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ON RIESZ HOMOMORPHISMS IN UNITAL f-ALGEBRAS

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Abstract. The main topic of the first section of this paper is the following theorem: let A be an Archimedean f-algebra with unit element e, and $T\colon A \to A$ a Riesz homomorphism such that $T^2(f) = T(fT(e))$ for all $f \in A$. Then every Riesz homomorphism extension $\widetilde{T}$ of T from the Dedekind completion A^δ of A into itself satisfies $\widetilde{T}^2(f) = \widetilde{T}(fT(e))$ for all $f \in A^\delta$. In the second section this result is applied in several directions. As a first application it is applied to show a result about extensions of positive projections to the Dedekind completion. A second application of the above result is a new approach to the Dedekind completion of commutative d-algebras.

Keywords: vector lattice, d-algebra, f-algebra

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1. Introduction

The well known extension theorem of Lipecki, Luxemburg and Schep concerning the Riesz homomorphism states that if A is a Riesz space with A^δ its Dedekind completion and $T\colon A \to A^\delta$ is a Riesz homomorphism, then T extends to a Riesz homomorphism $\widetilde{T}$ from A^δ into itself (see [1, Theorem 7.17]).

The principal aim of the present paper is to answer the following question: If in "Lipecki-Luxemburg-Schep's theorem" A is an f-algebra with unit element e and $T^2(f) = T(fT(e))$ for all $f \in A$, is $\widetilde{T}^2(f) = \widetilde{T}(fT(e))$ still true in A^δ.

Note that in general T need not be order continuous as is shown by the following example. Consider $A = C([0,1])$ with the pointwise addition, multiplication, scalar multiplication and partial ordering. It is easy to show that A is an f-algebra whose unit element is the constant function one e. Define the positive operator T on A by

$$T(f)(x) = f(\tfrac{1}{2}).$$

A straightforward computation shows that T is a Riesz homomorphism and $T^2(f) = T(fT(e))$ for all $f \in A$ but T is not order continuous.

This paper deals with two main results. First, we give an affirmative answer to the above question. This result will be applied in several directions. As a first application, we present a result about Riesz homomorphism extensions of the Riesz homomorphism and projection. Perhaps the most striking theorem in this direction, due to Triki [14, Theorem 1.7], is that if T is a positive projection on A, then T can be extended to a positive projection on A^δ. In the present paper, we intend to make some contributions to this area. We will prove that if in Triki's theorem A is an f-algebra with unit element e, T is a Riesz homomorphism such that $T(e) = e$, then every Riesz homomorphism extension of T to A^δ is a positive projection. In the survey paper [10], it was conjectured by Huijsmans that if A is an almost f-algebra, then there exists a multiplication in the Dedekind completion A^δ of A extending the multiplication in A, such that A^δ is an almost f-algebra with respect to this extended multiplication. This conjecture was proved by Buskes and van Rooij in [7, Theorem 4.1] for the class of almost f-algebras. In contrast to the original approach of Buskes and van Rooij, Boulabiar and the author gave in [6] an order theoretical and algebraic approach of the Buskes and van Rooij theorem. As an application, Boulabiar and the author gave in [6] an affirmative answer to Huijsmans's question for the class of commutative d-algebras. The proofs of the aforementioned results both rely on a representation theorem for almost f-algebras. As a second application, we present a new approach in the special case where A, in addition, is an f-algebra with unit element.

We assume that the reader is familiar with the notion of Riesz spaces (or vector lattices).

2. Preliminaries

For terminology and properties of Riesz spaces and order bounded operators not explained or proved in this paper, we refer to [1], [12], [15]. We refer to [2] for the elementary ℓ-algebra and d-algebra and to [1], [15] for the f-algebra and orthomorphism theories.

