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ON THE UNIQUENESS OF MEROMORPHIC FUNCTIONS THAT SHARE THREE SETS

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Abstract. With the aid of the notion of weighted sharing and pseudo sharing of sets we prove three uniqueness results on meromorphic functions sharing three sets, all of which will improve a result of Lin-Yi in Complex Var. Theory Appl. (2003).

Keywords: meromorphic functions, uniqueness, weighted sharing, shared set

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1. Introduction and main results

In this paper by meromorphic functions we will always mean meromorphic functions in the complex plane. We adopt the standard notation in the Nevanlinna theory of meromorphic functions as explained in [8]. It will be convenient to let E denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For a nonconstant meromorphic function h we denote by $T(r,h)$ the Nevanlinna characteristic of h and by $S(r,h)$ any quantity satisfying

$$S(r,h) = o(T(r,h)) \quad (r \to \infty,\ r \notin E).$$

Let f and g be two non-constant meromorphic functions and let a be a finite complex number. We say that f and g share a CM, provided that $f-a$ and $g-a$ have the same zeros with the same multiplicities. Similarly, we say that f and g share a IM, provided that $f-a$ and $g-a$ have the same zeros ignoring multiplicities. In addition we say that f and g share ∞ CM if $1/f$ and $1/g$ share 0 CM, and we say that f and g share ∞ IM if $1/f$ and $1/g$ share 0 IM.

Let S be a set of distinct elements of $\mathbb{C}\cup\{\infty\}$ and $E_f(S) = \bigcup\limits_{a\in S}\{z\colon\ f(z)-a=0\}$, where each zero is counted according to its multiplicity. If we do not count the multiplicity the set $\bigcup_{a\in S}\{z\colon\ f(z)-a=0\}$ is denoted by $\overline{E}_f(S)$. If $E_f(S)=E_g(S)$ we say that f and g share the set S CM. On the other hand, if $\overline{E}_f(S)=\overline{E}_g(S)$, we say that f and g share the set S IM.

Let m be a positive integer or infinity and $a\in\mathbb{C}\cup\{\infty\}$. We denote by $E_{m)}(a;f)$ the set of all a-points of f with multiplicities not exceeding m, where an a-point is counted according to its multiplicity. If $E_{\infty)}(a;f)$=$E_{\infty)}(a;g)$ for some $a\in\mathbb{C}\cup\{\infty\}$, we say that f, g share the value a CM. For a set S of distinct elements of $\mathbb{C}$ we define $E_{m)}(S,f)=\bigcup_{a\in S}E_{m)}(a,f)$.

The uniqueness problem for entire or meromorphic functions sharing sets was initiated by a famous question of F. Gross in [7]. In 1976 he posed the following question:

Question A. Can one find two finite sets S_j $(j=1,2)$ such that any two non-constant entire functions f and g satisfying $E_f(S_j)=E_g(S_j)$ for $j=1,2$ must be identical?

In [7], Gross said that if the answer of Question A is affirmative it would be interesting to know how large both sets would have to be?

In 1994, H. X. Yi posed the following question for meromorphic functions.

Question B [19]. Can one find three finite sets S_j $(j=1,2,3)$ such that any two non-constant meromorphic functions f and g satisfying $E_f(S_j)=E_g(S_j)$ for $j=1,2,3$ must be identical?

In 1994 Yi [19] gave an affirmative answer to Question B and proved that there exist three finite sets S_1 (with 7 elements), S_2 (with 2 elements) and S_3 (with 1 element) such that any two non-constant meromorphic functions f and g satisfying $E_f(S_j)=E_g(S_j)$ for $j=1$, 2, 3 must be identical.

Gradually the research on Question A corresponding to meromorphic functions as well as Question B gained pace and today it has become one of the most prominent branches of the uniqueness theory. Among a number of situations depending on the nature and the number of shared sets, the uniqueness of two meromorphic functions was studied by many authors. Especially during the last few years a considerable amount of work has been done to investigate the possible answer to Question B. (cf. [1], [2]–[5], [6], [9], [13], [16], [17], [18], [19], [20], [21], [23]). In 2001 the idea of gradation of sharing known as weighted sharing was introduced in [11], [12] which measures how close a shared value is to being shared CM or to being shared IM. In the following definition we explain the notion.

Definition 1.1 [11], [12]. Let k be a nonnegative integer or infinity. For $a\in\mathbb{C}\cup\{\infty\}$ we denote by $E_k(a;f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m\leqslant k$ and $k+1$ times if $m>k$. If $E_k(a;f)=E_k(a;g)$, we say that f,g share the value a with weight k.

We write f, g share (a,k) meaning that f, g share the value a with weight k. Clearly, if f, g share (a,k) then f,g share (a,p) for any integer p, $0 \leqslant p < k$. Also we note that f,g share a value a IM or CM if and only if f,g share $(a,0)$ or (a,∞), respectively.

Definition 1.2 [11]**.** Let S be a set of distinct elements of $\mathbb{C} \cup \{\infty\}$ and k a nonnegative integer or ∞. We denote by $E_f(S,k)$ the set $\bigcup\limits_{a\in S} E_k(a;f)$.

Clearly $E_f(S) = E_f(S,\infty)$ and $\overline{E}_f(S) = E_f(S,0)$.

Recently the present author [1] has provided the affirmative answer to Question B by applying the notion of weighted sharing. He has proved that if two non constant meromorphic functions share one set S_1 (containing 1 element) CM, and two other sets S_2 (containing 1 element) and S_3 (containing 4 elements) with finite weight, then $f \equiv g$ with some restriction on the ramification index of f and g at ∞. In this paper, by using the idea of weighted sharing, we will investigate the possible answer to Question B where solely the set sharing of the meromorphic functions will be given as in the follows.

$$P(w) = aw^n - n(n-1)w^2 + 2n(n-2)bw - (n-1)(n-2)b^2 \tag{1.1}$$

where $n \geqslant 3$ is an integer and a and b are two nonzero complex numbers satisfying $ab^{n-2} \neq 2$. We claim that the polynomial $P(w)$ has only simple zeros.

In fact we consider the rational function

$$R(w) = \frac{aw^n}{n(n-1)(w-\alpha_1)(w-\alpha_2)}, \tag{1.2}$$

where α_1 and α_2 are two distinct roots of

$$n(n-1)w^2 - 2n(n-2)bw + (n-1)(n-2)b^2 = 0.$$

From (1.2) we have

$$R'(w) = \frac{(n-2)aw^{n-1}\,(w-b)^2}{n(n-1)\,(w-\alpha_1)^2\,(w-\alpha_2)^2}. \tag{1.3}$$

From (1.3) we know that $w = 0$ is a root with multiplicity n of the equation $R(w) = 0$ and $w = b$ is a root with multiplicity 3 of the equation $R(w) - c = 0$, where $c = \frac{1}{2}ab^{n-2}$.

