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# OPERATORS ON LORENTZ SEQUENCE SPACES

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*Abstract.* Description of multiplication operators generated by a sequence and composition operators induced by a partition on Lorentz sequence spaces $l(p,q)$, $1 < p \leqslant \infty$, $1 \leqslant q \leqslant \infty$ is presented.

*Keywords*: composition operator, distribution function, Fredholm operator, Lorentz space, Lorentz sequence space, multiplication operator, non-increasing rearrangement

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## 1. Introduction

Let $f$ be a complex-valued measurable function defined on a $\sigma$-finite measure space $(X,\mathcal{A},\mu)$. For $s \geqslant 0$, define the *distribution function* $\mu_f$ of $f$ as

$$\mu_f(s) = \mu\{x \in X\colon\ |f(x)| > s\}.$$

By $f^*$ we mean the *non-increasing rearrangement* of $f$ given as

$$f^*(t) = \inf\{s > 0\colon\ \mu_f(s) \leqslant t\}, \quad t \geqslant 0.$$

The *Lorentz space* $L(p,q)$, $1 < p \leqslant \infty$, $1 \leqslant q \leqslant \infty$, is the set of all complex-valued measurable functions $f$ on $X$ such that $\|f\|^*_{pq} < \infty$, where

$$\|f\|^*_{pq} = \begin{cases} \left\{\dfrac{q}{p}\displaystyle\int_0^\infty (t^{1/p} f^*(t))^q \frac{\mathrm{d}t}{t}\right\}^{1/q}, & 1 < p < \infty,\ 1 \leqslant q < \infty,\\ \sup\limits_{t>0} t^{1/p} f^*(t), & 1 < p \leqslant \infty,\ q = \infty. \end{cases}$$

$L(p,q)$ spaces are linear spaces and $\|\cdot\|^*_{pq}$ is a quasi-norm which is a norm for $1 \leqslant q < p < \infty$. For $t > 0$, let

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\,\mathrm{d}s.$$

Now the functional defined as

$$\|f\|_{pq} = \begin{cases} \left\{ \dfrac{q}{p} \displaystyle\int_0^\infty (t^{1/p} f^{**}(t))^q \frac{\mathrm{d}t}{t} \right\}^{1/q}, & 1 < p < \infty,\ 1 \leqslant q < \infty, \\ \sup\limits_{t>0} t^{1/p} f^{**}(t), & 1 < p \leqslant \infty,\ q = \infty \end{cases}$$

is equivalent to $\|\cdot\|^*_{pq}$ and $L(p,q)$ is a normed linear space with respect to $\|\cdot\|_{pq}$. The $L(p,q)$ space is moreover a Banach space. The $L^p$-spaces for $1 < p \leqslant \infty$ are equivalent to the spaces $L(p,p)$. For more details on Lorentz spaces one can refer to [2], [7] and [8] and references therein. For $X = \mathbb{N}$ with $\mathcal{A} = 2^{\mathbb{N}}$, the power set of $X$, and $\mu =$ counting measure, the distribution function of any complex-valued function $a = \{a(n)\}_{n\geqslant 1}$ can be written as

$$\mu_a(s) = \mu\{n \in \mathbb{N}\colon\ |a(n)| > s\}, \quad s \geqslant 0.$$

The *non-increasing rearrangement* $a^*$ of $a$ is given as

$$a^*(t) = \inf\{s > 0\colon\ \mu_a(s) \leqslant t\}, \quad t \geqslant 0.$$

We can interpret the non-increasing rearrangement of $a$ with $\mu_a(s) < \infty$, $s > 0$, as a sequence $\{a^*(n)\}$ if we define for $n-1 \leqslant t < n$

$$a^*(n) = a^*(t) = \inf\{s > 0\colon\ \mu_a(s) \leqslant n-1\}.$$

Then the sequence $a^* = \{a^*(n)\}$ is obtained by permuting $\{|a(n)|\}_{n\in S}$, $S = \{n\colon\ a(n) \neq o\}$, in the decreasing order with $a^*(n) = 0$ for $n > \mu(S)$ if $\mu(S) < \infty$.