A lattice ordered group (briefly an ℓ-group) A is called *Archimedean* if for each non zero $f \in A$ the set $\{nf\colon n = \pm 1, \pm 2, \ldots\}$ has no upper bound in A. *In order to avoid unnecessary repetition we will assume throughout that all ℓ-groups under consideration are Archimedean.* An ℓ-group A which is simultaneously a ring with the property that $fg \in A^+$ for all $f, g \in A^+$ (equivalently, $|fg| \leqslant |f||g|$ for all $f, g \in A$) (where A^+ is the positive cone of A) is called a *lattice ordered ring* (briefly, an *ℓ-ring*). If, in addition, A is a real Riesz space, then A is called an *ℓ-algebra.*

Abstract f-ring theory and f-algebra theory have been studied by many authors (see e.g. [1], [3], [5], [15]). Some of these authors (see e.g. §8 in the paper [5] by G. Birkhoff and R. S. Pierce) define an *f-ring* as a lattice ordered ring with the property that $f \wedge g = 0$ and $h \in A^+$ implies $fh \wedge g = hf \wedge g = 0$. Others (see e.g. Definition 9.1.1 in the book [3] by A. Bigard, K. Keimel and S. Wolfenstein) define an f-ring as a lattice ordered ring which is isomorphic to a subdirect union of totally ordered rings. It is often desirable to have the equivalence of the two definitions available. However, any known equivalence proof is based on arguments using Zorn's lemma. If one uses the second definition, it is possible to prove a certain number of standard theorems on f-rings by means of the "metamathematical" observation that any identity holding in every totally ordered ring holds in every f-ring.

We adopt in this paper the original Birkhoff-Pierce definition of an *f-ring* (i.e. an ℓ-ring with the additional property that $f \wedge g = 0$ and $h \in A^+$ implies $fh \wedge g = hf \wedge g = 0$) as our starting point. An f-ring A is said to be an f-algebra if A is also a real Riesz space. It was shown by Birkhoff and Pierce in [5, Section 8] that any (Archimedean) f-ring is commutative, but for more elementary proof, due to Zaanen, we refer to [15, Theorem 140.10] or [3, Theorem 12.3.2]. If A is an f-ring then A has positive squares and $|fg| = |f||g|$ for all $f, g \in A$. Every f-ring A with identity element is *semiprime* (*or reduced*), that is, 0 is the only nilpotent in A. If f, g are elements in a semiprime f-ring A, then $|f| \wedge |g| = 0$ if and only if $fg = 0$. All f-ring properties listed above are satisfied by any f-algebra. An *almost f-algebra* A is an ℓ-algebra with the additional property that $f \wedge g = 0$ in A implies $fg = 0$. The ℓ-algebra A is called a *d-algebra* whenever $f \wedge g = 0$ and $h \in A^+$ imply $fh \wedge gh = hf \wedge hg = 0$ (equivalently, $|fg| = |f||g|$ for all $f, g \in A$). Any f-algebra is an almost f-algebra and a d-algebra, but not conversely (see [2]). An almost f-algebra need not be a d-algebra and vice versa. Both the f-algebras and almost f-algebras are automatically commutative and have positive squares but these properties fail in d-algebras. A d-algebra which is commutative or has positive squares is an almost f-algebra. More about almost f-algebras and d-algebras can be found in [2].

The relatively uniform topology on Riesz spaces plays a key role in the context of this work. Let us therefore recall the definition and some elementary properties of this topology. By $\mathbb{N}$ we mean the set $\{1, 2, \ldots\}$. Let A be an archimedean Riesz space and $u \in A^+$. A sequence $(f_n)_{n\in\mathbb{N}}$ of elements of A is said to *converge u-uniformly* to $f \in A$ whenever for every $\varepsilon > 0$ there exists a natural number N_ε such that $|f - f_n| \leqslant \varepsilon u$ for all $n \geqslant N_\varepsilon$. This is denoted by $f_n \to f(u)$. The element u is called the *regulator of convergence*. The sequence $(f_n)_{n\in\mathbb{N}}$ is said to *converge relatively uniformly* to $f \in A$ if $f_n \to f(u)$ for some $u \in A^+$. We shall write $f_n \to f\ (r \cdot u)$ if we do not want to specify the regulator. Relatively uniform limits are unique if and only if A is Archimedean [12, Theorem 63.2]. We refer to [13] for the relatively uniform topology. In the end of this paragraph, we recall an important fact about unital f-algebras. Let A be an Archimedean f-algebra with unit element $0 < e$. For every $0 \leqslant f \in A$, the increasing sequence $0 \leqslant f_n = f \wedge ne$ converges relatively uniformly to f in A (for details on this see, e.g., [1, Theorem 8.22]).