Then

$$R(w) - c = \frac{a(w-b)^3\,Q_{n-3}(w)}{n(n-1)(w-\alpha_1)(w-\alpha_2)}, \tag{1.4}$$

where $Q_{n-3}(w)$ is a polynomial of degree $n-3$.

Moreover, from (1.1) and (1.2) we have

$$R(w) - 1 = \frac{P(w)}{n(n-1)(w-\alpha_1)(w-\alpha_2)}. \tag{1.5}$$

Noting that $c = \frac{1}{2}ab^{n-2} \neq 1$, from (1.3) and (1.5) we obtain that

$$P(w) = aw^n - n(n-1)w^2 - 2n(n-2)bw + (n-1)(n-2)b^2$$

has only simple zeros.

In 2003, Lin and Yi proved the following result which answered Question B and improved the corresponding theorem in [19].

Theorem A [16]. *Let $S_1 = \{0\}$, $S_2 = \{\infty\}$ and $S_3 = \{w \mid P(w) = 0\}$, where $P(w)$ is given by (1.1) and $n \geqslant 5$. Suppose that f and g are two non-constant meromorphic functions satisfying $E_f(S_j, \infty) = E_g(S_j, \infty)$ $(j = 1, 2, 3)$. Then $f \equiv g$.*

In [16], Yi and Lin made the following remark.

Remark 1.1. If the condition $E_f(S_2, \infty) = E_g(S_2, \infty)$ is replaced by a weaker condition $E_f(S_2, 0) = E_g(S_2, 0)$ the conclusion of Theorem A remains true.

In this paper, we will prove the following three theorems which improve Theorem A.

Theorem 1.1. *Let S_1, S_2 and S_3 be defined as in Theorem A and $n \geqslant 5$. Suppose that f and g are two non-constant meromorphic functions satisfying $E_f(S_1, 4) = E_g(S_1, 4)$, $E_f(S_2, 0) = E_g(S_2, 0)$ and $E_{5)}(S_3, f) = E_{5)}(S_3, g)$. Then $f \equiv g$.*

Theorem 1.2. *Let S_1, S_2 and S_3 be defined as in Theorem A and $n \geqslant 5$. Suppose that f and g are two non-constant meromorphic functions satisfying $E_f(S_1, \infty) = E_g(S_1, \infty)$, $E_f(S_2, 0) = E_g(S_2, 0)$ and $E_{4)}(S_3, f) = E_{4)}(S_3, g)$. Then $f \equiv g$.*

Theorem 1.3. *Let S_1, S_2 and S_3 be defined as in Theorem A and $n \geqslant 5$. Suppose that f and g are two non-constant meromorphic functions satisfying $E_f(S_1, 2) = E_g(S_1, 2)$, $E_f(S_2, 0) = E_g(S_2, 0)$ and $E_{6)}(S_3, f) = E_{6)}(S_3, g)$. Then $f \equiv g$.*

We also need the following definitions.

Definition 1.3 [10]. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $N(r, a; f \mid= 1)$ the counting function of simple a-points of f. For a positive integer m we denote by $N(r, a; f \mid\leqslant m)$ $(N(r, a; f \mid\geqslant m))$ the counting function of those a points of f whose multiplicities are not greater(less) than m where each a point is counted according to its multiplicity; denote by $N(r,a;f \mid< m)$ $(N(r,a;f \mid> m))$ the counting function of those a-points of f whose multiplicities are less (greater) than m; denote by $\overline{N}(r,a;f \mid\leqslant m)$, $\overline{N}(r,a;f \mid\geqslant m)$, $\overline{N}(r,a;f \mid< m)$ and $\overline{N}(r,a;f \mid> m)$ the reduced forms of $N(r,a;f \mid\leqslant m)$, $N(r,a;f \mid\geqslant m)$, $N(r,a;f \mid< m)$ and $N(r,a;f \mid> m)$, respectively.

Definition 1.4 [1]**.** We denote by $\overline{N}(r,a;f \mid= k)$ the reduced counting function of those a-points of f whose multiplicity is exactly k, where $k \geqslant 2$ is an integer.

Definition 1.5. Let f and g be two non-constant meromorphic functions such that f and g share a value a IM where $a \in \mathbb{C} \cup \{\infty\}$. Let z_0 be an a-point of f with multiplicity p, an a-point of g with multiplicity q. We denote by $\overline{N}_L(r,a;f)$ $(\overline{N}_L(r,a;g))$ the counting function of those a-points of f and g where $p > q$ $(q > p)$, each a-point being counted only once.

Definition 1.6. Let f and g be two non-constant meromorphic functions and m be a positive integer such that $E_{m)}(a;f) = E_{m)}(a;g)$ where $a \in \mathbb{C} \cup \{\infty\}$. Let z_0 be an a-point of f with multiplicity $p > 0$, an a-point of g with multiplicity $q > 0$. We denote by $\overline{N}_L^{m)}(r,a;f)$ $(\overline{N}_L^{m)}(r,a;g))$ the counting function of those a-points of f and g where $p > q$ $(q > p)$, each a-point is counted only once.

Definition 1.7. For a positive integer p we denote $N_p(r,a;f) = \overline{N}(r,a;f) + \overline{N}(r,a;f \mid\geqslant 2) + \ldots + \overline{N}(r,a;f \mid\geqslant p)$. Clearly $\overline{N}(r,a;f) = N_1(r,a;f)$.

Definition 1.8. Let m be a positive integer. Also let z_0 be a zero of $f(z) - a$ of multiplicity p and a zero of $g(z) - a$ of multiplicity q. We denote by $\overline{N}_{f\geqslant m+1}(r,a;f \mid g \neq a)$ $(\overline{N}_{g\geqslant m+1}(r,a;g \mid f \neq a))$ the reduced counting functions of those a-points of f and g for which $p \geqslant m+1$ and $q = 0$ $(q \geqslant m+1$ and $p = 0)$.

Definition 1.9 [11], [12]**.** Let f, g share $(a,0)$. We denote by $\overline{N}_*(r,a;f,g)$ the reduced counting function of those a-points of f whose multiplicities differ from the multiplicities of the corresponding a-points of g.