The *Lorentz sequence space* $l(p,q)$, $1 < p \leqslant \infty$, $1 \leqslant q \leqslant \infty$, is the set of all complex sequences $a = \{a(n)\}$ such that $\|a\|_{(p,q)} < \infty$, where

$$\|a\|_{(p,q)} = \begin{cases} \left\{ \displaystyle\sum_{n=1}^\infty (n^{1/p} a^*(n))^q n^{-1} \right\}^{1/q}, & 1 < p < \infty,\ 1 \leqslant q < \infty, \\ \sup\limits_{n\geqslant 1} n^{1/p} a^*(n), & 1 < p \leqslant \infty,\ q = \infty. \end{cases}$$

The *Lorentz sequence space* $l(p,q)$, $1 < p \leqslant \infty$, $1 \leqslant q \leqslant \infty$, is a linear space and $\|\cdot\|_{(p,q)}$ is a quasi-norm. Moreover, $l(p,q)$, $1 < p \leqslant \infty$, $1 \leqslant q \leqslant \infty$, is complete with respect to the quasi-norm $\|\cdot\|_{(p,q)}$ and $l(p,q)$, $1 \leqslant q \leqslant p < \infty$ is a complete normed linear space with respect to $\|\cdot\|_{(p,q)}$. Throughout this paper we consider the spaces $l(p,q)$, $1 < p \leqslant \infty$, $1 \leqslant q \leqslant \infty$, with respect to $\|\cdot\|_{(p,q)}$. Such spaces $l(p,q)$ fall in the category of $L(p,q)$ spaces [8] as well as in the category of functional Banach spaces [7]. The $l^p$-spaces for $1 < p \leqslant \infty$ are equivalent to the spaces $l(p,p)$. In [7], [9], a description of the duals, isomorphic $l^p$-subspaces of *Orlicz-Lorentz sequence spaces* $L_{\varphi,w}$ is given and in [12] isomorphic properties of Orlicz-Lorentz sequence spaces are discussed.

The *Lorentz sequence space* $l(p,q)$ coincides with $L_{\varphi,w}$ when $\varphi(t) = t^q$ and the weight sequence is $w(n) = n^{(q/p)-1}$. In the case of the Lorentz sequence space $l(p,q)$ one can have a better feeling of the behavior of multiplication, composition operators and the inducing sequences while in the case of the abstract Lorentz space $L(p,q)$ as well as the Banach function spaces [6] it becomes difficult. Multiplication and composition operators are studied in various function spaces in [1], [3], [5], [6], [13] and [14]. In [15], Singh studied these operators on the weak Lebesgue space $l^p$.

Let $u = \{u(n)\}$ be a complex sequence. We define a linear transformation $M_u$ on the Lorentz sequence space $l(p,q)$, $1 < p \leqslant \infty$, $1 \leqslant q \leqslant \infty$, into the linear space of all complex sequences by

$$M_u(a) = ua = \{u(n)a(n)\}, \text{ where } a = \{a(n)\}.$$

If $M_u$ is bounded with range in $l(p,q)$, then it is called a multiplication operator on $l(p,q)$. For a mapping $T\colon \mathbb{N} \to \mathbb{N}$ we define a linear transformation $C_T$ on the Lorentz sequence space $l(p,q)$, $1 < p \leqslant \infty$, $1 \leqslant q \leqslant \infty$, into the linear space of all complex sequences by

$$C_T(a) = a \circ T = \{a(T(n))\}, \text{ where } a = \{a(n)\}.$$

If $C_T$ is bounded with range in $l(p,q)$, then it is called a composition operator on $l(p,q)$. By $\mathcal{B}(l(p,q))$ we mean the algebra of all bounded linear operators on $l(p,q)$. An operator $A \in \mathcal{B}(l(p,q))$ is said to be *Fredholm* if it has closed range, $\dim(\operatorname{Ker}(A))$ and $\operatorname{codim}(R(A))$ are finite, where $\dim(\operatorname{Ker}(A))$ is the dimension of the kernel of $A$ and $\operatorname{codim}(R(A))$ is the co-dimension of the range of $A$, namely the dimension of any subspace complementary to the range of $A$.