Let A and B be Riesz spaces. An operator $\pi\colon A \to B$ is called *order bounded* if the image under π of an order bounded set in A is again an order bounded set in B. The operator π is called *positive* if $\pi(A^+) \subset B^+$. The operator π is called a *Riesz homomorphism* (or *lattice homomorphism)* whenever $f \wedge g = 0$ implies $\pi(f) \wedge \pi(g) = 0$. Obviously, every Riesz homomorphism is positive. The order bounded operator $\pi\colon A \to A$ is called an *orthomorphism* if $|f| \wedge |g| = 0$ implies $|\pi(f)| \wedge |g| = 0$. The collection $\mathrm{Orth}(A)$ of all orthomorphisms on A is, with respect to the usual Riesz spaces and composition as multiplication, an Archimedean f-algebra with the identity mapping I_A on A as a unit element (for details on this see, e.g., [1, Theorem 8.24]). Every orthomorphism π of A is order continuous ([1, Theorem 8.10]). If A is supposed to be, in addition, an f-algebra then, for every $f \in A$, the map π_f defined by $\pi_f(g) = fg$ for all $g \in A$ is an orthomorphism of A. Furthermore, if A is an f-algebra with unit element then the mapping $f \to \pi_f$ from A into $\mathrm{Orth}(A)$ is a Riesz and algebra isomorphism. Therefore, for every $\pi \in \mathrm{Orth}(A)$ there exists a unique element $f \in A$ such that $\pi = \pi_f$ and π is a positive orthomorphism if and only if $f \geqslant 0$ (for details on this see, e.g., [1, Theorem 8.27]).

A bilinear map $\psi\colon A \times A \to B$ is said to be positive if $\psi(f, g) \in B^+$ for all $f, g \in A^+$. The positive bilinear map $\psi\colon A \times A \to B$ is called an *orthosymmetrical* map if $f \wedge g = 0$ implies $\psi(f, g) = 0$. The proof of commutativity of Archimedean almost f-algebras, given by Bernau and Huijsmans in [2, Theorem 2.15], does not make use of associativity. In fact, Bernau and Huijsmans proved that every orthosymmetrical map is symmetrical.

A Dedekind complete Riesz space is called *universally complete* whenever every set of pairwise disjoint positive elements has a supremum. Every Riesz space A has a *universal completion* A^u, i.e. there exists a unique (up to a lattice homomorphism) universally complete (and therefore Dedekind complete) Riesz space A^u such that A can be identified with an order dense vector sublattice of A^u. Moreover, A^u is furnished with a multiplication under which A^u is an f-algebra with unit element. Since we wish to avoid representation in this paper, we adhere to the purely algebraic approach for the existence of a universal completion presented in [1, §8, Exercise 13].

Let A be a Riesz space and $f \in A$. We denote by $\{f\}^{dd}$ the principal band generated by f and by $\{f\}^d$ its disjoint complement.

3. Main results

We have gathered now all ingredients for the main result of this paper.

Theorem 1. *Let A be an Archimedean f-algebra with unit element e, and $T\colon A \to A$ a Riesz homomorphism such that*

$$T^2(f) = T(fT(e))$$

for all $f \in A$. Then every Riesz homomorphism extension $\widetilde{T}$ of T from A^δ into itself satisfies

$$\widetilde{T}^2(f) = \widetilde{T}(fT(e))$$

for all $f \in A^\delta$.

P r o o f. As is well-known, multiplication in an Archimedean f-algebra A can be extended (uniquely) to an f-algebra multiplication in A^δ, so we can assume without loss of generality that A^δ is an f-algebra with unit element e. Now, define the bilinear map

$$\begin{aligned} \varphi\colon\ & A \times A \to A, \\ & (x, y) \to T(xT(e))T(y). \end{aligned}$$

Let $x, y \in A$ be such that $x \wedge y = 0$. From the fact that A is an f-algebra it follows that $xT(e) \wedge y = 0$ so $T(xT(e)) \wedge T(y) = 0$ (because T is a Riesz homomorphism). Consequently, $T(xT(e))T(y) = 0$ and thus

$$\varphi(x, y) = 0.$$

So φ is orthosymmetrical, which implies that φ is symmetrical and thus

$$T(xT(e))T(y) = T(yT(e))T(x).$$

Now, using the above equality with $y = e$ and the fact that $T(T(x)) = T(xT(e))$ (hypothesis of T), we observe that

