Ladislav Matejíčka

Some remarks on I-faster convergent infinite series

Mathematica Bohemica, Vol. 134 (2009), No. 3, 275–284

Persistent URL: http://dml.cz/dmlcz/140661

SOME REMARKS ON I-FASTER CONVERGENT INFINITE SERIES

Ladislav Matejíčka, Púchov

(Received May 13, 2008)

Abstract. A structure of terms of I-faster convergent series is studied in the paper. Necessary and sufficient conditions for the existence of I-faster convergent series with different types of their terms are proved. Some consequences are discussed.

Keywords: I-convergence, I-faster convergent series, terms of I-convergent series

MSC 2010: 65B10, 40A10

1. Introduction

Many papers have been devoted to the study of methods for faster convergence of sequences, of course also to the acceleration of convergence of sequence of partial sums of infinite series [2], [7], [10], [11], [14]. One of the methods how to accelerate the convergence of the sequence of partial sums of an infinite series consists in the substitution of the given series by another faster convergent series with the same sum. This way was suggested by Kummer. Knowledge of the structure of terms of faster convergent series plays an important role in the study of Kummer's series. In the paper [5] it is proved that under certain conditions a series $\sum\limits_{n=1}^{\infty} a_n$ is faster convergent than a series $\sum\limits_{n=1}^{\infty} b_n$ if and only if $\lim\limits_{n\to\infty} a_n/b_n = 0$. A similar result is proved for Kummer's series and for I-faster convergent series. More about I-convergence is in [1], [4], [8], [9]. Statistically faster convergent sequences as a special case of I-convergent sequences are studied in [3], [12], [13]. In the present paper we study the existence of I-convergent series $\sum\limits_{n=1}^{\infty} a_n$ I-faster convergent than I-convergent series $\sum\limits_{n=1}^{\infty} b_n$ with $I\text{-}\lim\limits_{n\to\infty} a_n/b_n = c \neq 0$ or $I\text{-}\lim\limits_{n\to\infty} a_n/b_n$ not existing. Results similar to those in [6] for I-convergent series are proved.

We denote by $\mathbb{N}$ the set of all positive integers and by $\mathbb{R}$ the set of all real numbers. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers. In what follows, if we say that $I\text{-}\lim\limits_{n\to\infty} a_n$ exists, we admit also the cases $I\text{-}\lim\limits_{n\to\infty} a_n = +\infty\ (-\infty)$. We will suppose that the terms of all infinite series are real nonzero numbers.

2. Basic definitions and concepts

In what follows we will suppose that I is an admissible ideal of subsets of $\mathbb{N}$.

Definition 1 [4]. We say that I is an admissible ideal of subsets of $\mathbb{N}$ if it satisfies:
(a) if $A, B \in I$, then $A \cup B \in I$ (additivity),
(b) if $B \subset A \in I$, then $B \in I$ (heredity),
(c) I contains all singletons and does not contain $\mathbb{N}$.

Definition 2 [2]. Let $\sum\limits_{n=1}^{\infty} a_n$, $\sum\limits_{n=1}^{\infty} b_n$ be convergent series such that $b_n + b_{n+1} + \ldots \neq 0$, $n \in \mathbb{N}$. Then the series $\sum\limits_{n=1}^{\infty} a_n$ is called faster convergent than $\sum\limits_{n=1}^{\infty} b_n$ if $\lim\limits_{n\to\infty} (a_n + a_{n+1} + \ldots)/(b_n + b_{n+1} + \ldots) = 0$.

Definition 3 [4]. We say that a sequence $\{a_n;\ n \in \mathbb{N}\}$ has the I-limit equal to a real number L and we write $I\text{-}\lim\limits_{n\to\infty} a_n{=}L$, if for each $\varepsilon > 0$ the set $A(\varepsilon) = \{n;\ |a_n - L| \geqslant \varepsilon\}$ belongs to the ideal I.

Definition 4 [4]. We say that $\sum\limits_{n=1}^{\infty} a_n$ I-converges to a real number L and we write $I\text{-}\sum\limits_{n=1}^{\infty} a_n{=}L$ if for each $\varepsilon > 0$ the set $A(\varepsilon) = \Big\{n;\ \Big|\sum\limits_{k=1}^{n} a_k - L\Big| \geqslant \varepsilon\Big\}$ belongs to the ideal I.

Definition 5 [5]. Let $\sum\limits_{n=1}^{\infty} a_n$, $\sum\limits_{n=1}^{\infty} b_n$ be I-convergent series such that $b_n + b_{n+1} + \ldots \neq 0$, $n \in \mathbb{N}$. A series $\sum\limits_{n=1}^{\infty} a_n$ is called I-faster convergent than $\sum\limits_{n=1}^{\infty} b_n$ if $I\text{-}\lim\limits_{n\to\infty} (a_{n+1} + a_{n+2} + \ldots)/(b_{n+1} + b_{n+2} + \ldots) = 0$.

Remark. The expression $b_n + b_{n+1} + \ldots$, $n \in \mathbb{N}$, $n > 1$ means $s - b_1 - b_2 - \ldots - b_{n-1}$, where $s = I\text{-}\sum\limits_{n=1}^{\infty} a_n$, I being an ideal.

In what follows we will write "I-fcst" instead of "I-faster convergent series than".

Definition 6 [9]. Let I be an ideal. A number $x \in \mathbb{R}$ is said to be an I-cluster point of $\sum\limits_{n=1}^{\infty} x_n$ if for each $\varepsilon > 0$ the set $\Big\{n \in \mathbb{N};\ \Big|\sum\limits_{k=1}^{n} x_k - x\Big| < \varepsilon\Big\}$ is not from I.