In this paper we are interested in the study of compactness, Fredholmness, invertibility etc. of multiplication and composition operators on the Lorentz sequence spaces $l(p,q)$, $1 < p \leqslant \infty$, $1 \leqslant q \leqslant \infty$. It is shown in this paper that there exists a plenty of compact multiplication operators on $l(p,q)$. Multiplication and composition operators having closed ranges are also characterized.

## 2. Characterizations: Multiplication operators

The section is devoted to the study of multiplication operators $M_u$ on the space $l(p,q)$, $1 < p \leqslant \infty$, $1 \leqslant q \leqslant \infty$, induced by a sequence $u = \{u(n)\}$. It follows immediately from [6] Theorem 2.4 that the only compact multiplication operator on the non-atomic Lorentz space is the zero operator. In the case of the Lorentz sequence space we show the existence of plenty of compact non-zero multiplication operators on $l(p,q)$, $1 < p \leqslant \infty$, $1 \leqslant q \leqslant \infty$, and compact multiplication operators are characterized.

**Theorem 2.1.** *Let $u = \{u(n)\}$ be a complex sequence. Then $M_u$ induced by $u$ is bounded on $l(p,q)$, $1 < p \leqslant \infty$, $1 \leqslant q \leqslant \infty$, if and only if $\{u(n)\}$ is bounded.*

Proof. If $M_u$ is a bounded operator, then there exists $K > 0$ such that

$$\|M_u a\|_{(p,q)} \leqslant K\|a\|_{(p,q)} \text{ for all } a = \{a(n)\} \in l(p,q).$$

For each $n \in \mathbb{N}$ and $e_n = \{e_n(m)\}_m$ in $l(p,q)$, where

$$e_n(m) = \begin{cases} 1 & \text{if } m = n, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad e_n^*(m) = \begin{cases} 1 & \text{if } m = 1, \\ 0 & \text{otherwise,} \end{cases}$$

we have $\|e_n\|_{(p,q)} = 1$ and so

$$\|M_u e_n\|_{(p,q)}^q \leqslant K^q \|e_n\|_{(p,q)}^q.$$

This gives, for $1 < p < \infty$, $1 \leqslant q < \infty$,

$$\sum_{m=1}^{\infty} ((ue_n)^*(m))^q m^{(q/p)-1} \leqslant K^q \sum_{m=1}^{\infty} (e_n^*(m))^q m^{(q/p)-1}$$
$$\Rightarrow (ue_n)^*(1) \leqslant K e_n^*(1), \text{ that is, } |u(n)| \leqslant K,$$

and for $q = \infty$, $1 < p \leqslant \infty$,

$$\sup_{m \geqslant 1} m^{1/p}((ue_n)^*(m)) \leqslant K \sup_{m \geqslant 1} m^{1/p}(e_n^*(m))$$
$$\Rightarrow (ue_n)^*(1) \leqslant K e_n^*(1), \text{ that is, } |u(n)| \leqslant K.$$

Thus in any case $\{u(n)\}$ is a bounded sequence.

