that (X, Y) and (U, V) are pairs of greedy order ideals such that (U, V) is packed to the right and $|X| + |Y| = |U| + |V|$. Since the role of X and Y is symmetric, we can assume that $|X| \leqslant |Y|$. We claim that if (X, Y) is not packed to the right, then there exists a pair (X', Y') of greedy order ideals such that $|X'| + |Y'| = |X| + |Y|$, $|Y'| > |Y|$ and $h(X) + h(Y) \leqslant h(X') + h(Y')$. By a repeated application of this claim the theorem will clearly follow.

Since (X, Y) is not packed to the right, $X \neq \emptyset$ and $Y \neq D$. Let $X_i = X \cap G_i$ and $Y_i = Y \cap G_i$. Both X and Y are greedy, whence there exist $r, s \in \{0, 1, \ldots, k_1 - 1\}$

such that

$$X_i = G_i \ \text{ for } \ i < r,\ X_r \neq \emptyset,\ X_i = \emptyset \ \text{ for } \ i > r,$$
$$Y_i = G_i \ \text{ for } \ i < s,\ Y_s \neq G_s,\ Y_i = \emptyset \ \text{ for } \ i > s.$$

We know from $|X| \leqslant |Y|$ that $r \leqslant s$. Using that $G_r \cong G \cong G_s$, we can pack the pair (X_r, Y_s) to the right. That is, we obtain a pair (X'_r, Y'_s) of greedy order ideals such that $X'_r \lhd G_r$, $Y'_s \lhd G_s$, $|X_r| + |Y_s| = |X'_r| + |Y'_s|$, and either $X'_r = \emptyset$ or $Y'_s = G_s$. In both cases, $|Y_s| < |Y'_s|$. Let $X' = (X \setminus X_r) \cup X'_r$ and $Y' = (Y \setminus Y_s) \cup Y'_s$. Then X' and Y' are greedy order ideals of D, $|X'| + |Y'| = |X| + |Y|$ and $|Y'| > |Y|$. Let us compute:

$$\begin{aligned}
&h(X') + h(Y') - (h(X) + h(Y)) = h(X'_r) + h(Y'_s) - h(X_r) - h(Y_s) \\
&\quad = h'_G(|X'_r|) + r|X'_r| + h'_G(|Y'_s|) + s|Y'_s| - h'_G(|X_r|) - r|X_r| - h'_G(|Y_s|) - s|Y_s| \\
&\quad = (h'_G(|X'_r|) + h'_G(|Y'_s|) - (h'_G(|X_r|) + h'_G(|Y_s|))) \\
&\qquad + (r|X'_r| + s|Y'_s| - (r|X_r| + s|Y_s|)).
\end{aligned}$$

The first summand at the end of the above formula is non-negative, because the theorem holds for G by the induction hypothesis. The second summand is non-negative by Lemma 9. Hence $h(X') + h(Y') \geqslant h(X) + h(Y)$. This proves the claim and settles the case $m = 2$.

Finally, let $m > 2$. Let us repeat the following transformation on the sequence $(X_1, \ldots, X_m)$ of greedy order ideals:

- Rearrange the sequence so that $|X_1| \leqslant \ldots \leqslant |X_m|$. If $X_2 \neq D$, then let $l = \max\{i;\ X_l \neq D\}$, and perform a "packing the pair (X_{l-1}, X_l) to the right".

At each step, the sum of total heights cannot decrease. At the end of these transformations we obtain a sequence $(Z_1, \ldots, Z_m)$ of greedy order ideals packed to the right. Since $(Z_1, \ldots, Z_m)$ is determined by $\sum_{i=1}^{m} |Z_i| = \sum_{i=1}^{m} |X_i|$, we have $(Z_1, \ldots, Z_m) = (Y_1, \ldots, Y_m)$. □

From now on, for the sake of a forthcoming induction, we assume that

(3) Theorem 5 holds for direct products of less then t finite chains.

Definition 10.