3. I-Faster convergent series

Let $\sum_{n=1}^{\infty} b_n$ be an I-convergent series such that $B_n = b_n + b_{n+1} + \ldots \neq 0$, $n \in \mathbb{N}$. It is easy to see that there exists $\sum_{n=1}^{\infty} a_n$ I-fcst $\sum_{n=1}^{\infty} b_n$. Namely, if we put $A_n = B_n^3$, $n \in \mathbb{N}$ then $I\text{-}\lim_{n\to\infty} A_{n+1}/B_{n+1} = 0$ and $A_n \neq A_{n+1}$, $n \in \mathbb{N}$. Put $a_n = A_n - A_{n+1}$, $n \in \mathbb{N}$, then $\sum_{n=1}^{\infty} a_n$ is I-fcst $\sum_{n=1}^{\infty} b_n$.

Lemma 1. *Let $\sum_{n=1}^{\infty} b_n$ be an I-convergent series such that $B_n = b_n + b_{n+1} + \ldots \neq 0$, $n \in \mathbb{N}$. Then there exists $\sum_{n=1}^{\infty} a_n$ I-fcst $\sum_{n=1}^{\infty} b_n$ such that $I\text{-}\lim_{n\to\infty} a_{n+1}/b_{n+1} = 0$.*

Proof. Define $\{\gamma_n\}_{n=1}^{\infty}$, $\gamma_n = (B_n - B_{n+1})/B_n$, $n \in \mathbb{N}$. Let $\{r_n\}_{n=1}^{\infty}$, $\{\alpha_n\}_{n=1}^{\infty}$ be sequences such that $|r_n| = \min\{1/n^2, 1/n^2|\gamma_n|\}$, $r_n\gamma_n > 0$, $n \in \mathbb{N}$ and $\alpha_{n+1} = \frac{1}{2}\min\{\alpha_n, |B_n/B_{n+1}|\alpha_n, n^{-1}\min\{|\gamma_{n+1}|, |\gamma_n|\}\}$, $n \in \mathbb{N}$, $\alpha_1 > 0$. This implies $\lim_{n\to\infty} r_n = 0$. Denote $\varepsilon_n = \sum_{k=n}^{\infty} r_k\gamma_k + p\alpha_n$, $n \in \mathbb{N}$ where $p \geqslant 0$ and such that $\varepsilon_n B_n \neq \varepsilon_{n+1}B_{n+1}$, $n \in \mathbb{N}$. Such p exists because the number of identities $\varepsilon_n B_n = \varepsilon_{n+1}B_{n+1}$, $n \in \mathbb{N}$ is countable. Since $\alpha_{n+1} < |B_n/B_{n+1}|\alpha_n$ we have $\alpha_n B_n \neq \alpha_{n+1}B_{n+1}$, $n \in \mathbb{N}$. We also get $\lim_{n\to\infty} \varepsilon_n = 0$ and $\varepsilon_n - \varepsilon_{n+1} = r_n\gamma_n + p(\alpha_n - \alpha_{n+1})$. Put $A_n = \varepsilon_n B_n$, $a_n = A_n - A_{n+1}$, $n \in \mathbb{N}$. This implies $I\text{-}\lim_{n\to\infty} A_{n+1} = 0$, so $\sum_{n=1}^{\infty} a_n$ is an I-convergent series and I-fcst $\sum_{n=1}^{\infty} b_n$. From

$$
\begin{aligned}
\Big|\frac{a_{n+1}}{b_{n+1}}\Big| &= \Big|\frac{A_{n+1} - A_{n+2}}{B_{n+1} - B_{n+2}}\Big| = \Big|\frac{B_{n+1}}{B_{n+1} - B_{n+2}}(\varepsilon_{n+1} - \varepsilon_{n+2}) + \varepsilon_{n+2}\Big| \\
&= \Big|r_{n+1} + p\frac{\alpha_{n+1} - \alpha_{n+2}}{\gamma_{n+1}} + \varepsilon_{n+2}\Big| \leqslant |r_{n+1}| + p\,\frac{|\alpha_{n+1}| + |\alpha_{n+2}|}{|\gamma_{n+1}|} + |\varepsilon_{n+2}| \\
&\leqslant |r_{n+1}| + \frac{2p}{n} + |\varepsilon_{n+2}|
\end{aligned}
$$

we get $\lim_{n\to\infty} a_{n+1}/b_{n+1} = 0$ and so $I\text{-}\lim_{n\to\infty} a_{n+1}/b_{n+1} = 0$. □

Lemma 2. *Let $\sum_{n=1}^{\infty} b_n$ be an I-convergent series such that $b_n + b_{n+1} + \ldots \neq 0$, $n \in \mathbb{N}$. Let $\{r_n\}_{n=1}^{\infty}$ be a nonzero sequence such that $I\text{-}\lim_{n\to\infty} r_{n+1} = r \in \mathbb{R} \setminus \{0\}$ and $\sum_{n=1}^{\infty} b_n r_n/(b_{n+1} + b_{n+2} + \ldots)$ is an I-convergent series. Then for every $c \in \mathbb{R} \setminus \{0\}$*

there exists an I-convergent series $\sum\limits_{n=1}^{\infty} a_n$ which is I-fcst $\sum\limits_{n=1}^{\infty} b_n$ and such that I-$\lim\limits_{n\to\infty} a_{n+1}/b_{n+1} = c$.