Conversely, if $u = \{u(n)\}$ satisfies $|u(n)| \leqslant K$ for all $n \in \mathbb{N}$ and some $K > 0$, then for any $a = \{a(n)\}$ in $l(p,q)$, $ua = \{u(n)a(n)\}$ satisfies

$$|u(n)a(n)| \leqslant K|a(n)|.$$

This gives $(ua)^*(n) \leqslant Ka^*(n)$ for each $n \in \mathbb{N}$, and so we obtain

$$\|M_u a\|_{(p,q)} = \begin{cases} \left\{ \sum_{n=1}^{\infty} ((ua)^*(n))^q n^{(q/p)-1} \right\}^{1/q}, & 1 < p < \infty,\ 1 \leqslant q < \infty, \\ \sup_{n \geqslant 1} n^{1/p} (ua)^*(n), & 1 < p \leqslant \infty,\ q = \infty \end{cases}$$
$$\leqslant K\|a\|_{(p,q)}.$$

Thus $M_u$ is bounded on $l(p,q)$, $1 < p \leqslant \infty$, $1 \leqslant q \leqslant \infty$. □

**Theorem 2.2.** *Let $M_u \in \mathcal{B}(l(p,q))$, $1 < p \leqslant \infty$, $1 \leqslant q < \infty$. Then $M_u$ is invertible if and only if there is $\delta > 0$ such that*

$$|u(n)| \geqslant \delta \quad \textit{for all } n \in \mathbb{N}.$$

P r o o f. If $M_u$ is invertible then we find $\delta > 0$ satisfying

$$\|M_u a\|_{(p,q)} \geqslant \delta \|a\|_{(p,q)} \quad \text{for all } a \in l(p,q).$$

In particular, for $e_n = \{e_n(m)\}$ this gives $|u(n)| \geqslant \delta$.

Conversely, if $|u(n)| \geqslant \delta$ for all $n \in \mathbb{N}$ and some $\delta > 0$, then define another sequence $v = \{v(n)\}$ where $v(n) = 1/u(n)$. Clearly, in view of Theorem 2.1, $M_v$ is bounded on $l(p,q)$ and $M_v = M_u^{-1}$. □

**Theorem 2.3.** *Let $M_u \in \mathcal{B}(l(p,q))$, $1 < p \leqslant \infty$, $1 \leqslant q \leqslant \infty$. Then $M_u$ has closed range if and only if for some $\delta > 0$,*

$$|u(n)| \geqslant \delta \quad \textit{for all } n \in S,$$

*where $S = \{n \in \mathbb{N} \colon\ u(n) \neq 0\}$.*

P r o o f. Suppose $|u(n)| \geqslant \delta$ for all $n \in S$ and some $\delta > 0$. We claim that $M_u|_{l_{pq}(S)}$ has closed range where

$$l_{pq}(S) = \{a = \{a(n)\} \in l(p,q) \colon\ a(n) = 0 \text{ for } n \in \mathbb{N} \setminus S\}.$$

Let $f, f_k \in l_{pq}(S)$ where $f = \{f(n)\}$ and for each $k \geqslant 1$, $f_k = \{f_k(n)\}$ are such that $M_u f_k \to f$ as $k \to \infty$. Then we have, as $n, m \to \infty$,

$$\|M_u f_n - M_u f_m\|_{(p,q)} \to 0.$$

Put $a_{nm} = f_n - f_m$, then for each $s > 0$,

$$\{k \in \mathbb{N}\colon\ |u(k)a_{nm}(k)| > s\} \supseteq \{k \in \mathbb{N}\colon\ |a_{nm}(k)| > s/\delta\}.$$

This gives $\delta a^*_{nm}(k) \leqslant (ua_{nm})^*(k)$ for each $k \in \mathbb{N}$. Therefore

$$\begin{aligned}
\|ua_{nm}\|_{(p,q)} &= \|M_u f_n - M_u f_m\|_{(p,q)} \\
&= \begin{cases} \left\{\sum\limits_{k\in S}((ua_{nm})^*(k))^q k^{(q/p)-1}\right\}^{1/q}, & 1 < p < \infty,\ 1 \leqslant q < \infty, \\ \sup\limits_{k\in S} k^{1/p}(ua^*_{nm}(k), & 1 < p \leqslant \infty,\ q = \infty \end{cases} \\
&\geqslant \begin{cases} \left\{\sum\limits_{k\in S}\delta^q((a_{nm})^*(k))^q k^{(q/p)-1}\right\}^{1/q}, & 1 < p < \infty,\ 1 \leqslant q < \infty, \\ \sup\limits_{k\in S} k^{1/p}\delta(a^*_{nm}(k), & 1 < p \leqslant \infty,\ q = \infty \end{cases} \\
&= \delta\|a_{nm}\|_{(p,q)}.
\end{aligned}$$