$$T(T(x))T(e) = T(xT(e))T(e) = T(x)T(T(e)). \tag{1.1}$$

Let $F = T(A)$, then F is a vector sublattice of A. Now, we define $\pi\colon F \to A^\delta$ by $\pi(x) = T(x)T(e)$. So, $\pi(T(x)) = T^2(x)T(e)$ for all $x \in A$. On the other hand, by using (1.1) we observe that

$$\pi(x) = xT^2(e) \quad \text{for all } x \in F.$$

Hence π is order continuous. According to [1, Theorem 4.12], the formula

$$\tilde{\pi}(x) = \sup\{\pi(y)\colon\ y \in F \text{ and } 0 \leqslant y \leqslant x\}$$

defines a unique order continuous extension of π to F^{δ}. Then

$$\tilde{\pi}(x) = xT^2(e) \quad \text{ for all } x \in F^{\delta}.$$

Now, in view of [1, Theorem 7.19], it is not hard to prove that $\widetilde{T}(A^{\delta}) \subset F^{\delta}$. We infer that

$$\tilde{\pi}(\widetilde{T}(x)) = \widetilde{T}(x)T^2(e) \quad \text{ for all } x \in A^{\delta}.$$

On the other hand,

$$\tilde{\pi}(\widetilde{T}(x)) = \sup\{\pi(T(y))\colon\ y \in A \text{ and } 0 \leqslant T(y) \leqslant \widetilde{T}(x)\} \quad \text{ for all } x \in (A^{\delta})^{+}.$$

So

$$\tilde{\pi}(\widetilde{T}(x)) = \sup\{T(e)T^2(y)\colon\ y \in A \text{ and } 0 \leqslant T(y) \leqslant \widetilde{T}(x)\} \quad \text{ for all } x \in (A^{\delta})^{+}.$$

This implies that

$$0 \leqslant \tilde{\pi}(\widetilde{T}(x)) \leqslant T(e)\widetilde{T}^2(x) \quad \text{ for all } x \in (A^{\delta})^{+}$$

and thus

$$0 \leqslant T^2(e)\widetilde{T}(x) \leqslant T(e)\widetilde{T}^2(x) \quad \text{ for all } x \in (A^{\delta})^{+}.$$

At this point, we define $T^{\#}\colon A^{\delta} \to A^{\delta}$ by $T^{\#}(x) = T(e)\widetilde{T}^2(x)$, and $S\colon A^{\delta} \to A^{\delta}$ by $S(x) = \widetilde{T}(x)T^2(e)$. Obviously, $T^{\#}$ is a Riesz homomorphism. Moreover, $0 \leqslant S \leqslant T^{\#}$. Hence, by [1, Theorem 8.16], there exists a positive orthomorphism $R \in \operatorname{Orth}(A^{\delta})$ satisfying $S = RT$. This yields that there exists $\omega \in A^{\delta}$ such that

$$S(x) = \omega T^{\#}(x) \quad \text{ for all } x \in A^{\delta}.$$

Therefore

$$\widetilde{T}(x)T^2(e) = \omega T(e)\widetilde{T}^2(x) \tag{2.1}$$

holds for all $x \in A^{\delta}$. This implies that

$$T(e)T^2(e) = \omega T(e)T^2(e).$$

Consequently, multiplying (2.1) by $T^2(e)$, we obtain

$$T^2(e)\widetilde{T}(x)T^2(e) = T^2(e)T(e)\widetilde{T}^2(x).$$

On the other hand, $\widetilde{T}(x)T^2(e) - T(e)\widetilde{T}^2(x) \in \{T(e)\}^{dd}$, so

$$\widetilde{T}(x)T^2(e) = T(e)\widetilde{T}^2(x) \quad \text{for all } x \in A^\delta.$$

Further,

$$T^2(e)\widetilde{T}(x) = T(T(e)e)\widetilde{T}(x) = \widetilde{T}(T(e)x)T(e)$$

(because the bilinear map $(x, y) \to \widetilde{T}(x)\widetilde{T}(T(e)y)$ is orthosymmetrical so symmetrical). So

$$T(e)\widetilde{T}^2(x) = T(e)\widetilde{T}(T(e)x),$$

and thus

$$\widetilde{T}^2(x) - \widetilde{T}(T(e)x) \in \{T(e)\}^d$$

(because A is semiprime). On the other hand, $\widetilde{T}^2(x) - \widetilde{T}(T(e)x) \in \{T(e)\}^{dd}$. Hence

$$\widetilde{T}^2(x) = \widetilde{T}(T(e)x)$$

for all $x \in A^\delta$, which is the desired result. □

In [14] Triki proved that if T is a positive projection on A, then T can be extended to a positive projection on A^δ (see [14, Theorem 1.7]). In this regard, we give the following result.