P r o o f. Suppose that there exists a nonzero sequence $\{r_n\}_{n=1}^{\infty}$ such that I-$\lim\limits_{n\to\infty} r_{n+1} = r \in \mathbb{R} \setminus \{0\}$ and $\sum\limits_{n=1}^{\infty} b_n r_n/(b_{n+1} + b_{n+2} + \ldots)$ is an I-convergent series. Denote $B_n = b_n + b_{n+1} + \ldots$, $\gamma_n = (B_n - B_{n+1})/B_n$, $\varepsilon_n = \sum\limits_{j=n}^{\infty} (B_j - B_{j+1}) r_j c/rB_{j+1}$, $n \in \mathbb{N}$. By induction we construct $\{D_n\}_{n=1}^{\infty}$ such that

$$0 < D_{n+1} < D_n,\ D_n < \min\Big\{\frac{1}{n^2}, \frac{|\gamma_n|}{n^2}\Big\},\ n \in \mathbb{N}, \tag{1}$$

$$D_{n-1} \neq (1+\gamma_n) D_n, \quad n > 1. \tag{2}$$

Choose D_1 such that $0 < D_1 < \min\{1, |\gamma_1|\}$. By the continuity of $f_1(x) = \min\{1, |\gamma_1|\} - |D_1 - x|$ at 0 there exists a neighbourhood O_1 of 0 such that if $x \in O_1$ then $|D_1 - x| < \min\{1, |\gamma_1|\}$. It follows that there is $D_2 \in O_1$, $0 < D_2 < D_1$, $D_2 < \min\{1/2^2, |\gamma_2|/2^2\}$, $|D_1 - D_2| < \min\{1, |\gamma_1|\}$, $D_1 \neq (1+\gamma_1) D_2$. Suppose $D_1, D_2, \ldots, D_n$ with (1), (2) are constructed. By the continuity of $f_n(x) = \min\{1/n^2, |\gamma_n|/n^2\} - |D_n - x|$ at 0 there exists D_{n+1} such that $0 < D_{n+1} < D_n$, $D_{n+1} < \min\{1/(n+1)^2, |\gamma_{n+1}|/(n+1)^2\}$, $|D_n - D_{n+1}| < \min\{1/n^2, |\gamma_n|/n^2\}$, $D_n \neq (1+\gamma_n) D_{n+1}$. It is evident that $\lim\limits_{n\to\infty} D_n = 0$. Denote $\varepsilon_n^\star = \varepsilon_n + D_n q$, $A_n = \varepsilon_n^\star B_n$, $n \in \mathbb{N}$ where $q \neq 0$, $q \in \mathbb{R}$ is such that $q(D_n B_n - D_{n+1} B_{n+1}) \neq \varepsilon_{n+1} B_{n+1} - \varepsilon_n B_n$, $\varepsilon_n^\star \neq 0$, $n \in \mathbb{N}$ (such q exists because the number of conditions for q is countable). From $\varepsilon_n^\star B_n - \varepsilon_{n+1}^\star B_{n+1} = \varepsilon_n B_n - \varepsilon_{n+1} B_{n+1} + q(D_n B_n - D_{n+1} B_{n+1})$ it follows that $A_n \neq A_{n+1}$, $n \in \mathbb{N}$. It is evident that I-$\lim\limits_{n\to\infty} \varepsilon_{n+1}^\star = 0$ and this implies I-$\lim\limits_{n\to\infty} A_{n+1} = 0$, I-$\lim\limits_{n\to\infty} A_{n+1}/B_{n+1} = 0$. From

$$\frac{A_{n+1} - A_{n+2}}{B_{n+1} - B_{n+2}} = \frac{\varepsilon_{n+1}^\star B_{n+1} - \varepsilon_{n+2}^\star B_{n+2}}{B_{n+1} - B_{n+2}} = \varepsilon_{n+1}^\star + \frac{B_{n+2}}{B_{n+1} - B_{n+2}} (\varepsilon_{n+1}^\star - \varepsilon_{n+2}^\star)$$

and from

$$\frac{B_{n+2}}{B_{n+1} - B_{n+2}} (\varepsilon_{n+1}^\star - \varepsilon_{n+2}^\star) = \frac{\varepsilon_{n+1} - \varepsilon_{n+2}}{\gamma_{n+1}} + \frac{q(D_{n+1} - D_{n+2})}{\gamma_{n+1}},$$

$$|D_{n+1} - D_{n+2}| \leqslant \frac{|\gamma_{n+1}|}{n^2}, \qquad \varepsilon_{n+1} - \varepsilon_{n+2} = \frac{\gamma_{n+1} r_{n+1} c}{r}$$

we obtain I-$\lim\limits_{n\to\infty} (A_{n+1} - A_{n+2})/(B_{n+1} - B_{n+2}) = c$. Put $a_n = A_n - A_{n+1}$, $n \in \mathbb{N}$. The proof is complete. $\square$

Lemma 3. *Let* $\sum_{n=1}^{\infty} b_n$ *be an* I*-convergent series such that* $b_n + b_{n+1} + \ldots \neq 0$, $n \in \mathbb{N}$. *If for some* $c \in \mathbb{R}\setminus\{0\}$ *there exists an* I*-convergent series* $\sum_{n=1}^{\infty} a_n$ *which is* I*-fcst* $\sum_{n=1}^{\infty} b_n$ *such that* $I\text{-}\lim_{n\to\infty} a_{n+1}/b_{n+1} = c$ *then there exists a sequence* $\{r_n\}_{n=1}^{\infty}$, $r_n \in \mathbb{R}\setminus\{0\}$, $I\text{-}\lim_{n\to\infty} r_{n+1} \neq 0$ *such that* $\sum_{n=1}^{\infty} b_n r_n/(b_{n+1} + b_{n+2} + \ldots)$ *is an* I*-convergent series.*