Since $\|ua_{nm}\|_{(p,q)} \to 0$ as $n, m \to \infty$, this implies $a_{nm} \to 0$ as $n, m \to \infty$. This means $\{f_k\}$ is a Cauchy sequence in $l_{pq}(S)$, which is a closed subspace of $l(p,q)$.

Hence we can find $g \in l_{pq}(S)$ such that $f_k \to g$ as $k \to \infty$. By virtue of the continuity of $M_u$, $M_u f_k \to M_u g$. Hence $f = M_u g$ and thus $M_u|_{l_{pq}(S)}$ has closed range. Since $\mathrm{Ker}(M_u) = l_{pq}(\mathbb{N}\setminus S)$, we find that $M_u$ has closed range.

Conversely, if the condition does not hold, then for each $n \in \mathbb{N}$ we can find $k_n \in S$ satisfying

$$|u(k_n)| < 1/n.$$

For each $n$, the sequence $e_{k_n} = \{e_{k_n}(m)\}$, where

$$e_{k_n}(m) = \begin{cases} 1 & \text{if } m = k_n, \\ 0 & \text{otherwise}, \end{cases}$$

satisfies $\|e_{k_n}\|_{(p,q)} = 1$ and

$$\begin{aligned}
\|M_u e_{k_n}\|_{(p,q)} &= \|ue_{k_n}\|_{(p,q)} \\
&= \begin{cases} \left\{\sum\limits_{m=1}^{\infty}((ue_{k_n})^*(m))^q m^{(q/p)-1}\right\}^{1/q}, & 1 < p < \infty,\ 1 \leqslant q < \infty, \\ \sup\limits_{m\geqslant 1} m^{1/p}(ue_{k_n})^*(m), & 1 < p \leqslant \infty,\ q = \infty \end{cases} \\
&= (ue_{k_n})^*(1) = |u(k_n)| < \frac{1}{n}\|e_{k_n}\|_{(p,q)}.
\end{aligned}$$

Thus $M_u$ is not bounded away from zero, a contradiction. Hence the result. □

**Theorem 2.4.** *Let $M_u \in \mathcal{B}(l(p,q)), 1 < p \leqslant \infty$, $1 \leqslant q \leqslant \infty$. A necessary and sufficient condition for $M_u$ to be compact is that $|u(n)| \to 0$ as $n \to \infty$.*

Proof. Suppose $u(n)$ does not tend to 0 as $n \to \infty$. Then $|u(n)| \geqslant \delta$ for infinitely many values of $n$ and some $\delta > 0$. Let

$$A = \{n \in \mathbb{N} \colon |u(n)| \geqslant \delta\} \quad \text{and} \quad B = \{e_k = \{e_k(n)\} \colon k \in A\}.$$

Then $B$ is a bounded set in $l(p,q)$. Moreover, for each $n, k, l \in A$,

$$|(ue_k - ue_l)(n)| \geqslant \delta |(e_k - e_l)(n)|$$

and so

$$(ue_k - ue_l)^*(n) \geqslant \delta (e_k - e_l)^*(n).$$

Thus

$$\|M_u e_k - M_u e_l\|_{(p,q)} \geqslant \delta \|e_k - e_l\|_{(p,q)}$$

or

$$\|M_u e_k - M_u e_l\|_{(p,q)} \geqslant \delta \quad \text{for } k \neq l,$$

which shows that $M_u$ is not compact.