- For $0 \leqslant m \leqslant k_1 + k_2 - 2$, the subset $\{(i, j);\ i + j = m\}$ of E is called the *m-th layer* of E, and is denoted by L_m, see Figure 2.
- In figures, C_1 is always left from C_2. The elements of L_m are on a horizontal line. Hence "left" and "right" in L_m make sense. Notice that (i, j) is left from (u, v) iff $j < v$ and, of course, $i + j = u + v$.

- If $(i,j),(i-1,j+1) \in E$, then $(i-1,j+1)$ is the *right neighbor* of (i,j) and (i,j) is the *left neighbor* of $(i-1,j+1)$. Sometimes we use this terminology for objects indexed by elements of E.
- An order ideal X of D will be called an *eastern order ideal*, if for every $0 \leqslant m \leqslant k_1+k_2-2$ and every $(i,j),(k,l) \in L_m$ the following condition holds: if $(i,j) \neq (k,l)$, (k,l) is right from (i,j) and $X \cap F_{ij} \neq \emptyset$, then $F_{kl} \subseteq X$. (In other words, if the sequence $(|X \cap F_{ij}|;\ (i,j) \in L_m)$ is packed to the right in $\{0,\dots,f\}$).
- For $0 \leqslant m \leqslant k_1+k_2-2$, let $a=(a_1,a_2)$ and $b=(b_1,b_2)$ be the leftmost and the rightmost element of L_m, respectively. If (a,b) is a (join-reducible, join-reducible), (join-irreducible, join-irreducible) or (join-reducible, join-irreducible) pair of elements of E, then L_m is called a layer of type A, type V or type N, respectively. In Figure 2, L_1 and L_2 are of type V, L_3 is of type N, L_4 and L_5 are of type A, and L_0 has no type.

Greedy order ideals of D are clearly eastern order ideals, but the converse is not true. Hence the following lemma is just an intermediate step towards Theorem 5.

Lemma 11. *Let X be an order ideal of D. Then there exists an eastern order ideal Y of D such that $|X|=|Y|$ and $h(X) \leqslant h(Y)$.*

P r o o f. First, observe that

$$\text{(4)} \qquad \text{if } \ i \prec j \in C_1, \ \text{ then } \ |G_i \cap X| \geqslant |G_j \cap X|.$$

Indeed, this follows from the implication $(j,x_2,\dots,x_t) \in X \Longrightarrow (i,x_2,\dots,x_t) \in X$. Replace $G_i \cap X$ by a greedy order ideal of G_i of the same size, for all $i \in C_1$. By (3), this way the total height does not decrease. Moreover, we obtain an order ideal of D by Lemma 7. Hence we can assume that every $G_i \cap X$ is a greedy order ideal of G_i.

For $(i,j) \in E$, let $X_{ij} = F_{ij} \cap X$ and $x_{ij} = |X_{ij}|$. Since $G_i \cap X$ is greedy, we conclude that for every $i \in C_1$,

$$\text{(5)} \quad (x_{i0},\dots,x_{i,k_2-1}) \ \text{ is packed to the left in } \ \{0,\dots,f\} \qquad \text{(a “}\swarrow\text{ property”).}$$

Similarly to (4), we conclude easily that

$$\text{(6)} \qquad \text{if } \ (i,j) \in E \ \text{ and } \ i \neq 0 \ \text{ then } \ x_{ij} \leqslant x_{i-1,j} \qquad \text{(a “}\searrow\text{ property”).}$$

Now we define a subset Z of D as follows. For each $(i,j) \in E$, the intersection $Z_{ij} = F_{ij} \cap Z$ will be a greedy order ideal of F_{ij}, whence it suffices to define $z_{ij} = |Z_{ij}|$. Let (i_1,j_1), $\dots$, (i_p,j_p) be an enumeration of L_m from left to right.

