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A CHARACTERIZATION OF $C^{1,1}$ FUNCTIONS VIA LOWER DIRECTIONAL DERIVATIVES

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Abstract. The notion of $\tilde{\ell}$-stability is defined using the lower Dini directional derivatives and was introduced by the authors in their previous papers. In this paper we prove that the class of $\tilde{\ell}$-stable functions coincides with the class of $C^{1,1}$ functions. This also solves the question posed by the authors in SIAM J. Control Optim. 45 (1) (2006), pp. 383–387.

Keywords: second-order derivative, $C^{1,1}$ function, ℓ-stable function, $\tilde{\ell}$-stability

MSC 2010: 49K10, 26B05

1. Introduction

The notion of $\tilde{\ell}$-stability was introduced by the authors in [2]. Mainly, we are concerned in the problem whether the class of $C^{1,1}$ functions can be characterized in terms of $\tilde{\ell}$-stability.

Throughout the paper, we will work with functions $f\colon \mathbb{R}^N \to \mathbb{R}$ defined on an open subset of the N-dimensional Euclidean space $\mathbb{R}^N$. By $S_{\mathbb{R}^N}$ we denote the unit sphere $\{x \in \mathbb{R}^N; \|x\| = 1\}$. The (first-order) lower Dini right hand directional derivative of f at $x \in \mathbb{R}^N$ in a direction $h \in \mathbb{R}^N$ is defined by

$$f^\ell(x;h) = \liminf_{t\to 0+} \frac{f(x+th)-f(x)}{t}.$$

The classical bilateral directional derivative of f at $x \in \mathbb{R}^N$ in the direction $h \in \mathbb{R}^N$ is then denoted by $f'(x;h)$. Recall that f is said to belong to the class of $C^{1,1}$ functions on an open subset U of $\mathbb{R}^N$ provided that f has the Fréchet derivative (which we

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denote by $Df(x)$) at each point $x \in U$ and the mapping $x \mapsto Df(x)$ is locally Lipschitz on U.

Definition 1.1. Let A be a nonempty open subset of $\mathbb{R}^N$, f a real valued function defined on A, $x_0 \in A$. We say that f is $\tilde{\ell}$-stable at x_0 if there are a neighborhood U of x_0 and a constant $L > 0$ such that for every $y, z \in U$:

$$|f^\ell(z; z-y) - f^\ell(y; z-y)| \leqslant L\|z-y\|^2. \tag{1.1}$$

It follows immediately from the definitions that each function of the class $C^{1,1}$ on a neighbourhood of x_0 must be also $\tilde{\ell}$-stable at x_0.The goal of the article is then to prove the reverse implication, which is done in Theorem 2.1. To this end we have used the notion of semiconcavity.

Definition 1.2. Assume that $U \subset \mathbb{R}^N$ is an open and convex set, and let $C \geqslant 0$. We say f is semiconcave on U (with linear modulus of semiconcavity C) provided that the function $x \mapsto f(x) - C\|x\|^2$ is concave on U, and f is said to be semiconvex on U provided that $-f$ is semiconcave on U.

For more details the reader should consult e.g. [4]. We will need the following lemma.

Lemma 1.1 [4, Corollary 3.3.8]**.** *Let U be an open convex subset of $\mathbb{R}^N$ and let $f\colon U \to \mathbb{R}$ be a function which is both semiconcave and semiconvex with a linear modulus C. Then f is a $C^{1,1}$ function.*

2. Differentiability properties of $\tilde{\ell}$-stable functions

At first, we will work with functions of one variable. Consider a function $f\colon I \to \mathbb{R}$ defined on an open subinterval I of $\mathbb{R}$. Recall the definitions of two of the well known unilateral Dini derivatives:

$$D^- f(x) = \limsup_{t \to 0-} \frac{f(x+t) - f(x)}{t},$$
$$D_+ f(x) = \liminf_{t \to 0+} \frac{f(x+t) - f(x)}{t}.$$

Note that then we have $f^\ell(x; 1) = D_+ f(x)$ and $f^\ell(x; -1) = -D^- f(x)$.

Let us now state some useful auxiliary results.

Lemma 2.1 [6, page 134]. *Suppose that f satisfies the following conditions:*
(i) $\liminf_{t\to 0+} f(x-t) \leqslant f(x),\ \forall x \in I,$
(ii) $D_+ f(x) \geqslant 0$ *for almost all* $x \in I$,
(iii) $D_+ f(x) > -\infty,\ \forall x \in I.$
Then f is a nondecreasing function on I.

Lemma 2.2. *Suppose that the following conditions hold:*
(i) $-\infty < D^- f(x) < \infty,\ \forall x \in I,$
(ii) $x \mapsto D_+ f(x)$ *is finite and continuous on* I.
Then f is a continuous function on I.

Proof. Let us choose $x \in I$ arbitrarily. The condition (ii) then implies the existence of $\delta > 0$, $K > 0$ such that $(x-\delta, x+\delta) \subset I$, and $|D_+ f(y)| \leqslant K$ whenever $y \in (x-\delta, x+\delta)$. Next we put

$$g(y) = f(y) + K(y - x), \quad y \in (x-\delta, x+\delta).$$

Now it easily follows that $g(x) = f(x)$, and $D_+ g(y) = D_+ f(y) + K \geqslant 0$ for every $y \in (x-\delta, x+\delta)$. It also follows that for every $y \in (x-\delta, x+\delta)$ we have $\liminf_{t\to 0+} g(y-t) \leqslant g(y)$. Otherwise we would have $D^- g(y) = D^- f(y) + K = -\infty$ which contradicts the assumption (i). Thus due to Lemma 2.1, g is nondecreasing on $(x-\delta, x+\delta)$ and hence

$$\lim_{t\to 0+} g(x-t) \leqslant g(x) \leqslant \lim_{t\to 0+} g(x+t). \tag{2.1}$$

We claim that the above inequalities (2.1) are actually equalities. Indeed, otherwise $D^- g(x) = \infty$ or $D_+ g(x) = \infty$ which contradicts (i) or (ii).