Conversely, if $u(n) \to 0$ as $n \to \infty$, we can find $\delta > 0$ and $n_0 \in \mathbb{N}$ such that $|u(n)| < \delta$ for all $n \geqslant n_0$. For each $n \in \mathbb{N}$, define $u_n \equiv \{u_n(k)\}$, where

$$u_n(k) = \begin{cases} u(k) & \text{if } k \leqslant n, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\{u_n(k)\}$ is a bounded sequence so that $M_{u_n}$ is bounded on $l(p,q)$. Moreover, each $M_{u_n}$ is compact and one can check that $M_{u_n} \to M_u$ uniformly. This yields that $M_u$ is compact. □

As one can easily find that if $\mathbb{N} \setminus S$ is a finite set then $\operatorname{Ker}(M_u)$ and range of $M_u$ are subspaces generated by $\{e_n \colon n \in \mathbb{N} \setminus S\}$ and $\{e_m \colon m \in S\}$ respectively, we have

**Theorem 2.5.** *Let $M_u \in \mathcal{B}(l(p,q)), 1 < p \leqslant \infty$, $1 \leqslant q \leqslant \infty$. Then $M_u$ is Fredholm if and only if $\mathbb{N} \setminus S$ is finite and there exists $\delta > 0$ such that*

$$|u(n)| \geqslant \delta \quad \text{for all } n \in \mathbb{N}.$$

## 3. Characterizations: Composition operators

In this section, isometric and Fredholm composition operators are characterized. The study of boundedness, compactness and closed range of composition operators on $l(p,q)$, $1 < p \leqslant \infty$, $1 \leqslant q \leqslant \infty$, is also included.

**Theorem 3.1.** *A mapping* $T\colon \mathbb{N}\to\mathbb{N}$ *induces a bounded composition operator*

$$C_T\colon\ a\mapsto a\circ T$$

*on* $l(p,q)$, $1<p\leqslant\infty$, $1\leqslant q\leqslant\infty$, *if and only if there exists* $M>0$ *such that*

$$\mu T^{-1}(\{n\})\leqslant M \text{ for all } n\in\mathbb{N}.$$

P r o o f. In case $C_T$ is bounded, we have for some $R>0$

$$\|C_T a\|_{(p,q)}\leqslant R\|a\|_{(p,q)} \text{ for all } a\in l(p,q).$$

Let $n\in\mathbb{N}$ be such that $T^{-1}(\{n\})$ is not empty.
Then $e_n=\{e_n(k)\}\in l(p,q)$ and hence

$$\|C_T e_n\|_{(p,q)}\leqslant R\|e_n\|_{(p,q)}=R,$$

that is,

$$\|e_{T^{-1}(\{n\})}\|_{(p,q)}\leqslant R.$$

However, $e_{T^{-1}(\{n\})}=\{e_{T^{-1}(\{n\})}(k)\}$ where

$$e_{T^{-1}(\{n\})}(k)=\begin{cases}1 & \text{if } k\in T^{-1}(\{n\}),\\ 0 & \text{otherwise.}\end{cases}$$

Then

$$e^*_{T^{-1}(\{n\})}(k)=\begin{cases}1 & \text{if } k=1,2,\dots,\mu T^{-1}(\{n\}),\\ 0 & \text{otherwise.}\end{cases}$$