By packing the sequence $(x_{i_1 j_1}, \ldots, x_{i_p j_p})$ to the right in $\{0, \ldots, f\}$ we obtain the sequence $(z_{i_1 j_1}, \ldots, z_{i_p j_p})$. We do this for each m, and this defines z_{ij} for each $(i,j) \in E$. Hence Z is defined and $|Z| = |X|$. We obtain from Theorem 3 that $h(X) \leqslant h(Z)$.

We have to show that Z is an order ideal; then it is clearly an eastern order ideal. In virtue of Lemma 7, it suffices to show that for every $(u,v) \in E \setminus \{(0,0)\}$,

$$(7) \qquad u \neq 0 \Longrightarrow z_{uv} \leqslant z_{u-1,v} \ (\text{“}\searrow\text{”}) \ \text{ and } \ v \neq 0 \Longrightarrow z_{uv} \leqslant z_{u,v-1} \ (\text{“}\swarrow\text{”}).$$

Let (u,v) belong to L_m. That is, $m = u + v > 0$. Since $|C_1| \leqslant |C_2|$, we conclude that L_m is of type V, N or A.

C a s e 1. L_m is of type V. Then L_m, enumerated from the left to the right, is $\{(m,0), (m-1,1), \ldots, (0,m)\}$. For $k \in \{m-1, m\}$ define

$$w_k = \sum \{x_{ij};\ (i,j) \in L_k\}.$$

First assume that $x_{ij} \neq 0$ for all $(i,j) \in L_m \setminus \{(m,0)\}$. Then we get from (5) that $x_{ij} = f$ and, therefore, $z_{ij} = f$ for all $(i,j) \in L_{m-1}$. Then (7) holds evidently.

Secondly, assume that $x_{m-s,s} = 0$ for some $(m-s,s) \in L_m \setminus \{(m,0)\}$, see Figure 3. By (6), we get $x_{m-r,r} \leqslant x_{m-r-1,r}$ if $(m-r,r) \in L_m$ is to the left from $(m-s,s)$, that is, $r < s$. We obtain from (5) that $x_{m-r,r} \leqslant x_{m-r,r-1}$ if $r > s$. Hence we conclude that $w_m \leqslant w_{m-1}$.

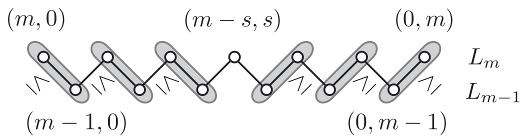

Figure 3. Case 1, $x_{m-s,s} = 0$

If $0 < i \leqslant m$, then $w_m \leqslant w_{m-1}$ implies $z_{m-i,i} \leqslant z_{m-i,i-1}$. For $0 < i < m$ we obtain $z_{m-i,i} \leqslant z_{m-i-1,i}$ from $z_{m-i,i} \leqslant z_{m-i,i-1} \leqslant z_{m-i-1,i}$. Finally, $z_{m,0} = 0 \leqslant z_{m-1,0}$. This shows (7).

C a s e 2. L_m is of type N. Due to $|L_m| = |L_{m-1}|$, this case is much simpler. We have $L_m = \{(k_1 - 1, m - k_1 + 1), (k_1 - 2, m - k_1 + 2), \ldots, (0,m)\}$. Since $x_{r,m-r} \leqslant x_{r,m-r-1}$ for $r \in C_1$ by (5), we get $w_m \leqslant w_{m-1}$. Hence $z_{i,m-i} \leqslant z_{i,m-i-1}$ for $i \in C_1$ and $z_{i,m-i} \leqslant z_{i,m-i-1} \leqslant z_{i-1,m-i}$ for $i \in C_1 \setminus \{0\}$. This shows (7).