Remark 1.2. Clearly $\overline{N}_*(r,a;f,g) = \overline{N}_*(r,a;g,f) = \overline{N}_L(r,a;f) + \overline{N}_L(r,a;g)$. If $E_{m)}(a;f) = E_{m)}(a;g)$, then $\overline{N}_*(r,a;f,g) = \overline{N}_L^{m)}(r,a;f) + \overline{N}_L^{m)}(r,a;g) + \overline{N}_{f\geqslant m+1}(r,a;f \mid g \neq a) + \overline{N}_{g\geqslant m+1}(r,a;g \mid f \neq a)$.

Definition 1.10 [14]**.** Let $a, b \in \mathbb{C} \cup \{\infty\}$. We denote by $N(r,a;f \mid g = b)$ the counting function of those a-points of f, counted according to their multiplicity, which are b-points of g.

Definition 1.11 [14]**.** Let $a, b_1, b_2, \ldots, b_q \in \mathbb{C} \cup \{\infty\}$. We denote by $N(r,a;f \mid g \neq b_1, b_2, \ldots, b_q)$ the counting function of those a-points of f, counted according to their multiplicity, which are not the b_i-points of g for $i = 1, 2, \ldots, q$.

2. Lemmas

In this section we present some lemmas which will be needed in the sequel. Let F and G be two non-constant meromorphic functions defined in $\mathbb{C}$. Henceforth we will denote by H, Φ and V the following three functions:

$$H = \Big(\frac{F''}{F'} - \frac{2F'}{F-1}\Big) - \Big(\frac{G''}{G'} - \frac{2G'}{G-1}\Big),$$
$$\Phi = \frac{F'}{F-1} - \frac{G'}{G-1}$$

and

$$V = \Big(\frac{F'}{F-1} - \frac{F'}{F}\Big) - \Big(\frac{G'}{G-1} - \frac{G'}{G}\Big) = \frac{F'}{F(F-1)} - \frac{G'}{G(G-1)}.$$

Lemma 2.1 [15]**.** *For $E_{m)}(1;F) = E_{m)}(1;G)$ and $H \not\equiv 0$ we have*

$$N(r,1;F \mid= 1) = N(r,1;G \mid= 1) \leqslant N(r,H) + S(r,F) + S(r,G).$$

Lemma 2.2. *If $N(r,0;f^{(k)} \mid f \neq 0)$ denotes the counting function of those zeros of $f^{(k)}$ which are not the zeros of f, where a zero of $f^{(k)}$ is counted according to its multiplicity, then*

$$N(r,0;f^{(k)} \mid f \neq 0) \leqslant k\overline{N}(r,\infty;f) + N_k(r,0;f) + S(r,f).$$

P r o o f. By the first fundamental theorem and Milloux theorem ([see [8], Theorem 3.1]) we get

$$\begin{aligned} N(r,0;f^{(k)} \mid f \neq 0) &\leqslant N\Big(r,0;\frac{f^{(k)}}{f}\Big) \leqslant N\Big(r,\infty;\frac{f^{(k)}}{f}\Big) + m\Big(r,\frac{f^{(k)}}{f}\Big) + O(1) \\ &\leqslant N(r,0;f \mid< k) + k\overline{N}(r,0;f \mid\geqslant k) + k\overline{N}(r,\infty;f) + S(r,f) \\ &= N_k(r,0;f) + k\overline{N}(r,\infty;f) + S(r,f). \end{aligned}$$

□

Lemma 2.3. *Let F and G be two meromorphic functions such that $E_{m)}(1;F) = E_{m)}(1;G)$, where $1 \leqslant m < \infty$. Then*

$$\overline{N}(r,1;F) + \overline{N}(r,1;G) - N(r,1;F \mid= 1) + \Big(\frac{m}{2} - \frac{1}{2}\Big)\{\overline{N}_{F\geqslant m+1}(r,1;F \mid G \neq 1)$$

$$+\overline{N}_{G\geqslant m+1}(r,1;G\mid F\neq 1)\}+\left(m-\frac{1}{2}\right)\{\overline{N}_L^{m)}(r,1;F)+\overline{N}_L^{m)}(r,1;G)\}$$
$$\leqslant \frac{1}{2}\left[N(r,1;F)+N(r,1;G)\right].$$

P r o o f. Since $E_{m)}(1;F)=E_{m)}(1;G)$, we note that common zeros of $F-1$ and $G-1$ upto multiplicity m are the same. Let z_0 be a 1-point of F with multiplicity p and a 1-point of G with multiplicity q. If $p=m+1$ the possible values of q are (i) $q=m+1$, (ii) $q\geqslant m+2$, (iii) $q=0$. Similarly, when $p=m+2$ the possible values of q are (i) $q=m+1$, (ii) $q=m+2$, (iii) $q\geqslant m+3$, (iv) $q=0$. If $p\geqslant m+3$ we can similarly find the possible values of q. Now the lemma follows from the above explanation. □

Let f and g be two non-constant meromorphic functions and

$$F=R(f),\quad G=R(g), \tag{2.1}$$

where $R(w)$ is given by (1.2). From (1.2) and (2.1) it is clear that

$$T(r,f)=\frac{1}{n}T(r,F)+S(r,f),\quad T(r,g)=\frac{1}{n}T(r,G)+S(r,g). \tag{2.2}$$

□

Lemma 2.4. *Let F, G be given by* (2.1) *and let $\omega_1,\omega_2\ldots\omega_n$ be the roots of $P(w)=0$.*

If $E_{m)}(1;F)=E_{m)}(1;G)$, where $1\leqslant m<\infty$, then

(i) $\overline{N}_{F\geqslant m+1}(r,1;F\mid G\neq 1)\leqslant m^{-1}[\overline{N}(r,0;f)+\overline{N}(r,\infty;f)-N_{\otimes}(r,0;f')]+S(r,f)$

(ii) $\overline{N}_{G\geqslant m+1}(r,1;G\mid F\neq 1)\leqslant m^{-1}[\overline{N}(r,0;g)+\overline{N}(r,\infty;g)-N_{\otimes}(r,0;g')]+S(r,g)$,

where $N_{\otimes}(r,0;f')=N(r,0;f'\mid f\neq 0,\omega_1,\omega_2\ldots\omega_n)$. $N_{\otimes}(r,0;g')$ is defined similarly.