Hence

$$\begin{aligned}
R &\geqslant \|e_{T^{-1}(\{n\})}\|_{(p,q)}\\
&=\begin{cases}\Big\{\displaystyle\sum_{k=1}^{\mu T^{-1}(\{n\})} k^{(q/p)-1}\Big\}^{1/q}, & 1<p<\infty,\ 1\leqslant q<\infty,\\[2ex] \displaystyle\sup_{k=1,2,\dots,\mu T^{-1}(\{n\})} k^{1/p}e^*_{T^{-1}(\{n\})}(k), & 1<p\leqslant\infty, q=\infty\end{cases}\\
&=\begin{cases}\Big\{(1)+\Big(\dfrac{1}{2^{1-q/p}}\Big)+\dots+\Big(\dfrac{1}{(\mu T^{-1}(\{n\}))^{1-q/p}}\Big)\Big\}^{1/q}, & 1<p<\infty,\ 1\leqslant q<\infty,\\ (\mu T^{-1}(\{n\})), & 1<p\leqslant\infty,\ q=\infty\end{cases}\\
&\geqslant\begin{cases}\Big\{(\mu T^{-1}(\{n\})\Big(\dfrac{1}{(\mu T^{-1}(\{n\}))^{1-q/p}}\Big)\Big\}^{1/q}, & 1\leqslant q<p<\infty,\\ (\mu T^{-1}(\{n\})^{1/q}, & 1<p\leqslant q<\infty,\\ (\mu T^{-1}(\{n\})^{1/p}, & 1<p\leqslant\infty,\ q=\infty\end{cases}\\
&=\begin{cases}(\mu T^{-1}(\{n\})^{1/p}, & 1\leqslant q<p<\infty \text{ or } 1<p\leqslant\infty,\ q=\infty,\\ (\mu T^{-1}(\{n\})^{1/q}, & 1<p\leqslant q<\infty.\end{cases}
\end{aligned}$$

Hence in any case we can find $M > 0$ such that $\mu T^{-1}(\{n\}) \leqslant M$ for each $n \in \mathbb{N}$.

Conversely, if $\mu T^{-1}(\{n\}) \leqslant M$ for some $M \in \mathbb{N}$ then for any $a = \{a(n)\}$ in $l(p,q)$ and $a \circ T = \{(a \circ T)(n)\}$ we have for all $t > 0$

$$(a \circ T)^*(Mt) \leqslant a^*(t),$$

and so for all $k \in \mathbb{N} \cup \{0\}$ and $m = 1, 2, \ldots, M$ we have

$$(a \circ T)^*(kM + m) \leqslant a^*(k+1).$$

Hence, for $1 < p < \infty$, $1 \leqslant q < \infty$, taking $r = 1 - q/p$ we obtain

$$\begin{aligned}
&\|a \circ T\|^q_{(p,q)} \\
&\quad = \sum_{k=1}^{\infty} ((a \circ T)^*(k))^q k^{(q/p)-1} \\
&\quad = \Big[((a \circ T)^*(1))^q + ((a \circ T)^*(2))^q \frac{1}{2^r} + \ldots + ((a \circ T)^*(M))^q \frac{1}{M^r}\Big] \\
&\qquad + \Big[((a \circ T)^*(M+1))^q \frac{1}{(M+1)^r} + \ldots + ((a \circ T)^*(2M))^q \frac{1}{(2M)^r}\Big] + \ldots \\
&\quad \leqslant \Big[1 + \frac{1}{2^r} + \ldots + \frac{1}{M^r}\Big](a^*(1))^q + \Big[\frac{1}{(M+1)^r} + \ldots + \frac{1}{(2M)^r}\Big](a^*(2))^q \\
&\qquad + \Big[\frac{1}{(2M+1)^r} + \ldots + \frac{1}{(3M)^r}\Big](a^*(3))^q + \ldots \\
&\quad \leqslant \begin{cases} M\Big[(a^*(1))^q + \frac{1}{2^r}(a^*(2))^q + \frac{1}{3^r}(a^*(3))^q + \ldots\Big], & 1 \leqslant q < p < \infty, \\ M^{(1-r)}\Big[(a^*(1))^q + \frac{1}{2^r}(a^*(2))^q + \frac{1}{3^r}(a^*(3))^q + \ldots\Big], & 1 < p \leqslant q < \infty \end{cases} \\
&\quad = \begin{cases} M\|a\|^q_{(p,q)}, & 1 \leqslant q < p < \infty, \\ M^{q/p}\|a\|^q_{(p,q)}, & 1 < p \leqslant q < \infty \end{cases}
\end{aligned}$$

and for $q = \infty$, $1 < p \leqslant \infty$ we have

$$\|a \circ T\|^q_{(p,q)} \leqslant M\|a\|^q_{(p,q)}.$$

Thus $C_T$ is bounded on $l(p,q)$, $1 < p \leqslant \infty$, $1 \leqslant q \leqslant \infty$. $\square$