C a s e 3. L_m is of type A. If $x_{i,m-i} = 0$ for all $(i, m-i) \in L_m$, then $z_{i,m-i} = 0$ for all $(i, m-i) \in L_m$, and (7) is evident. Otherwise, let us choose an element $(q, m-q)$ in L_m such that $x_{q,m-q} \neq 0$, see Figure 4. If $(i, m-i) \in L_m$ is to the left from $(q, m-q)$, then $x_{i,m-i} \leqslant x_{i,m-i-1}$ by (5). If $(i, m-i) \in L_m$ is to the right from or equals $(q, m-q)$, then $x_{i,m-i} \leqslant x_{i-1,m-i}$ by (6). These two inequalities yield that $w_m \leqslant w_{m-1} - x_{q,m-q-1}$. Since $x_{q,m-q-1} = f$ by (5), we conclude that $w_m \leqslant w_{m-1} - f$. Therefore at the rightmost element (u, v) of L_{m-1} we have $z_{uv} = f$. Let $w'_{m-1} = \sum\{x_{ij};\ (i,j) \in L_{m-1},\ (i,j) \neq (u,v)\}$. Then $w'_{m-1} = w_{m-1} - x_{uv} \geqslant w_{m-1} - f \geqslant w_m$, and disregarding (u, v) of L_{m-1} we conclude (7) the same way as in Case 2. $\square$

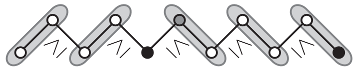

Figure 4. Case 3, $x_{q,m-q} \neq 0$

Lemma 12. *Let X be a nonempty eastern order ideal of D. Then there exists an eastern order ideal Y of D satisfying* (5) *such that $|X| = |Y|$ and $h(X) \leqslant h(Y)$.*

P r o o f. Suppose that (5) fails, and let i be the largest element of C_1 such that $G_i \cap X$ is not a greedy order ideal of G_i. Let Y_i be the $|G_i \cap X|$-element greedy ideal of G_i, and let $Y = (X \setminus G_i) \cup Y_i$. The induction hypothesis applies to G_i, and we conclude that $h(X) \leqslant h(Y)$. We have to show that Y is an eastern order ideal. In virtue of Lemma 7, only the pairs of $\{i-1, i, i+1\} \times C_2$ have to be considered.

Let $i < k_1 - 1$, $j < k_2$, and assume that $y_{i+1,j} = x_{i+1,j} > 0$. If $j < k_2 - 1$, then $x_{i,j+1}$, the right neighbor of $x_{i+1,j}$, equals f, because X is eastern. Then $y_{iu} = x_{iu} = f$ for $0 \leqslant u \leqslant j+1$, since X is an order ideal. The case $j = k_2 - 1$ is a bit different. Then the $x_{i+1,u}$ are equal to f for all $u < j = k_2 - 1$. The same holds for their right neighbors, that is, $x_{i,u} = f$ for all $1 \leqslant u \in C_2$. Since X is an order ideal, we obtain $y_{iu} = x_{iu} = f$ for all $u \in C_2$.

Let $0 < i \in C_1$, and assume that $y_{ij} > 0$. Then $x_{ij} > 0$. If $j < k_2 - 1$, then $x_{i-1,j+1} = y_{i-1,j+1}$, the right neighbor of x_{ij}, equals f. Further, this implies $y_{i-1,j} = x_{i-1,j} = f$. If $j = k_2 - 1$, then x_{i,k_2-2} and its right neighbor, x_{i-1,k_2-1} are equal to f, whence $y_{i-1,u} = x_{i-1,u} = f$ for all $u \in C_2$. Hence Y is an eastern order ideal such that $G_v \cap Y$ is a greedy order ideal for every $v \geqslant i$. In other words, we have got rid of i, the largest failure of (5). Repeating the above argument, we can get rid of all the failures. $\square$

Lemma 13. *Let X be a nonempty eastern order ideal of D satisfying* (5). *Then there exists an eastern order ideal Y of D satisfying* (5) *such that $|X| = |Y|$, $h(X) \leqslant h(Y)$ and, moreover, $0 < y_{ij} < f$ for at most one $(i, j) \in E$.*

P r o o f. Let $K = \{(i,j) \in E;\ 0 < x_{ij} < f\}$. We intend to show that $|K| \leqslant 1$ can be supposed. The assumptions on X allow a (unique) enumeration