Corollary 2. *Let A be an Archimedean f-algebra with unit element e, and $T\colon A \to A$ a Riesz homomorphism such that $T(e) = e$ and T is a projection. Then every Riesz homomorphism extension $\widetilde{T}$ of T from A^δ into itself is a positive projection.*

Corollary 3. *Let A be an Archimedean f-algebra with unit element e, and $T\colon A \to A$ a Riesz homomorphism such that*

$$T(T(fg)h) = T(fT(gh))$$

for all $f, g, h \in A$. Then every Riesz homomorphism extension $\widetilde{T}$ of T from A^δ into itself satisfies

$$\widetilde{T}(\widetilde{T}(fg)h) = \widetilde{T}(f\widetilde{T}(gh))$$

for all $f, g, h \in A^\delta$.

P r o o f. It is not hard to see that it suffices to establish the equality for $f, g, h \in (A^\delta)^+$. Let $f, h \in (A^\delta)^+$ and let $\widetilde{T}$ be an extension of T from A^δ into itself. Since $\widetilde{T}$ is a Riesz homomorphism and A^δ is an f-algebra, it is not hard to prove that the bilinear map $(x, y) \to \widetilde{T}(\widetilde{T}(fx)h)\widetilde{T}^2(y)$ is orthosymmetrical, hence symmetrical. Therefore

$$\widetilde{T}(\widetilde{T}(fx)h)\widetilde{T}^2(y) = \widetilde{T}(\widetilde{T}(fy)h)\widetilde{T}^2(x)$$

for all $x, y \in A^\delta$. Now, for $x = g$ and $y = e$, we obtain

$$\widetilde{T}(\widetilde{T}(fg)h)\widetilde{T}^2(e) = \widetilde{T}(\widetilde{T}(f)h)\widetilde{T}^2(g).$$

So

$$\widetilde{T}(\widetilde{T}(fg)h)\widetilde{T}^2(e)T(e) = \widetilde{T}(\widetilde{T}(f)h)\widetilde{T}^2(g)T(e).$$

On the other hand,

$$\widetilde{T}(\widetilde{T}(f)h)T(e) = \widetilde{T}(\widetilde{T}(f))\widetilde{T}(h)$$

(because the bilinear map $(x, y) \to \widetilde{T}(\widetilde{T}(f)x)\widetilde{T}(y)$ is orthosymmetrical, hence symmetrical). Consequently,

$$\widetilde{T}(\widetilde{T}(fg)h)T^2(e)T(e) = \widetilde{T}(\widetilde{T}(f))\widetilde{T}^2(g)\widetilde{T}(h).$$

On the other hand

$$\widetilde{T}(h)T^2(e) = \widetilde{T}(he)T(T(e)) = \widetilde{T}(hT(e))T(e)$$

(because the bilinear map $(x, y) \to \widetilde{T}(hx)\widetilde{T}(y)$ is orthosymmetrical, hence symmetrical). Now, by using Theorem 1 we have

$$\widetilde{T}(hT(e)) = \widetilde{T}^2(h)$$

and

$$\widetilde{T}^2(f) = \widetilde{T}(fT(e)),$$

which implies

$$\widetilde{T}(\widetilde{T}(fg)h)T^2(e)T(e)T^2(e) = \widetilde{T}(fT(e))\widetilde{T}^2(g)\widetilde{T}^2(h)T(e).$$

On the other hand, $\widetilde{T}^2(g)\widetilde{T}^2(h) = \widetilde{T}^2(gh)\widetilde{T}^2(e)$ (because the bilinear map $(x, y) \to \widetilde{T}^2(gx)\widetilde{T}^2(y)$ is orthosymmetrical, hence symmetrical). Then

$$\begin{aligned}\widetilde{T}(\widetilde{T}(fg)h)T^2(e)T(e)T^2(e) &= \widetilde{T}(fT(e))\widetilde{T}^2(gh)T^2(e)T(e)\\ &= \widetilde{T}(f\widetilde{T}(gh))T^2(e)T^2(e)T(e).\end{aligned}$$