Proof. Suppose $\sum_{n=1}^{\infty} b_n$ is an I-convergent series such that $B_n = b_n + b_{n+1} + \ldots \neq 0$, $n \in \mathbb{N}$. Let $c \in \mathbb{R}\setminus\{0\}$ and let $\sum_{n=1}^{\infty} a_n$ be an I-convergent series which is I-fcst $\sum_{n=1}^{\infty} b_n$ such that $I\text{-}\lim_{n\to\infty} a_{n+1}/b_{n+1} = c$. Denote $A_n = a_n + a_{n+1} + \ldots$, $n \in \mathbb{N}$. From $a_{n+1}/b_{n+1} = (A_{n+1} - A_{n+2})/(B_{n+1} - B_{n+2})$ it follows that $I\text{-}\lim_{n\to\infty} (A_{n+1} - A_{n+2})/(B_{n+1} - B_{n+2}) = c \neq 0$. Since $\sum_{n=1}^{\infty} a_n$ is I-fcst $\sum_{n=1}^{\infty} b_n$ we have $I\text{-}\lim_{n\to\infty} A_{n+1}/B_{n+1} = 0$. Denote $\varepsilon_n = A_n/B_n$, $n \in \mathbb{N}$, then $(A_{n+1} - A_{n+2})/(B_{n+1} - B_{n+2}) = \varepsilon_{n+1} + B_{n+2}/(B_{n+1} - B_{n+2})(\varepsilon_{n+1} - \varepsilon_{n+2})$. Put $r_n = B_{n+1}/(B_n - B_{n+1})(\varepsilon_n - \varepsilon_{n+1} + \alpha_n)$, $n \in \mathbb{N}$ where

$$0 < \alpha_n < \min\Big\{\Big|\frac{B_n - B_{n+1}}{B_{n+1}}\Big|\frac{1}{n^2}, \frac{1}{n^2}\Big\}$$

are such that $\varepsilon_n - \varepsilon_{n+1} + \alpha_n \neq 0$ for $n \in \mathbb{N}$. Then $r_n \neq 0$, $n \in \mathbb{N}$ and $I\text{-}\lim_{n\to\infty} r_{n+1} = c \neq 0$. From $\varepsilon_{n+1} = \varepsilon_n + \alpha_n - r_n((B_n - B_{n+1})/B_{n+1})$, $n \in \mathbb{N}$ we get $\varepsilon_{n+1} = \varepsilon_1 - \sum_{j=1}^{n} r_j((B_j - B_{j+1})/B_{j+1}) + \sum_{j=1}^{n} \alpha_j$ and this implies the I-convergence of $\sum_{j=1}^{\infty} r_j((B_j - B_{j+1})/B_{j+1})$. □

Definition 7. We say that $\sum_{n=1}^{\infty} a_n$ I-converges to $+\infty$ $(-\infty)$ and we write $I\text{-}\sum_{n=1}^{\infty} a_n = +\infty$ $(-\infty)$ if for each $K > 0$ the set $A(K) = \Big\{n;\ \sum_{k=1}^{n} a_k \leqslant K\Big\}$ $\Big(A(K) = \Big\{n;\ \sum_{k=1}^{n} a_k \geqslant -K\Big\}\Big)$ belongs to the ideal I.

Lemma 4. *Let* $\sum_{n=1}^{\infty} b_n$ *be an* I*-convergent series such that* $b_n + b_{n+1} + \ldots \neq 0$, $n \in \mathbb{N}$. *Let* $\{r_n\}_{n=1}^{\infty}$, *be a nonzero sequence such that* $I\text{-}\lim_{n\to\infty} r_{n+1} = +\infty$ $(I\text{-}\lim_{n\to\infty} r_{n+1} = -\infty)$ *and let* $\sum_{n=1}^{\infty} b_n r_n/(b_{n+1} + b_{n+2} + \ldots)$ *be an* I*-convergent series. Then there* *exists an I-convergent series $\sum\limits_{n=1}^{\infty} a_n$ I-fcst $\sum\limits_{n=1}^{\infty} b_n$ such that $I\text{-}\lim\limits_{n\to\infty} a_{n+1}/b_{n+1} = +\infty$ ($I\text{-}\lim\limits_{n\to\infty} a_{n+1}/b_{n+1} = -\infty$).*

P r o o f. If we put $\varepsilon_n = \sum\limits_{j=n}^{\infty} (B_j - B_{j+1}) r_j / B_{j+1}$, $n \in \mathbb{N}$ then the proof of this lemma is similar to the proof of Lemma 2. $\square$

Lemma 5. *Let $\sum\limits_{n=1}^{\infty} a_n$, $\sum\limits_{n=1}^{\infty} b_n$ be I-convergent series such that $b_n + b_{n+1} + \ldots \neq 0$, $n \in \mathbb{N}$. Let $\sum\limits_{n=1}^{\infty} a_n$ be I-fcst $\sum\limits_{n=1}^{\infty} b_n$ such that $I\text{-}\lim\limits_{n\to\infty} a_{n+1}/b_{n+1} = +\infty$ ($I\text{-}\lim\limits_{n\to\infty} a_{n+1}/b_{n+1} = -\infty$). Then there exists a nonzero sequence $\{r_n\}_{n=1}^{\infty}$ such that $I\text{-}\lim\limits_{n\to\infty} r_{n+1} = +\infty$ ($I\text{-}\lim\limits_{n\to\infty} r_{n+1} = -\infty$) and $\sum\limits_{n=1}^{\infty} b_n r_n / (b_{n+1} + b_{n+2} + \ldots)$ is an I-convergent series.*

P r o o f. The proof is similar to that of Lemma 3. $\square$

Definition 8. We say that I has the property (sh) if the following implication holds:

(sh) If $M \in I$ then $M + 1 = \{n + 1;\ n \in M\} \in I$ and $M - 1 = \{n \in \mathbb{N};\ n = m - 1$, for some $m \in M\} \in I$.

Lemma 6. *The following assertions are equivalent:*

(a) *I has the property* (sh).