$\{(i_1, j_1) \ldots, (i_s, j_s)\}$ of K such that $i_1 > \ldots > i_s$ and $j_1 < \ldots < j_s$. We claim that

$$i_1 + j_1 < \ldots < i_s + j_s. \tag{8}$$

By way of contradiction, assume that $1 \leqslant u < v \leqslant s$ but $i_u + j_u \geqslant i_v + j_v$. Since $j_u < j_v$, we have $i_u + j_u - j_v < i_u$. If we had $i_u + j_u - j_v < 0$, then $i_u + j_u < j_v \leqslant i_v + j_v$ would contradict the assumption. Hence $i_u + j_u - j_v \in C_1$ and $(i_u + j_u - j_v, j_v) \in E$. Since $(i_u + j_u - j_v, j_v)$ and (i_u, j_u) belong to the same level, $j_u < j_v$ and X is eastern, we conclude that $x_{i_u + j_u - j_v, j_v} = f$. From $i_u + j_u \geqslant i_v + j_v$ we see that $i_u + j_u - j_v \geqslant i_v$. Since X is an order ideal, we obtain that $x_{i_v j_v} \geqslant x_{i_u + j_u - j_v, j_v} = f$, which contradicts $(i_v, j_v) \in K$. This shows (8).

Next, we consider the (uniquely determined) sequence $\vec{y} = (y_{i_1,j_1}, \ldots, y_{i_s,j_s})$ that is packed to the right in $\{0, \ldots, f\}$ such that $x_{i_1,j_1} + \ldots + x_{i_s,j_s} = y_{i_1,j_1} + \ldots + y_{i_s,j_s}$.

(9) We intend to replace $\vec{x} = (x_{i_1,j_1}, \ldots, x_{i_s,j_s})$ with $\vec{y}$;

this way we obtain Y from X. The contribution of $\vec{x}$ to $h(X)$ is

$$\sum_{r=1}^{s} h'(x_{i_r,j_r}) + \sum_{r=1}^{s} (i_r + j_r) x_{i_r,j_r}.$$

If we replace $\vec{x}$ with $\vec{y}$, then the first sum does not decrease by Theorem 3 and the second does not decrease by Lemma 9. Hence $h(X) \leqslant h(Y)$. Since K is an antichain in E, that is, its elements are pairwise incomparable, Y is an eastern order ideal satisfying (5). $\square$

Proof of Theorem 5. By Lemmas 11, 12 and 13, we can assume that X is a non-empty eastern order ideal of D satisfying (5) and at most one of the x_{ij} belongs to $\{1, \ldots, f-1\}$.

By an "inner" induction on $k_1 = |C_1|$, we are going to show how X can be transformed to a greedy order ideal of D. We can assume that $G_{k_1-1} \cap X \neq \emptyset$ and $G_0 \not\subseteq X$, because otherwise we could use the induction hypothesis on k_1 (if $k_1 > 2$) or the induction hypothesis on t (if $k_1 = 2$). Let $m \in \{0, \ldots, k_1 + k_2 - 2\}$ be the largest subscript such that $B_m = \{(i,j) \in L_m ;\ x_{ij} \neq 0\}$ is non-empty.

If $m > k_2 - 1$, then $G_0 \subseteq X$ and we are done.

Suppose that $m = k_2 - 1$. If $x_{0,m}$, corresponding to the rightmost element of B_m, equals f, then $G_0 \subseteq X$ again. Since X is eastern, the definition of m yields that $1 \leqslant x_{0,m} \leqslant f - 1$ and $B_m = \{(0,m)\}$. Since $G_{k_1-1} \cap X \neq \emptyset$, there exists a largest index r such that $x_{k_1-1,r} \neq 0$. Notice that $x_{k_1-1,r} = f$ by Lemma 13. Notice also that $k_1 - 1 + r \leqslant m$ by the definition of m, whence $k_1 - 1 + r < m$ since $B_m = \{(0,m)\}$.