**Theorem 3.2.** *Let $C_T$ be a bounded linear composition operator on $l(p,q)$, $1 < p \leqslant \infty$, $1 \leqslant q \leqslant \infty$. Then the following conditions are equivalent:*
(1) *$T$ is invertible,*
(2) *$C_T$ is invertible,*
(3) *$C_T$ is an isometry.*

P r o o f. The proofs of (1) $\Leftrightarrow$ (2) follow the lines of the proof given in [15] in the case of $l^p$, which is independent of any other result except Theorem 3.1. Here we just prove the equivalence of (1) and (3). In case (1) holds, then for every $E \subseteq \mathbb{N}$

$$\mu\{T^{-1}(E)\} = \mu(E).$$

Then for each $a = \{a(n)\}$ in $l(p,q)$ and $a \circ T = \{(a \circ T)(n)\}$ we have for all $s > 0$

$$\mu_{a\circ T}(s) = \mu_a(s) \Rightarrow (a \circ T)^*(n) = a^*(n) \quad \text{for all } n \in \mathbb{N}.$$

Hence $\|C_T\|_{(p,q)} = \|a\|_{(p,q)}$ so that $C_T$ is an isometry.

Conversely, if $C_T$ is an isometry, then for each $n \in \mathbb{N}$ we have

$$\|C_T e_n\|_{(p,q)} = \|e_n\|_{(p,q)} = 1.$$

This implies $\mu T^{-1}(\{n\}) = 1$. Thus $T^{-1}(\{n\})$ is a singleton for each $n \in \mathbb{N}$. Hence $T$ is invertible. □

**Theorem 3.3.** *Let $C_T$ be a bounded linear composition operator on $l(p,q)$, $1 < p \leqslant \infty$, $1 \leqslant q \leqslant \infty$. Then $C_T$ is Fredholm if and only if both $\{n \in \mathbb{N} \colon \mu T^{-1}(\{n\}) \geqslant 2\}$ and $\mathbb{N} \setminus T(\mathbb{N})$ are finite.*

P r o o f. Suppose $C_T$ is Fredholm. If $E = \{n \in \mathbb{N} \colon \mu T^{-1}(\{n\}) \geqslant 2\}$ is not finite, then for each $k \in E$ let $n_k, m_k \in \mathbb{N}$ be such that $T(n_k) = T(m_k), n_k \neq m_k$. For each $k \in E$, define $f_k = \{f_k(m)\}$ where

$$f_k(m) = \begin{cases} 1 & \text{if } m = n_k, \\ -1 & \text{if } m = m_k, \\ 0 & \text{otherwise.} \end{cases}$$

Then each $f_k$ lies in $l(p,q)$ but not in range of $C_T$. Moreover, $\{f_k \colon k \in E\}$ being linearly independent implies $l(p,q) \setminus R(C_T)$ is infinite dimensional, a contradiction. Thus the set $E$ must be finite. Similarly, $\mathbb{N} \setminus T(\mathbb{N})$ being an infinite set implies that $\mathrm{Ker}(C_T)$ is infinite dimensional, a contradiction.

The converse is easy to prove. Hence the result follows. □

Along the lines of the proof carried out in [15] for $l_p$-spaces, we arrive at the following results:

(1) Let $C_T$ be a bounded linear composition operator on $l(p,q)$, $1 < p \leqslant \infty$, $1 \leqslant q \leqslant \infty$. Then $C_T$ has closed range but not a compact one.

(2) An operator $A$ on $l(p,q), 1 < p \leqslant \infty$, $1 \leqslant q \leqslant \infty$, is a composition operator if and only if there exists a partition $\{P_n\}$ of $\mathbb{N}$ such that

$$A(e_n) = \sum_{m \in P_n} e_m.$$

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