(b) *For all $\sum\limits_{n=1}^{\infty} a_n$ I-convergent series we have $I\text{-}\lim\limits_{n\to\infty} a_n = I\text{-}\lim\limits_{n\to\infty} a_{n+1} = 0$.*

P r o o f. (a)$\Rightarrow$(b). Let I be an admissible ideal with (sh). Let $\sum\limits_{n=1}^{\infty} a_n$ be an I-convergent series to the number $a \in \mathbb{R}$. For every $\varepsilon > 0$ we have $M_\varepsilon = \{n \in \mathbb{N};\ |s_n - a| \geqslant \varepsilon\} \in I$, $s_n = a_1 + a_2 + \ldots + a_n$, $n \in \mathbb{N}$. Denote $A_{n+1} = a - s_n$, $n \in \mathbb{N}$, $A_1 = I - \lim\limits_{n\to\infty} s_n$. From (sh) it follows that $M_\varepsilon - 1 \in I$ and $M_\varepsilon + 1 \in I$. So $\{n \in \mathbb{N};\ |s_n - a| \geqslant \varepsilon\} \in I$, $\{n \in \mathbb{N};\ |s_{n+1} - a| \geqslant \varepsilon\} \in I$, $\{n \in \mathbb{N};\ |s_{n-1} - a| \geqslant \varepsilon\} \in I$. Consequently $I\text{-}\lim\limits_{n\to\infty} A_n = 0$, $I\text{-}\lim\limits_{n\to\infty} A_{n+1} = 0$, $I\text{-}\lim\limits_{n\to\infty} A_{n+2} = 0$. This implies $I\text{-}\lim\limits_{n\to\infty} a_n = I\text{-}\lim\limits_{n\to\infty} (A_n - A_{n+1}) = 0$, $I\text{-}\lim\limits_{n\to\infty} a_{n+1} = I\text{-}\lim\limits_{n\to\infty} (A_{n+1} - A_{n+2}) = 0$.

(b)$\Rightarrow$(a). Suppose (b) is true. Let M be an infinite set from I. Put $A_1 = 1$, and

$$A_{m+1} = \begin{cases} \dfrac{1}{m+1} & \text{for} \quad m \in \mathbb{N},\ m \notin M, \\[2ex] 1 + \dfrac{1}{m+1} & \text{for} \quad m \in M. \end{cases}$$

We have $\{n \in \mathbb{N};\ |A_{n+1}| \geqslant \varepsilon\} = B_\varepsilon \cup M$, where B_ε is a finite subset of $\mathbb{N}$ for every ε such that $0 < \varepsilon < 1$. Because of $A_n \neq A_{n+1}$ for $n \in \mathbb{N}$ the series $\sum\limits_{n=1}^{\infty}(A_n - A_{n+1})$ is a nonzero I-convergent series. From (b) we have $I\text{-}\lim\limits_{n\to\infty}(A_n - A_{n+1}) = 0$, $I\text{-}\lim\limits_{n\to\infty}(A_{n+1} - A_{n+2}) = 0$. If $M + 1 \notin I$ then $R = ((M + 1) \setminus M) \notin I$. If $r \in R \setminus \{1\}$ then $r \notin M$ and $r = m + 1$ for some $m \in M$, so $r - 1 \in M$. Thus implies $|A_r - A_{r+1}| = |1 + 1/r - 1/(r + 1)| > 1$ and so $r \in \{n \in \mathbb{N};\ |A_n - A_{n+1}| \geqslant 1\}$. Hence $R \in I$, a contradiction. So $M + 1 \in I$. If $M - 1 \notin I$ then from $M \in I$ we get $S = ((M - 1) \setminus M) \notin I$. If $s \in S$ then $s \notin M$ and from $s = m - 1$, $m \in M$ we have $s + 1 \in M$. So $|A_{s+1} - A_{s+2}| = |1 - 1/(s + 1)(s + 2)| \geqslant \frac{5}{6}$. Consequently $s \in \{n \in \mathbb{N};\ |A_{n+1} - A_{n+2}| \geqslant \frac{5}{6}\}$. This implies $S \in I$, a contradiction. □

Definition 9 [8]**.** We say that I has the property (AP) if the following condition is fulfilled:

(AP) For any countable family $\{A_i;\ i \in \mathbb{N}\}$ of mutually disjoint sets ($A_i \cap A_j = \emptyset$, $i \neq j$) from I there exists a countable family $\{B_i;\ i \in \mathbb{N}\}$ such that the symmetric difference $A_i \triangle B_i$ is finite for every $i \in \mathbb{N}$ and $B = \bigcup\limits_{i=1}^{\infty} B_i$ belongs to I.

Lemma 7. *Let I have the property* (sh)*. Let $\sum\limits_{n=1}^{\infty} b_n$ be an I-convergent series such that $b_n + b_{n+1} + \ldots \neq 0$, $n \in \mathbb{N}$. Let $\sum\limits_{n=1}^{\infty} a_n$ be an I-convergent series I-fcst $\sum\limits_{n=1}^{\infty} b_n$ such that $I\text{-}\lim\limits_{n\to\infty} a_{n+1}/b_{n+1}$ does not exist. Then $I\text{-}\liminf\limits_{n\to\infty} |b_{n+1}/(b_{n+1} + b_{n+2} + \ldots)| = 0$.*

P r o o f. Suppose I has the property (sh). This implies that if $\{\alpha_n\}_{n=1}^{\infty}$ is a sequence such that $I\text{-}\lim\limits_{n\to\infty} \alpha_n = \alpha$, $\alpha \in R_0 = \mathbb{R} \cup \{+\infty, -\infty\}$ then $I\text{-}\lim\limits_{n\to\infty} \alpha_{n+1} = \alpha$, $I\text{-}\lim\limits_{n\to\infty} \alpha_{n-1} = \alpha$. Let $\sum\limits_{n=1}^{\infty} a_n$, $\sum\limits_{n=1}^{\infty} b_n$ be I-convergent series such that $B_n = b_n + b_{n+1} + \ldots \neq 0$, $n \in \mathbb{N}$ and let $\sum\limits_{n=1}^{\infty} a_n$ be I-fcst $\sum\limits_{n=1}^{\infty} b_n$. Assume $I\text{-}\lim\limits_{n\to\infty} a_{n+1}/b_{n+1}$ does not exist. Denote $\gamma_n = B_n/(B_n - B_{n+1})$, $A_n = a_n + a_{n+1} + \ldots$, $n \in \mathbb{N}$. Since $I\text{-}\lim\limits_{n\to\infty}(a_{n+1} + a_{n+2} + \ldots)/(b_{n+1} + b_{n+2} + \ldots) = \lim\limits_{n\to\infty} A_{n+1}/B_{n+1} = 0$, $a_{n+1}/b_{n+1} = (A_{n+1} - A_{n+2})/(B_{n+1} - B_{n+2}) = A_{n+2}/B_{n+2} + \gamma_{n+1}(A_{n+1}/B_{n+1} - A_{n+2}/B_{n+2})$ and the $I\text{-}\lim\limits_{n\to\infty} a_{n+1}/b_{n+1}$ does not exist, the sequence $\{\gamma_{n+1}\}_{n=1}^{\infty}$ is not I-bounded (see the proof of Lemma 3.9 [5]), i.e., for arbitrary $K \in \mathbb{R}$ we have $\{n \in \mathbb{N};\ |\gamma_{n+1}| > K\} \notin I$. This implies $I\text{-}\liminf\limits_{n\to\infty} |1/\gamma_{n+1}| = I\text{-}\liminf\limits_{n\to\infty} |b_{n+1}/(b_{n+1} + b_{n+2} + \ldots)| = 0$.