"Exchange $x_{k_1-1,r} = f$ and $x_{0,m} < f$," that is, exchange $X \cap F_{k_1-1,r}$ and $X \cap F_{0,m}$ in the ideal X. This way $h(X)$ increases and, clearly, we obtain a new eastern order ideal satisfying the same conditions that are given in the first sentence of the proof.

Next, assume that $m < k_2 - 1$. It suffices to show that m can be enlarged.

First, assume that $|B_m| \geqslant 2$, and let (i, j) be the leftmost element of B_m. Let $q = x_{ij} > 0$. Since X is eastern, the rightmost element of B_m is $(0, m)$. Note that $(0, m + 1) \in E$ and $x_{0,m+1} = 0$. Replace x_{ij} and $x_{0,m+1}$ with 0 and q, respectively. This way $h(X)$ increases (by q), m increases (by 1), and, clearly, we obtain a new eastern order ideal.

Secondly, assume that $|B_m| = 1$. Since X is eastern, $B_m = \{(0, m)\}$. Since we have assumed that $G_{k_1-1} \cap X \neq \emptyset$, there is a (unique) largest $v \in C_2$ such that $q := x_{k_1-1,v}$ is positive. If we had $0 < x_{0,m} < f$, then $q = f$ and we could repeat the trick (9) to $\vec{x} = (x_{k_1-1,v}, x_{0,m})$. Hence we can assume that $x_{0,m} = f$. Then, replacing $x_{k_1-1,v}$ and $x_{0,m+1} = 0$ with 0 and q, we obtain a new eastern order ideal with larger total height and larger m. □

P r o o f of Corollary 4. Let $b_0, b_1, \ldots, b_{d-1}$ be the lexicographic enumeration of D. Then $\alpha_i = h(b_i)$. Let $s = \alpha_j + \ldots + \alpha_{j+n-1}$ where $0 \leqslant j$ and $j + n - 1 \leqslant d - 1$.

Let $X_1 = \{b_0, \ldots, b_{n-1}\}$, $X_2 = \{b_0, \ldots, b_{j-1}\}$, $Y_1 = \emptyset$ and $Y_2 = \{b_0, \ldots, b_{j-1}, b_j, \ldots, b_{j+n-1}\}$. The conditions of Theorem 3 hold, whence we obtain

$$\begin{aligned}(\alpha_0 + \ldots + \alpha_{n-1}) + (\alpha_0 + \ldots + \alpha_{j-1}) &= h(X_1) + h(X_2)\\ &\leqslant h(Y_1) + h(Y_2) = 0 + \alpha_0 + \ldots + \alpha_{j-1} + s,\end{aligned}$$

which yields the first inequality of Corollary 4.

Let $X_1 = \{b_0, \ldots, b_{j-1}, b_j, \ldots, b_{j+n-1}\}$, $X_2 = \{b_0, \ldots, b_{d-n-1}\}$, $Y_1 = \{b_0, \ldots, b_{j-1}\}$ and $Y_2 = D = \{b_0, \ldots, b_{d-n-1}, b_{d-n}, \ldots, b_{d-1}\}$. Then Theorem 3 yields

$$\begin{aligned}(\alpha_0 + \ldots + \alpha_{j-1}) + s + (\alpha_0 + \ldots + \alpha_{d-n-1}) &= h(X_1) + h(X_2) \leqslant h(Y_1) + h(Y_2)\\ &= (\alpha_0 + \ldots + \alpha_{j-1}) + (\alpha_0 + \ldots + \alpha_{d-n-1}) + (\alpha_{d-n} + \ldots + \alpha_{d-1}),\end{aligned}$$

which implies the second inequality of Corollary 4. □

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Authors' addresses: *Gábor Czédli*, *Miklós Maróti*, University of Szeged, Bolyai Institute, Aradi vértanúk tere 1, Szeged, 6720 Hungary, e-mail: `czedli@math.u-szeged.hu`, `mmaroti@math.u-szeged.hu`.