P r o o f. We prove (i) since (ii) can be proved in a similar way. Using Lemma 2.2 we get from (1.5) and (2.1) that

$$\begin{aligned}\overline{N}_{F\geqslant m+1}(r,1;F\mid G\neq 1)&\leqslant \overline{N}(r,1;F\mid\geqslant m+1)\\&\leqslant \frac{1}{m}\left(N(r,1;F)-\overline{N}(r,1;F)\right)\\&\leqslant \frac{1}{m}\left[\sum_{j=1}^{n}\left(N(r,\omega_j;f)-\overline{N}(r,\omega_j;f)\right)\right]\\&\leqslant \frac{1}{m}(N(r,0;f'\mid f\neq 0)-N_{\otimes}(r,0;f'))\\&\leqslant \frac{1}{m}[\overline{N}(r,0;f)+\overline{N}(r,\infty;f)-N_{\otimes}(r,0;f')]+S(r,f).\end{aligned}$$

□

Lemma 2.5. *Let F, G be given by* (2.1) *and let $\omega_1, \omega_2, \ldots, \omega_n$ be the roots of $P(w) = 0$. If $E_{m)}(1; F) = E_{m)}(1; G)$, where $1 \leqslant m < \infty$, then*

(i) $\overline{N}_{F \geqslant m+1}(r, 1; F \mid G \neq 1) + \overline{N}_L(r, 1; F) \leqslant m^{-1}[\overline{N}(r, 0; f) + \overline{N}(r, \infty; f) - N_{\otimes}(r, 0; f')] + S(r, f)$,

(ii) $\overline{N}_{G \geqslant m+1}(r, 1; G \mid F \neq 1) + \overline{N}_L(r, 1; G) \leqslant m^{-1}[\overline{N}(r, 0; g) + \overline{N}(r, \infty; g) - N_{\otimes}(r, 0; g')] + S(r, g)$.

P r o o f. We prove (i) since (ii) can be proved in a similar way.

Since $\overline{N}_{F \geqslant m+1}(r, 1; F \mid G \neq 1) + \overline{N}_L(r, 1; F) \leqslant \overline{N}(r, 1; F \mid \geqslant m+1)$ the lemma can be proved following the line of proof of Lemma 2.4. $\square$

Lemma 2.6. *Let F and G be given by* (2.1) *and assume f, g share* $(0, 0)$ *and* 0 *is not a Picard exceptional value of f and g. Then $\Phi \equiv 0$ implies $F \equiv G$.*

P r o o f. Suppose $\Phi \equiv 0$. Then by integration we obtain

$$F - 1 = C(G - 1).$$

It is clear that if z_0 is a zero of f then it is a zero of g. So from (1.2) and (2.1) it follows that $F(z_0) = 0$ and $G(z_0) = 0$. So $C = 1$ and hence $F \equiv G$. $\square$

Lemma 2.7. *Let F, G be given by* (2.1) *and let $H \not\equiv 0$. If $E_{m)}(1; F) = E_{m)}(1; G)$ and f, g share (∞, k) and $(0, p)$, where $1 \leqslant m < \infty$ and $0 \leqslant p < \infty$, then*

$$\begin{aligned}
[np + n - 1]\overline{N}(r, 0; f \mid \geqslant p + 1) &= [np + n - 1]\overline{N}(r, 0; g \mid \geqslant p + 1) \\
&\leqslant \overline{N}_L^{m)}(r, 1; F) + \overline{N}_L^{m)}(r, 1; G) + \overline{N}_{F \geqslant m+1}(r, 1; F \mid G \neq 1) \\
&\quad + \overline{N}_{G \geqslant m+1}(r, 1; G \mid F \neq 1) + \overline{N}_*(r, \infty; f, g) + \overline{N}(r, \alpha_1; f) \\
&\quad + \overline{N}(r, \alpha_2; f) + \overline{N}(r, \alpha_1; g) + \overline{N}(r, \alpha_2; g) \\
&\quad + S(r, f) + S(r, g).
\end{aligned}$$

P r o o f. Suppose 0 is a Picard exceptional value of f and g. Then the lemma follows immediately.

Next suppose 0 is not a Picard exceptional value of f and g. Since $H \not\equiv 0$ by Lemma 2.6 we can deduce $\Phi \not\equiv 0$. Let z_0 be a zero of f with multiplicity q and a zero of g with multiplicity r. From (1.2) and (2.1) we know that z_0 is a zero of F with multiplicity nq and a zero of G with multiplicity nr. Since f, g share $(0; p)$, it follows that F, G share $(0; np)$ and so a zero of F with multiplicity q $(\geqslant np + 1)$ is a zero of G of multiplicity r $(\geqslant np + 1)$ and vice versa. We note that F and G have no zero of multiplicity t where $np < t < n(p+1)$. So it is clear from the definition of Φ that z_0 is a zero of Φ with multiplicity at least $n(p+1)-1$. So we have

$$\begin{aligned}
[np+n-1]\overline{N}(r,0;f \mid\geqslant p+1) &= [np+n-1]\overline{N}(r,0;g \mid\geqslant p+1) \\
&= [np+n-1]\overline{N}\,(r,0;F \mid\geqslant n(p+1)) \\
&\leqslant N(r,0;\Phi) \\
&\leqslant N(r,\infty;\Phi) + S(r,f) + S(r,g) \\
&\leqslant \overline{N}_*(r,\infty;f,g) + \overline{N}(r,\alpha_1;f) + \overline{N}(r,\alpha_2;f) \\
&\quad + \overline{N}(r,\alpha_1;g) + \overline{N}(r,\alpha_2;g) + \overline{N}_L^{m)}(r,1;F) + \overline{N}_L^{m)}(r,1;G) \\
&\quad + \overline{N}_{F\geqslant m+1}(r,1;F \mid G \neq 1) + \overline{N}_{G\geqslant m+1}(r,1;G \mid F \neq 1) \\
&\quad + S(r,f) + S(r,g).
\end{aligned}$$

□

Lemma 2.8. *Let F and G be given by* (2.1) *and assume f, g share* $(\infty, 0)$ *and ∞ is not a Picard exceptional value of f and g. Then $V \equiv 0$ implies $F \equiv G$.*

P r o o f. Suppose

$$V \equiv 0.$$

Then by integration we obtain

$$1 - \frac{1}{F} = A\Big(1 - \frac{1}{G}\Big).$$

It is clear that if z_0 is a pole of f then it is a pole of g. Hence from the definition of F and G we have $1/F(z_0) = 0$ and $1/G(z_0) = 0$. So $A = 1$ and hence $F \equiv G$. □

Lemma 2.9. *Let F, G be given by* (2.1) *and let $H \not\equiv 0$. If $E_{m)}(1;F) = E_{m)}(1;G)$, f, g share* (∞, k) *and* $(0, p)$, *where $1 \leqslant m < \infty$, $0 \leqslant k < \infty$, then*

$$\begin{aligned}
[(n-2)k+n-3)]&\overline{N}(r,\infty;f \mid\geqslant k+1) \\
&= [(n-2)k+n-3)]\overline{N}(r,\infty;g \mid\geqslant k+1) \\
&\leqslant \overline{N}_*(r,0;f,g) + \overline{N}_L^{m)}(r,1;F) + \overline{N}_L^{m)}(r,1;G) \\
&\quad + \overline{N}_{F\geqslant m+1}(r,1;F \mid G \neq 1) \\
&\quad + \overline{N}_{G\geqslant m+1}(r,1;G \mid F \neq 1) + S(r,f) + S(r,g).
\end{aligned}$$

P r o o f. Suppose ∞ is a Picard exceptional value of f and g. Then the lemma follows immediately.