□

Lemma 8. *Let I have properties* (sh), (AP). *Let $\sum\limits_{n=1}^{\infty} b_n$ be I-convergent series such that $b_n + b_{n+1} + \ldots \neq 0$, $n \in \mathbb{N}$. Let $I\text{-}\lim\limits_{n\to\infty} |b_{n+1}/(b_{n+1} + b_{n+2} + \ldots)| = 0$. Then there exists $\sum\limits_{n=1}^{\infty} a_n$ I-fcst $\sum\limits_{n=1}^{\infty} b_n$ such that $I\text{-}\lim\limits_{n\to\infty} a_{n+1}/b_{n+1}$ does not exist.*

Proof. Suppose I has properties (sh), (AP). Let $\sum\limits_{n=1}^{\infty} b_n$ be an I-convergent series such that $B_n = b_n+b_{n+1}+\ldots \neq 0$, $n \in \mathbb{N}$ and $I\text{-}\lim\limits_{n\to\infty} |b_{n+1}/(b_{n+1} + b_{n+2} + \ldots)| = 0$. Denote $\gamma_n = B_n/(B_n - B_{n+1})$, $n \in \mathbb{N}$. (sh) implies $I\text{-}\lim\limits_{n\to\infty} 1/|\gamma_{n+1}| = I\text{-}\lim\limits_{n\to\infty} 1/|\gamma_n| = 0$ and so $I\text{-}\lim\limits_{n\to\infty} |\gamma_n| = +\infty$. From $\{n \in \mathbb{N};\ |\gamma_n| \leqslant 1\} \in I$ we get $K = \{n \in \mathbb{N};\ \gamma_n < -1\} \notin I$ or $T = \{n \in \mathbb{N};\ \gamma_n > 1\} \notin I$. There are two cases. First, $K \in I$ or $T \in I$ and thus $I\text{-}\lim\limits_{n\to\infty} \gamma_n = +\infty$ or $I\text{-}\lim\limits_{n\to\infty} \gamma_n = -\infty$. Second, $K \notin I$ and $T \notin I$ and thus $I\text{-}\lim\limits_{n\in K, n\to\infty} \gamma_n = -\infty$, $I\text{-}\lim\limits_{n\in T, n\to\infty} \gamma_n = +\infty$. From definition, $K \cap T = \emptyset$, $\mathbb{N} \setminus (K \cup T) \in I$. Consider the first case $I\text{-}\lim\limits_{n\to\infty} \gamma_n = +\infty$ or $I\text{-}\lim\limits_{n\to\infty} \gamma_n = -\infty$. Since the proof of the case $I\text{-}\lim\limits_{n\to\infty} \gamma_n = -\infty$ is similar to that of the case $I\text{-}\lim\limits_{n\to\infty} \gamma_n = +\infty$ we will suppose that $I\text{-}\lim\limits_{n\to\infty} \gamma_n = +\infty$. From (AP) (similarly to [8]) we get that there exists $M \subset N$, $M \notin I$, $N \setminus M \in I$ such that $\lim\limits_{n\in M, n\to\infty} \gamma_n = +\infty$. Hence $\lim\limits_{n\in M, n\to\infty} 1/\gamma_n = 0$, and there exists $n_0 \in \mathbb{N}$ such that $\gamma_n > 0$ for $n > n_0$, $n \in M$. We will show there are $M_1 \subset M$, $M_2 \subset M$ such that $M_1 \cap M_2 = \emptyset$, $M_1 \notin I$, $M_2 \notin I$. Denote $M_3 = \{n \in M;\ n-1 \notin M, n+1 \notin M\}$, $M_4 = M \setminus M_3$. If $M_4 \in I$ then $M_3 \notin I$. Hence $N \setminus M_3 \in I$ because $N \setminus M \in I$, $N \setminus M_3 = (N \setminus M) \cup M_4$. (sh) implies $(N \setminus M_3) + 1 \in I$, where $X + 1 = \{n \in \mathbb{N};\ n = x+1, x \in X\}$ for $\emptyset \neq X \subset \mathbb{N}$. From $M_3 \setminus \{1\} \subset (N \setminus M) + 1 \subset (N \setminus M_3) + 1$ we have $M_3 \in I$, a contradiction. So $M_4 \notin I$. Denote $M_5 = \{n \in M_4$; there exist $n_0 \in M$, $k_0 \in N$ such that $n_0 - 1 \notin M$, $n_0 + k_0 + 1 \notin M$, $n_0 + i \in M$ for $i = 1, \ldots, k_0$, and $n = n_0 + j$ for j an even number and $1 \leqslant j \leqslant k_0\}$, $M_6 = M_4 \setminus M_5$. If $M_6 \in I$ then (sh) yields $M_6 + 1 \in I$. Because $M_5 \subset M_6 + 1$ we have $M_5 \in I$ and so $M_4 \in I$, a contradiction. So $M_6 \notin I$. If $M_5 \in I$ then $M_5 + 1 \in I$ and $M_5 - 1 \in I$. This implies $M_6 \subset (M_5 + 1) \cup (M_5 - 1)$ and so $M_6 \in I$, a contradiction. So $M_5 \notin I$. If we put $M_1 = M_5$ and $M_2 = M_6$ we get the sought sets. First suppose that $\sum\limits_{n\in M_4} 1/\gamma_n = +\infty$. We construct a sequence $\{r_n\}_{n=1}^{\infty}$, where $r_n \in \{-\frac{1}{2}, -\frac{1}{3}, \frac{1}{3}, \frac{1}{2}\}$ for $n \in M_4$ are such that $\sum\limits_{j=1}^{\infty} r_j/\gamma_j$ is a convergent series. Let $L \subset \mathbb{N}$, $L = \{l_1 < l_2 < \ldots\}$ be such that $M_4 = \{n \in \mathbb{N};\ l_{2i-1} \leqslant n \leqslant l_{2i}$, where $l_{2i-1} - 1 \notin M, l_{2i} + 1 \notin M, i \in \mathbb{N}\}$. Put $q_j = \min\{|\gamma_j|/2^j, 1/2^j\} \operatorname{sign}(\gamma_j)$ for $l_{2i} < j < l_{2i+1}$, $l_{2i}, l_{2i+1} \in L$, $i, j \in \mathbb{N}$, $q_j = \frac{1}{3}$ for $j \in M_1$ and $q_j = \frac{1}{2}$ for $j \in M_2$. Moreover, if $l_1 > 1$ we put an extra $q_j = 1$ for $j = 1, \ldots, l_1 - 1$, where $\operatorname{sign}(a) = 1$ if $a > 0$, $\operatorname{sign}(a) = -1$ if $a < 0$. Choose any $s \in \mathbb{R}$, $s > 1/\gamma_1$ in the case $l_1 = 1$ and $s > \sum_{j=1}^{l_1-1} 1/\gamma_j$ in the case $l_1 > 1$. Because of $\sum_{j=1}^{\infty} q_j/\gamma_j = +\infty$ we can define by induction the following increasing sequence $\{n_m\}_{m=1}^{\infty}$, $n_m \in \mathbb{N}$. Put $n_1 = \min\Big\{n \in \mathbb{N};\ \sum_{j=1}^{n} q_j/\gamma_j > s\Big\}$, $n_2 = \min\Big\{n > n_1;\ \sum_{j=1}^{n_1} q_j/\gamma_j - \sum_{j=n_1+1}^{n} q_j/\gamma_j < s\Big\}$. Suppose that we have $n_1 < n_2 < \ldots < n_{m-1}$, $m > 1$. If $m = 2k+1$ we put $n_m = \min\Big\{n > n_{2k};\ \sum_{j=1}^{n_1} q_j/\gamma_j - \sum_{j=n_1+1}^{n_2} q_j/\gamma_j + \ldots - \sum_{j=n_{2k-1}+1}^{n_{2k}} q_j/\gamma_j + \sum_{j=n_{2k}+1}^{n} q_j/\gamma_j > s\Big\}$, if $m = 2k+2$ we put $n_m = \min\Big\{n > n_{2k+1};\ \sum_{j=1}^{n_1} q_j/\gamma_j - \sum_{j=n_1+1}^{n_2} q_j/\gamma_j + \ldots + \sum_{j=n_{2k}+1}^{n_{2k+1}} q_j/\gamma_j - \sum_{j=n_{2k+1}+1}^{n} q_j/\gamma_j < s\Big\}$. Define $\{r_n\}_{n=1}^{\infty}$ as follows: denote $n_0 = 1$ and put $r_n = (-1)^{m+1} q_n$ if $n_{m-1} < n \leqslant n_m$, $n \in \mathbb{N}$. From $\lim_{n\to\infty} q_n/\gamma_n = 0$ we have that $\sum_{n=1}^{\infty} r_n/\gamma_n$ is a convergent series. Define $\varepsilon_n = \sum_{j=n}^{\infty} r_j/\gamma_j + pB_n^2$, $n \in \mathbb{N}$ where $p \in \mathbb{R} \setminus 0$ is such that $\varepsilon_n \neq 0$, $\varepsilon_n B_n \neq \varepsilon_{n+1} B_{n+1}$, $n \in \mathbb{N}$ (such p exists, see the proof of Lemma 2). Put $A_n = \varepsilon_n B_n$, $a_n = A_n - A_{n+1}$, $n \in \mathbb{N}$. Then $I\text{-}\lim_{n\to\infty} (a_{n+1} + a_{n+2} + \ldots)/(b_{n+1} + b_{n+2} + \ldots) = I\text{-}\lim_{n\to\infty} A_{n+1}/B_{n+1} = 0$ and since $a_{n+1}/b_{n+1} = (A_{n+1} - A_{n+2})/(B_{n+1} - B_{n+2}) = \varepsilon_{n+2} + r_{n+1} + pB_{n+1}(B_{n+1} + B_{n+2})$ and $Q = \{n \in \mathbb{N};\ |r_n - r| \geqslant \frac{1}{12}\} \notin I$, where $r \in \mathbb{R}$, ($Q \supset M_1$ or $Q \supset M_2$), I-$\lim_{n\to\infty} a_{n+1}/b_{n+1}$ does not exist. If $\sum_{n\in M_4} 1/\gamma_n$ is a convergent series then $\sum_{n\in M_4} q_n/\gamma_n$ is again a convergent series because $\gamma_n > 0$ for $n \geqslant n_0$, $n_0 \in \mathbb{N}$, $n \in M_4$. If we put $r_n = q_n$, $n \in \mathbb{N}$, the proof is similar to that of the case $\sum_{n\in M_4} 1/\gamma_n = +\infty$. Secondly, suppose $I\text{-}\lim_{n\in K, n\to\infty} \gamma_n = -\infty$, $I\text{-}\lim_{n\in T, n\to\infty} \gamma_n = +\infty$, $K \notin I$, $T \notin I$, $K \cap T = \emptyset$ and $\mathbb{N} \setminus (K \cup T) \in I$. From (AP) (similarly to [8]) put $A_i = \{n \in \mathbb{N};\ -i < \gamma_n \leqslant -(i-1)\} \cup \{n \in \mathbb{N};\ i-1 < \gamma_n \leqslant i\}$, $i \in \mathbb{N}$. Then $A_i \in I$, $A_i \cap A_j = \emptyset$, $i \neq j$, $i, j \in \mathbb{N}$. (AP) implies there is $\{B_i\}_{i=1}^{\infty}$, $B_i \subset \mathbb{N}$, $B_i \in I$, $A_i \triangle B_i$ is a finite set, $i \in \mathbb{N}$, $B = \Big(\bigcup_{i=1}^{\infty} B_i\Big) \in I$, $\mathbb{N} \setminus B \notin I$. Denote $L_1 = (\mathbb{N} \setminus B) \cap T$, $L_2 = (\mathbb{N} \setminus B) \cap K$. Because $B \cap T \in I$, $B \cap K \in I$ we get $L_1 \notin I$, $L_2 \notin I$, $L_1 \cap L_2 = \emptyset$, $\mathbb{N} \setminus (L_1 \cup L_2) \in I$ and $\lim_{n\in L_1, n\to\infty} \gamma_n = +\infty$, $\lim_{n\in L_2, n\to\infty} \gamma_n = -\infty$. There are three cases posible:

(a) $\sum_{n\in L_1, n=1}^{\infty} 1/\gamma_n$, $\sum_{n\in L_2, n=1}^{\infty} 1/\gamma_n$ are convergent series,

(b) one of $\sum_{n\in L_1, n=1}^{\infty} 1/\gamma_n$, $\sum_{n\in L_2, n=1}^{\infty} 1/\gamma_n$ is convergent, the other is a divergent series,

(c) both $\sum_{n\in L_1, n=1}^{\infty} 1/\gamma_n$, $\sum_{n\in L_2, n=1}^{\infty} 1/\gamma_n$ are divergent series.

In case (a) we put $r_n = g_1$, $n \in L_1$, $r_n = g_2$, $n \in L_2$, where $g_1, g_2 \in \mathbb{R} \setminus \{0\}$, $|g_1| \neq |g_2|$ and $r_j = \min\{|\gamma_j|/2^j, 1/2^j\} \operatorname{sign}(\gamma_j)$ for $j \in \mathbb{N} \setminus (L_1 \cup L_2)$. In the other cases (b), (c) we choose a divergent series, for example $\sum\limits_{n \in L_1, n=1}^{\infty} 1/\gamma_n$, and similarly to the first part of the proof we construct $\{r_n\}_{n=1}^{\infty}$ such that $\sum\limits_{n=1}^{\infty} r_n/\gamma_n$ is convergent. We put again $\varepsilon_n = \sum\limits_{j=n}^{\infty} r_j/\gamma_j + pB_n^2$ and proceed as in the previous part of our proof. $\square$

References

[1] *J. Činčura, M. Sleziak, T. Šalát, V. Toma*: Sets of statistical cluster points and I-cluster points. Real Anal. Exchange *30* (2005), 565–580.

[2] *D. F. Dawson*: Matrix summability over certain classes of sequences ordered with respect to rate of convergence. Pacific J. Math. *24* (1968), 51–56.

[3] *M. Dindoš, T. Šalát, V. Toma*: Statistical convergence of infinite series. Czech. Math. J. *53* (2003), 989–1000.

[4] *R. Filipów, I. Reclaw, N. Mrożek, P. Szuca*: Ideal convergence of bounded sequences. J. Symbolic Logic *72* (2007), 501–512.

[5] *D. Holý, L. Matejíčka, Ľ. Pinda*: On faster convergent infinite series. International Journal of Mathematics and Mathematical Sciences, ISSN 0161-1712, Vol. 2008, Article ID 753632 (2008).

[6] *D. Holý, L. Matejíčka, Ľ. Pinda*: Some remarks on faster convergent infinite series. Preprint.

[7] *T. A. Keagy, W. F. Ford*: Acceleration by subsequence transformation. Pacific J. Math. *132* (1988), 357–362.

[8] *P. Kostyrko, T. Šalát, W. Wilczyński*: $\mathbb{I}$-Convergence. Real Anal. Exchange *26* (2000/2001), 669–686.

[9] *P. Kostyrko, M. Mačaj, T. Šalát, M. Sleziak*: $\mathbb{I}$-Convergence and extremal $\mathbb{I}$-limit points. Math. Slovaca *55* (2005).

[10] *H. E. Salzer*: A simple method for summing certain slowly convergent series. J. Math. Physics *33* (1955), 356–359.

[11] *D. A. Smith, W. F. Ford*: Acceleration of linear and logarithmic convergence. Siam J. Numer. Anal. *16* (1979), 223–240.

[12] *T. Šalát, V. Toma*: A classical Olivier's theorem and statistical convergence. Annales Mathématiques Blaise Pascal *10* (2003), 305–313.

[13] *B. C. Tripathy*: On statistical convergence. Proc. Estonian Acad. Sci. Phys. Math. *47* (1998), 299–303.

[14] *J. Wimp*: Some transformations of monotone sequences. Math. Comput. *26* (1972), 251–254.

Author's address: *Ladislav Matejíčka*, Trenčín University of Alexander Dubček in Trenčín, Faculty of Industrial Technologies in Púchov, I. Krasku 491/30, 020 01 Púchov, Slovakia, e-mail: `matejicka@tnuni.sk`.