Thus, we infer that

$$\lim_{t\to 0+} g(x-t) = g(x) = \lim_{t\to 0+} g(x+t).$$

This proves the continuity of g and consequently the continuity of f at x. Since x was chosen arbitrarily in I, we completed the proof. □

It is worth noting that in the above proof we have used only local boundedness of the function $x \mapsto D_+ f(x)$ on I.

Lemma 2.3. *Let f satisfy the same assumptions as in Lemma 2.2. Then f is continuously differentiable on I.*

Proof. It suffices to use the previous lemma together with the classical Dini theorem, see [3, Ch. 4, Theorem 1.3] □

Now we are ready to state and prove the main result of this note.

Theorem 2.1. *Let $f\colon \mathbb{R}^N \to \mathbb{R}$ be a function which is $\tilde{\ell}$-stable at x_0. Then f is $C^{1,1}$ on a neighborhood of x_0.*

Proof. First we will show that there exists a constant $L > 0$ such that the functions $\pm f + L\|\cdot\|^2$ are convex on an open convex neighborhood of x_0.

Suppose that (1.1) is satisfied on an open convex neighbourhood U of x_0. Let us fix $x \in U$, $h \in S_{\mathbb{R}^N}$. Then there exists an open interval $I \subset \mathbb{R}$ such that

$$x + th \in U \iff t \in I,$$

i.e. $x + Ih = U \cap \{x + th\colon t \in \mathbb{R}\}$. Consider a function $\varphi_{x,h}\colon I \to \mathbb{R}$ defined as follows:

$$\varphi_{x,h}(t) = f(x + th), \quad t \in I.$$

Then for every $t \in I$ we have

$$D_+(\varphi_{x,h})(t) = \liminf_{s\downarrow o} \frac{\varphi_{x,h}(t+s) - \varphi_{x,h}(t)}{s} = f^\ell(x + th; h).$$

Fix $t', t'' \in I$. Then, if we plug $z = x + t'h$, $y = x + t''h$ into (1.1), we get

$$\begin{aligned} L|t' - t''|^2 = L\|z - y\|^2 &\geqslant |f^\ell(z; z - y) - f^\ell(y; z - y)| \\ &= |t' - t''||f^\ell(x + t'h; h) - f^\ell(x + t''h; h)| \\ &= |t' - t''||D_+(\varphi_{x,h})(t') - D_+(\varphi_{x,h})(t'')|. \end{aligned}$$

Consequently,

$$|D_+(\varphi_{x,h})(t') - D_+(\varphi_{x,h})(t'')| \leqslant L|t' - t''| \tag{2.2}$$

for every $t', t'' \in I$. Next we will show that $D^-(\varphi_{x,h})(t) \in \mathbb{R}$ for every $t \in I$. Now for an arbitrary fixed $t \in I$, due to (1.1) we have that

$$\begin{aligned} -D^-(\varphi_{x,h})(t) &= f^\ell(x + th; -h) = f^\ell(x + (-t)(-h); -h) \\ &= D_+\varphi_{x,-h}(-t) \in \mathbb{R}. \end{aligned}$$

Hence $D^-(\varphi_{x,h})(t) \in \mathbb{R}$ whenever $t \in I$.

By Lemma 2.3, $\varphi_{x,h}$ is continuously differentiable on I and due to (2.2) it has L-Lipchitzian derivative on I. Consequently, there is $\gamma > 0$ such that for all $t \in I$ we have $D_+(\pm\varphi'_{x,h})(t) \geqslant -\gamma$. Let us assume that $L > \gamma/2$ and consider two functions $F = f + L\|\cdot\|^2$, $G = -f + L\|\cdot\|^2$ defined on U. If we put $\psi(t) = F(x + th)$ for each $t \in I$, then we have

$$\psi(t) = \varphi_{x,h}(t) + L(\langle x, x\rangle + 2t\langle x, h\rangle + t^2).$$

Hence we have for each $t \in I$ that $\psi'(t) = \varphi'_{x,h}(t) + L(2\langle x, h\rangle + 2t)$. This implies for each $t \in I$

$$D_+(\psi')(t) = D_+(\varphi'_{x,h})(t) + 2L > -\gamma + 2L > 0,$$

and ψ' is a continuous function on I. As a consequence of the classical Dini theorem (see [3, Ch. 4, Theorem 1.2]), we get that ψ' is increasing on I and thus ψ is convex on I. This verifies the convexity of F on the set U. In a similar way it can be shown that G is also convex on U. Consequently, the functions $\pm f - L\|\cdot\|^2$ are concave. Now it follows that the functions $\pm f$ are semiconcave on U with the linear modulus of semiconcavity $C = L$. Finally, the assertion is now a consequence of Lemma 1.1. $\square$

Remark 1. We note that due to Theorem 2.1, the recent optimality result published by the authors (see [1, Theorem 7]) is now just an easy consequence of a previous result by I. Ginchev, A. Guerraggio and M. Rocca, see [5, Theorem 2].

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