Next suppose ∞ is not a Picard exceptional value of f and g. Since $H \not\equiv 0$, from Lemma 2.8 we have $V \not\equiv 0$. We suppose that z_0 is a pole of f with multiplicity q and a pole of g with multiplicity r. From (1.2) and (2.1) we know that z_0 is a pole of f with multiplicity $(n-2)q$ and a pole of g with multiplicity $(n-2)r$. Noting that f, g share $(\infty;k)$ from the definition of V it is clear that z_0 is a zero of V with multiplicity at least $(n-2)(k+1)-1$. So from the definition of V we have

$$\begin{aligned}
&[(n-2)k+n-3]\overline{N}(r,\infty;f\mid\geqslant k+1)\\
&\quad= [(n-2)k+n-3]\overline{N}(r,\infty;g\mid\geqslant k+1)\\
&\quad\leqslant N(r,0;V)\leqslant N(r,\infty;V)+S(r,f)+S(r,g)\\
&\quad\leqslant \overline{N}_*(r,0;f,g)+\overline{N}_L^{m)}(r,1;F)+\overline{N}_L^{m)}(r,1;G)+\overline{N}_{F\geqslant m+1}(r,1;F\mid G\neq 1)\\
&\qquad+\overline{N}_{G\geqslant m+1}(r,1;G\mid F\neq 1)+S(r,f)+S(r,g).
\end{aligned}$$

□

Lemma 2.10. *Let F, G be given by* (2.1) *and let $H\not\equiv 0$. If $E_{m)}(1;F)=E_{m)}(1;G)$ and f, g share $(\infty,0)$ and $(0,p)$, where $1\leqslant m<\infty$, $0\leqslant p<\infty$ then*

$$[m(n-3)-2]\overline{N}(r,\infty;f)\leqslant (m+2)\overline{N}(r,0;f)+S(r,f)+S(r,g).$$

P r o o f. First we note that since f, g share $(0,p)$ they share $(0,0)$. So using Lemma 2.5, we obtain from Lemma 2.9 with $k=0$ that

$$\begin{aligned}
&(n-3)\overline{N}(r,\infty;f)\\
&\quad\leqslant \overline{N}(r,0;f)+\overline{N}_L^{m)}(r,1;F)+\overline{N}_L^{m)}(r,1;G)+\overline{N}_{F\geqslant m+1}(r,1;F\mid G\neq 1)\\
&\qquad+\overline{N}_{G\geqslant m+1}(r,1;G\mid F\neq 1)+S(r,f)+S(r,g)\\
&\quad\leqslant \overline{N}(r,0;f)+\frac{1}{m}\big[\overline{N}(r,0;f)+\overline{N}(r,\infty;f)+\overline{N}(r,0;g)+\overline{N}(r,\infty;g)\big]\\
&\qquad+S(r,f)+S(r,g)\\
&\quad\leqslant \frac{m+2}{m}\,\overline{N}(r,0;f)+\frac{2}{m}\,\overline{N}(r,\infty;f)+S(r,f)+S(r,g).
\end{aligned}$$

Now the lemma follows. □

Lemma 2.11. *Let F, G be given by* (2.1) *and let $H\not\equiv 0$. If $E_{m)}(1;F)=E_{m)}(1;G)$ and f, g share $(\infty,0)$ and $(0,\infty)$, where $1\leqslant m<\infty$, then*

$$[m(n-3)-2]\overline{N}(r,\infty;f)\leqslant 2\overline{N}(r,0;f)+S(r,f)+S(r,g).$$

P r o o f. Since f, g share $(0,\infty)$, we observe that $\overline{N}_*(r,0;f,g)=0$. So using Lemma 2.5, we obtain from Lemma 2.9 with $k=0$ that

$$\begin{aligned}(n-3)\overline{N}(r,\infty;f) &\leqslant \overline{N}_L^{m)}(r,1;F)+\overline{N}_L^{m)}(r,1;G)+\overline{N}_{F\geqslant m+1}(r,1;F\mid G\neq 1)\\ &\quad+\overline{N}_{G\geqslant m+1}(r,1;G\mid F\neq 1)+S(r,f)+S(r,g)\\ &\leqslant \frac{1}{m}\big[\overline{N}(r,0;f)+\overline{N}(r,\infty;f)+\overline{N}(r,0;g)+\overline{N}(r,\infty;g)\big]\\ &\quad+S(r,f)+S(r,g)\\ &\leqslant \frac{2}{m}\overline{N}(r,0;f)+\frac{2}{m}\overline{N}(r,\infty;f)+S(r,f)+S(r,g).\end{aligned}$$

Now the lemma follows. □

Lemma 2.12. *Let F, G be given by* (2.1) *and let $H\not\equiv 0$. If $E_{m)}(1;F)=E_{m)}(1;G)$ and f, g share (∞,k), $(0,p)$ where $1\leqslant m<\infty$, then*

$$\begin{aligned}N(r,1;F\mid=1) &\leqslant \overline{N}_*(r,0;f,g)+\overline{N}_*(r,\infty;f,g)+\overline{N}_L^{m)}(r,1;F)+\overline{N}_L^{m)}(r,1;G)\\ &\quad+\overline{N}_{F\geqslant m+1}(r,1;F\mid G\neq 1)+\overline{N}_{G\geqslant m+1}(r,1;G\mid F\neq 1)\\ &\quad+\overline{N}(r,b;f)+\overline{N}(r.b;g)+\overline{N}_0(r,0;f')+\overline{N}_0(r,0;g'),\end{aligned}$$

where $\overline{N}_0(r,0;f')$ denotes the reduced counting function corresponding to the zeros of f' which are not the zeros of $f(f-b)$ and $F-1$, and $\overline{N}_0(r,0;g')$ is defined similarly.

P r o o f. From (1.2) and (2.1) we have

$$F'=\frac{(n-2)af^{n-1}(f-b)^2f'}{n(n-1)(f-\alpha_1)^2(f-\alpha_2)^2}, \tag{2.3}$$

$$G'=\frac{(n-2)ag^{n-1}(g-b)^2g'}{n(n-1)(g-\alpha_1)^2(g-\alpha_2)^2}. \tag{2.4}$$

It is obvious that the simple zeros of $f-\alpha_1$ and $f-\alpha_2$ are the simple poles of F, the simple zeros of $g-\alpha_1$ and $g-\alpha_2$ are the simple poles of G. It can be easily verified that the simple zeros of $f-\alpha_1$, $f-\alpha_2$, $g-\alpha_1$ and $g-\alpha_2$ are not the poles of H.