So

$$(\widetilde{T}(\widetilde{T}(fg)h)T^2(e)T(e) - \widetilde{T}(f\widetilde{T}(gh))T^2(e)T(e))T^2(e) = 0,$$

and thus

$$(\widetilde{T}(\widetilde{T}(fg)h)T^2(e)T(e) - \widetilde{T}(f\widetilde{T}(gh))T^2(e)T(e)) \in \{T^2(e)\}^{dd}.$$

So $(\widetilde{T}(\widetilde{T}(fg)h)T^2(e)T(e) - \widetilde{T}(f\widetilde{T}(gh))T^2(e)T(e)) = 0$. Analogously

$$(\widetilde{T}(\widetilde{T}(fg)h)T^2(e) - \widetilde{T}(f\widetilde{T}(gh))T^2(e)) \in \{T(e)\}^{dd},$$

hence

$$(\widetilde{T}(\widetilde{T}(fg)h)T^2(e) - \widetilde{T}(f\widetilde{T}(gh))T^2(e) = 0.$$

And, since $\widetilde{T}(\widetilde{T}(fg)h) - \widetilde{T}(f\widetilde{T}(gh)) = 0$ (because $(\widetilde{T}(\widetilde{T}(fg)h) - \widetilde{T}(f\widetilde{T}(gh)) \in \{T^2(e)\}^{dd}$), we obtain the desired result. $\square$

In [6] Boulabiar and the author proved that if $(A, *)$ is a commutative d-algebra, then there exist (i) a universally complete f-algebra B with unit element, multiplication of which is denoted by juxtaposition; (ii) a lattice homomorphism σ from A into B such that B is the universal completion of $\sigma(A)$; (iii) a lattice homomorphism T from $\sigma(A)_p$ into A; and (iv) a positive element ξ of B, such that

$$f * g = T(\sigma(f)\sigma(g))$$

and

$$(f * g) * h = T(\xi\sigma(f)\sigma(g)\sigma(h))$$

hold for all $f, g, h \in A$. As an application, they proved in [6] that there exists a d-algebra multiplication on the Dedekind completion of an Archimedean commutative d-algebra. Therefore, it seems natural to ask what would happen if, in addition, A were an f-algebra with unit element. Now, let A be an Archimedean f-algebra with unit element e and a commutative d-algebra with respect to a multiplication $*$. The operator T of A, defined by $T(x) = x * e$, is a Riesz homomorphism. Now, it is not hard to see that $fg * e = f * g$ (because the bilinear map $(x, y) \to zx * y$ is orthosymmetrical for all $z \in A^+$, hence symmetrical). Consequently,

$$f * g = T(fg) \quad \text{for all } f, g \in A.$$

Now, we are going to address a natural question: can we use the latter simple representation of $*$ to extend it to a d-algebra multiplication in A^δ.

We now give an affirmative answer to this question.

Theorem 4. *Let A be an Archimedean f-algebra with unit element e and a commutative d-algebra with respect to a multiplication $*$. Then $*$ extends to a d-algebra multiplication in A^δ.*

P r o o f. Let A be a commutative d-algebra with respect to the multiplication $*$. As mentioned before, there exists a lattice homomorphism T from A into itself such that

$$f * g = T(fg).$$

In view of [1, Theorem 7.17], T extends to a lattice homomorphism $\widetilde{T}$ from A^δ into itself. Define now a multiplication $*$ in A^δ by putting

$$f * g = \widetilde{T}(fg)$$

for all $f, g \in A^\delta$. We assert that A^δ is a d-algebra with respect to the multiplication $*$. The difficult part of the proof is to establish that $*$ is associative. But it is not hard to prove this by using associativity in A and the preceding corollary. Indeed, we have $(f * g) * h = f * (g * h)$ for all $f, g, h \in A$. Therefore

$$T(T(fg)h) = T(fT(gh)).$$

Now, by using the preceding corollary we have

$$\widetilde{T}(\widetilde{T}(fg)h) = \widetilde{T}(f\widetilde{T}(gh)).$$

Therefore

$$(f * g) * h = f * (g * h)$$

for all $f, g, h \in A^\delta$ and the proof of the theorem is now complete. $\square$

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