We note that the multiple zeros of $f-\alpha_1$, $f-\alpha_2$ and $g-\alpha_1$, $g-\alpha_2$ are the zeros of f' and g' respectively. Also the poles of H come from those poles (zeros) of f and g whose multiplicities are different and those 1 points of F whose multiplicities are different from those of the corresponding 1 points of G. Since all the poles of H are simple, using Lemma 2.1 we get the conclusion of the lemma from (1.2), (2.3) and (2.4). □

Lemma 2.13. *Let F, G be given by* (2.1) *and let $H \not\equiv 0$. If $E_{m)}(1;F) = E_{m)}(1;G)$ and f, g share (∞,k), $(0,p)$, where $3 \leqslant m < \infty$, then*

$$\begin{aligned}\Big(\frac{n}{2}+1\Big)\{T(r,f)+T(r,g)\} &\leqslant \overline{N}(r,0;f)+2\overline{N}(r,b;f)+\overline{N}(r,\infty;f)+\overline{N}(r,0;g)\\ &+2\overline{N}(r,b;g)+\overline{N}(r,\infty;g)+\overline{N}_*(r,0;f,g)+\overline{N}_*(r,\infty;f,g)\\ &-\Big(\frac{m}{2}-\frac{3}{2}\Big)\{\overline{N}_{F\geqslant m+1}(r,1;F\mid G\neq 1)+\overline{N}_{G\geqslant m+1}(r,1;G\mid F\neq 1)\}\\ &-\Big(m-\frac{3}{2}\Big)\{\overline{N}_L^{m)}(r,1;F)+\overline{N}_L^{m)}(r,1;G)\}+S(r,f)+S(r,g).\end{aligned}$$

P r o o f. By the second fundamental theorem we get

$$\begin{aligned}&(n+1)T(r,f)+(n+1)T(r,g)\\ &\leqslant \overline{N}(r,1;F)+\overline{N}(r,0;f)+\overline{N}(r,b;f)+\overline{N}(r,\infty;f)+\overline{N}(r,1;G)+\overline{N}(r,0;g)\\ &\quad+\overline{N}(r,b;g)+\overline{N}(r,\infty;g)-N_0(r,0;f')-N_0(r,0;g')+S(r,f)+S(r,g).\end{aligned}\tag{2.5}$$

Using Lemmas 2.3 and 2.12, we see that

$$\begin{aligned}\overline{N}(r,1;F)+\overline{N}(r,1;G) &\leqslant \frac{1}{2}\,[N(r,1;F)+N(r,1;G)]+N(r,1;F\mid=1)\\ &-\Big(\frac{m}{2}-\frac{1}{2}\Big)\{\overline{N}_{F\geqslant m+1}(r,1;F\mid G\neq 1)+\overline{N}_{G\geqslant m+1}(r,1;G\mid F\neq 1)\}\\ &-\Big(m-\frac{1}{2}\Big)\Big\{\overline{N}_L^{m)}(r,1;F)+\overline{N}_L^{m)}(r,1;G)\Big\}\\ &\leqslant \frac{n}{2}\,\{T(r,f)+T(r,g)\}+\overline{N}_*(r,0;f,g)+\overline{N}_*(r,\infty;f,g)\\ &+\overline{N}(r,b;f)+\overline{N}(r,b;g)+\overline{N}_L^{m)}(r,1;F)+\overline{N}_L^{m)}(r,1;G)\\ &+\overline{N}_{F\geqslant m+1}(r,1;F\mid G\neq 1)+\overline{N}_{G\geqslant m+1}(r,1;G\mid F\neq 1)\\ &-\Big(\frac{m}{2}-\frac{1}{2}\Big)\{\overline{N}_{F\geqslant m+1}(r,1;F\mid G\neq 1)+\overline{N}_{G\geqslant m+1}(r,1;G\mid F\neq 1)\}\\ &-\Big(m-\frac{1}{2}\Big)\Big\{\overline{N}_L^{m)}(r,1;F)+\overline{N}_L^{m)}(r,1;G)\Big\}\\ &+\overline{N}_0(r,0;f')+\overline{N}_0(r,0;g')+S(r,f)+S(r,g)\\ &\leqslant \frac{n}{2}\,\{T(r,f)+T(r,g)\}+\overline{N}_*(r,0;f,g)+\overline{N}_*(r,\infty;f,g)\\ &+\overline{N}(r,b;f)+\overline{N}(r,b;g)\\ &-\Big(\frac{m}{2}-\frac{3}{2}\Big)\Big\{\overline{N}_{F\geqslant m+1}(r,1;F\mid G\neq 1)+\overline{N}_{G\geqslant m+1}(r,1;G\mid F\neq 1)\Big\}\\ &-\Big(m-\frac{3}{2}\Big)\Big\{\overline{N}_L^{m)}(r,1;F)+\overline{N}_L^{m)}(r,1;G)\Big\}+\overline{N}_0(r,0;f')\\ &+\overline{N}_0(r,0;g')+S(r,f)+S(r,g).\end{aligned}\tag{2.6}$$

Using (2.6) in (2.5) the lemma follows. $\square$

Lemma 2.14 [22]. *If $H \equiv 0$, then F, G share $(1, \infty)$.*

Lemma 2.15. *Let F, G be given by* (2.1) *and let $H \equiv 0$. If f, g share $(0, 0)$ then f and g share $(0, \infty)$.*

Proof. If f and g have no zero then clearly f and g share $(0, \infty)$.

Next suppose that f and g have common zeros. Since $H \equiv 0$ we have

$$F = \frac{AG + B}{CG + D}, \tag{2.7}$$

where $AD - BC \neq 0$. Let z_0 be a common zero of f and g. From (2.1) it is clear that z_0 is a common zero of F and G. Consequently, from (2.7) we get $B = 0$. Hence from (2.7) we get

$$F = \frac{AG}{CG + D}.$$

So F and G share $(0, \infty)$, that is, f and g share $(0, \infty)$. $\square$

3. Proofs of the theorems

Proof of Theorem 1.1. Let F and G be given by (2.1). Since $E_{5)}(S_3, f) = E_{5)}(S_3, f)$ it follows from (1.5) and (2.1) that $E_{5)}(1; F) = E_{5)}(1; G)$. Suppose $H \not\equiv 0$. Then by Lemma 2.13 for $m = 5$, $k = 0$, $p = 4$ we get

$$\begin{aligned} &\Big(\frac{n}{2} - 2\Big)\{T(r, f) + T(r, g)\} \\ &\leqslant \overline{N}(r, 0; f \mid\geqslant 5) + 3\overline{N}(r, \infty; f) - \{\overline{N}_{F\geqslant 6}(r, 1; F \mid G \neq 1) \\ &\quad + \overline{N}_{G\geqslant 6}(r, 1; G \mid F \neq 1)\} - \frac{7}{2}\Big\{\overline{N}_L^{5)}(r, 1; F) + \overline{N}_L^{5)}(r, 1; G)\Big\} \\ &\quad + S(r, f) + S(r, g). \end{aligned} \tag{3.1}$$

Using Lemma 2.4, Lemma 2.5, Lemma 2.7 for $m = 5$, $k = 0$ and $p = 4$, Lemma 2.9 for $k = 0$ and noting that $n \geqslant 5$,

$$\overline{N}_*(r, \infty; f, g) \leqslant \overline{N}(r, \infty; f)$$

and

$$\overline{N}(r, 0; f) \leqslant \frac{1}{2}[\overline{N}(r, 0; f) + \overline{N}(r, 0; g)]$$

we get

$$
\begin{aligned}
(3.2)\qquad & \\
\Big(\frac{n}{2}-2\Big)\{T(r,f)+T(r,g)\} &\leqslant \overline{N}(r,0;f\mid\geqslant 5)+\frac{3}{n-3}[\overline{N}_L^{5)}(r,1;F)+\overline{N}_L^{5)}(r,1;G) \\
&\quad+\overline{N}_{F\geqslant 6}(r,1;F\mid G\neq 1)+\overline{N}_{G\geqslant 6}(r,1;G\mid F\neq 1)+\overline{N}(r,0;f\mid\geqslant 5)] \\
&\quad-\{\overline{N}_{F\geqslant 6}(r,1;F\mid G\neq 1)+\overline{N}_{G\geqslant 6}(r,1;G\mid F\neq 1)\} \\
&\quad-\frac{7}{2}\{\overline{N}_L^{5)}(r,1;F)+\overline{N}_L^{5)}(r,1;G)\}+S(r,f)+S(r,g) \\
&\leqslant \frac{n}{n-3}\overline{N}(r,0;f\mid\geqslant 5)+\frac{3}{n-3}[\overline{N}_L^{5)}(r,1;F) \\
&\quad+\overline{N}_L^{5)}(r,1;G)+\overline{N}_{F\geqslant 6}(r,1;F\mid G\neq 1)+\overline{N}_{G\geqslant 6}(r,1;G\mid F\neq 1)] \\
&\quad-\{\overline{N}_{F\geqslant 6}(r,1;F\mid G\neq 1)+\overline{N}_{G\geqslant 6}(r,1;G\mid F\neq 1)\} \\
&\quad-\frac{7}{2}\{\overline{N}_L^{5)}(r,1;F)+\overline{N}_L^{5)}(r,1;G)\}+S(r,f)+S(r,g) \\
&\leqslant \frac{n}{(n-3)(5n-1)}\Big[\,2T(r,f)+2T(r,g)+\overline{N}_*(r,\infty;f,g) \\
&\quad+\frac{2}{5}\{\overline{N}(r,0;f)+\overline{N}(r,\infty;f)\}\Big] \\
&\quad+\frac{2(6-n)}{5(n-3)}\{\overline{N}(r,0;f)+\overline{N}(r,\infty;f)\}+S(r,f)+S(r,g) \\
&\leqslant \frac{n}{(n-3)(5n-1)}\Big[\,2T(r,f)+\frac{1}{5}\overline{N}(r,0;f)+2T(r,g) \\
&\quad+\frac{1}{5}\overline{N}(r,0;g)+\frac{7}{5}\overline{N}(r,\infty;f)\Big] \\
&\quad+\frac{(6-n)}{(n-3)}\Big\{\frac{1}{5}\{\overline{N}(r,0;f)+\overline{N}(r,0;g)\}+\frac{2}{5}\overline{N}(r,\infty;f)\Big\}+S(r,f)+S(r,g) \\
&\leqslant \frac{n}{(n-3)(5n-1)}\Big[\frac{11}{5}\{T(r,f)+T(r,g)\}+\frac{7}{5}\overline{N}(r,\infty;f)\Big] \\
&\quad+\frac{(6-n)}{(n-3)}\Big\{\frac{1}{5}\{T(r,f)+T(r,g)\}+\frac{2}{5}\overline{N}(r,\infty;f)\Big\}+S(r,f)+S(r,g)
\end{aligned}
$$

Now using Lemma 2.10 for $m=5$ in (3.2) we obtain

$$
\begin{aligned}
(3.3)\qquad \Big(\frac{n}{2}-2\Big)&\{T(r,f)+T(r,g)\} \\
&\leqslant \frac{n}{(n-3)(5n-1)}\Big[\Big\{\frac{11}{5}+\frac{49}{10(5n-17)}\Big\}\{T(r,f)+T(r,g)\}\Big] \\
&\quad+\frac{(6-n)}{(n-3)}\Big[\Big\{\frac{1}{5}+\frac{14}{10(5n-17)}\Big\}\{T(r,f)+T(r,g)\}\Big]+S(r,f)+S(r,g),
\end{aligned}
$$

i.e.,

$$\Big(\frac{n}{2}-2-\frac{n(22n-65)}{2(n-3)(5n-1)(5n-17)}-\frac{(n-2)(6-n)}{(n-3)(5n-17)}\Big)\{T(r,f)+T(r,g)\} \leqslant S(r,f)+S(r,g),$$

which is a contradiction. So $H\equiv 0$. Hence Lemma 2.14 and Lemma 2.15 imply respectively that F and G share $(1,\infty)$ and f, g share $(0,\infty)$. So $E_f(S_3,\infty) = E_g(S_3,\infty)$ and the theorem follows from Theorem A and Remark 1.1. □

P r o o f of Theorem 1.2. Let F and G be given by (2.1). Since $E_{4)}(S_3,f) = E_{4)}(S_3,f)$ it follows from (1.5) and (2.1) that $E_{4)}(1;F)=E_{4)}(1;G)$. Suppose $H\not\equiv 0$. Then by Lemma 2.13 for $m=4$, $k=0$, $p=\infty$ we get

$$\begin{aligned}\Big(\frac{n}{2}-2\Big)\{T(r,f)+T(r,g)\} &\leqslant 3\overline{N}(r,\infty;f)\\ &-\frac{1}{2}\{\overline{N}_{F\geqslant 5}(r,1;F\mid G\neq 1)+\overline{N}_{G\geqslant 5}(r,1;G\mid F\neq 1)\}\\ &-\frac{5}{2}\{\overline{N}_L^{4)}(r,1;F)+\overline{N}_L^{4)}(r,1;G)\}+S(r,f)+S(r,g).\end{aligned} \tag{3.4}$$

Using Lemma 2.5 and Lemma 2.9 for $k=0$, $p=\infty$ we obtain

$$\begin{aligned}&\Big(\frac{n}{2}-2\Big)\{T(r,f)+T(r,g)\}\\ &\leqslant \frac{3}{n-3}[\overline{N}_L^{4)}(r,1;F)+\overline{N}_L^{4)}(r,1;G)\\ &\quad+\overline{N}_{F\geqslant 5}(r,1;F\mid G\neq 1)+\overline{N}_{G\geqslant 5}(r,1;G\mid F\neq 1)]\\ &\quad-\frac{1}{2}\{\overline{N}_{F\geqslant 5}(r,1;F\mid G\neq 1)+\overline{N}_{G\geqslant 5}(r,1;G\mid F\neq 1)\}\\ &\quad-\frac{5}{2}\{\overline{N}_L^{4)}(r,1;F)+\overline{N}_L^{4)}(r,1;G)\}+S(r,f)+S(r,g)\\ &\leqslant \frac{(9-n)}{2(n-3)}\,\big[\overline{N}(r,0;f)+\overline{N}(r,\infty;f)\big]+S(r,f)+S(r,g).\end{aligned} \tag{3.5}$$

Now using Lemma 2.11 for $m=4$ in (3.5) we obtain

$$\begin{aligned}&\Big(\frac{n}{2}-2\Big)\{T(r,f)+T(r,g)\}\\ &\leqslant \frac{(9-n)}{2(n-3)}\Big[\Big\{\frac{1}{4}+\frac{1}{2(4n-14)}\Big\}\Big\{T(r,f)+T(r,g)\Big\}\Big]+S(r,f)+S(r,g),\end{aligned} \tag{3.6}$$

i.e.,

$$\Big(\frac{n}{2}-2-\frac{9-n}{8(n-3)}-\frac{9-n}{4(4n-14)(n-3)}\Big)\{T(r,f)+T(r,g)\}\leqslant S(r,f)+S(r,g),$$

which is a contradiction for $n\geqslant 5$. So $H\equiv 0$. Hence by Lemma 2.14 we get that F and G share $(1,\infty)$. Now the theorem follows from Theorem A and Remark 1.1. □

P r o o f of Theorem 1.3. Let F and G be given by (2.1). Since $E_{6)}(S_3, f) = E_{6)}(S_3, f)$ it follows from (1.5) and (2.1) that $E_{6)}(1; F) = E_{6)}(1; G)$. Suppose $H \not\equiv 0$. Then by Lemma 2.13 for $m = 6$, $k = 0$, $p = 2$ we get

$$
\begin{aligned}
(3.7)\qquad \Big(\frac{n}{2} - 2\Big)\{T(r,f) + T(r,g)\} &\leqslant \overline{N}(r, 0; f \mid\geqslant 3) + 3\overline{N}(r, \infty; f) \\
&- \frac{3}{2}\{\overline{N}_{F\geqslant 7}(r, 1; F \mid G \neq 1) + \overline{N}_{G\geqslant 7}(r, 1; G \mid F \neq 1)\} \\
&- \frac{9}{2}\{\overline{N}_L^{6)}(r, 1; F) + \overline{N}_L^{6)}(r, 1; G)\} + S(r,f) + S(r,g).
\end{aligned}
$$

Using Lemma 2.5, Lemma 2.7 for $p = 2$, Lemma 2.9 for $k = 0$ we obtain

$$
\begin{aligned}
(3.8)\quad &\Big(\frac{n}{2} - 2\Big)\{T(r,f) + T(r,g)\} \\
&\leqslant \overline{N}(r, 0; f \mid\geqslant 3) + \frac{3}{n-3}[\overline{N}_L^{6)}(r, 1; F) + \overline{N}_L^{6)}(r, 1; G) \\
&\quad + \overline{N}_{F\geqslant 7}(r, 1; F \mid G \neq 1) + \overline{N}_{G\geqslant 7}(r, 1; G \mid F \neq 1) + \overline{N}(r, 0; f \mid\geqslant 3)] \\
&\quad - \frac{3}{2}\{\overline{N}_{F\geqslant 7}(r, 1; F \mid G \neq 1) + \overline{N}_{G\geqslant 7}(r, 1; G \mid F \neq 1)\} \\
&\quad - \frac{9}{2}\{\overline{N}_L^{6)}(r, 1; F) + \overline{N}_L^{6)}(r, 1; G)\} + S(r,f) + S(r,g) \\
&\leqslant \frac{n}{(n-3)(3n-1)}[2T(r,f) + 2T(r,g) + \overline{N}_*(r, \infty; f, g) \\
&\quad + \frac{1}{3}\{\overline{N}(r, 0; f) + \overline{N}(r, \infty; f)\}] + S(r,f) + S(r,g) \\
&\leqslant \frac{n}{(n-3)(3n-1)}\Big[\frac{13}{6}\{T(r,f) + T(r,g)\} + \frac{4}{3}\overline{N}(r, \infty; f)\Big] + S(r,f) + S(r,g)
\end{aligned}
$$

Now using Lemma 2.10 for $m = 6$ in (3.8) we obtain

$$
\begin{aligned}
(3.9)\qquad &\Big(\frac{n}{2} - 2\Big)\{T(r,f) + T(r,g)\} \\
&\leqslant \frac{n}{(n-3)(3n-1)}\Big[\Big\{\frac{13}{6} + \frac{16}{3(6n-20)}\Big\}\{T(r,f) + T(r,g)\}\Big] \\
&\quad + S(r,f) + S(r,g),
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
\Big(\frac{n}{2} - 2 - \frac{13n}{6(n-3)(3n-1)} - \frac{16n}{3(n-3)(3n-1)(6n-20)}\Big)\{T(r,f) + T(r,g)\} \\
\leqslant S(r,f) + S(r,g),
\end{aligned}
$$

which is a contradiction for $n \geqslant 5$. So $H \equiv 0$. Hence Lemma 2.14 and Lemma 2.15 imply respectively that F and G share $(1, \infty)$ and f, g share $(0, \infty)$. So $E_f(S_3, \infty) = E_g(S_3, \infty)$ and the theorem follows from Theorem A and Remark 1.1. $\square$

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