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PRIMENESS AND SEMIPRIMENESS IN POSETS

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Abstract. The concept of a semiprime ideal in a poset is introduced. Characterizations of semiprime ideals in a poset P as well as characterizations of a semiprime ideal to be prime in P are obtained in terms of meet-irreducible elements of the lattice of ideals of P and in terms of maximality of ideals. Also, prime ideals in a poset are characterized.

Keywords: semiprime ideal, prime ideal, meet-irreducible element, I-atom

MSC 2010: 06B10

1. Introduction

Y. Rav [10] introduced and studied semiprime ideals in lattices. An ideal I of a lattice L is said to be *semiprime* if $x \wedge y \in I$ and $x \wedge z \in I$ together imply $x \wedge (y \vee z) \in I$. Also, Beran [1] studied some properties of semiprimeness (see also Beran [2] and [4]) and the connection between primeness and semiprimeness in lattices. In fact, he proved that

Theorem A (L. Beran [2]). *Let I be a semiprime ideal of a lattice L. Then I is prime if and only if I is a meet irreducible element of* $\mathrm{Id}(L)$.

In this paper we introduce the concept of a semiprime ideal in a general poset. Characterizations of semiprime ideals in posets as well as characterizations of a semiprime ideal to be prime are obtained. Also, prime ideals in a poset are characterized. It is proved that a prime ideal I of P and its corresponding filter F_I make a separation of the poset P. Further, we prove some properties and characterizations of prime ideals and semiprime ideals in posets.

We begin with the necessary concepts and terminology. For undefined notation and terminology the reader is referred to Grätzer [5].

Let $A \subseteq P$. The set $A^u = \{x \in P;\ x \geqslant a \text{ for every } a \in A\}$ is called the *upper cone* of A. Dually, we have the concept of the *lower cone* A^l of A. We shall write A^{ul} instead of $\{A^u\}^l$ and dually. The upper cone $\{a\}^u$ is simply denoted by a^u and $\{a,b\}^u$ is denoted by $(a,b)^u$. Similar notation is used for lower cones. Further, for $A, B \subseteq P$, $\{A \cup B\}^u$ is denoted by $\{A, B\}^u$ and for $x \in P$, the set $\{A \cup \{x\}\}^u$ is denoted by $\{A, x\}^u$. Similar notation is used for lower cones. We note that $A \subseteq A^{ul}$ and $A \subseteq A^{lu}$. If $A \subseteq B$, then $B^l \subseteq A^l$ and $B^u \subseteq A^u$. Moreover, $A^{lul} = A^l$, $A^{ulu} = A^u$ and $\{a^u\}^l = \{a\}^l = a^l$.

Now, we consider a concept of an ideal and a prime ideal introduced by Halaš [6] and Halaš and Rachůnek [8].

Definition 1. A subset I of a poset P is called an *ideal* if $a, b \in I$ implies $(a,b)^{ul} \subseteq I$. A proper ideal I is called *prime* if $(a,b)^l \subseteq I$ implies that either $a \in I$ or $b \in I$.

Dually, we have the concepts of a *filter* and a *prime filter*. Given $a \in P$, the subset $\{x \in P;\ x \leqslant a\}$ is an ideal of P generated by a, denoted by $(a]$; we shall call $(a]$ a *principal ideal*. Dually, a filter $[a)$ generated by a is called a principal *filter*.

We generalize the concept of a semiprime ideal to a general poset as follows:

Definition 2. An ideal I of a poset P is called *semiprime* if $(a,b)^l \subseteq I$ and $(a,c)^l \subseteq I$ together imply $\{a, (b,c)^u\}^l \subseteq I$.

Dually, we have the concept of a *semiprime filter*. In what follows, $\mathrm{Id}(P)$ denotes the set of all ideals of a poset P which forms a complete lattice with respect to set inclusion (see Halaš and Rachůnek [8]).

The following result establishes a connection between primeness and semiprimeness:

Lemma 3. *Let I be an ideal of a poset P. If I is prime, then I is semiprime.*

P r o o f. Let I be a prime ideal and for $a, b, c \in P$, let $(a,b)^l \subseteq I$ and $(a,c)^l \subseteq I$. Since I is prime, we have two cases:

(i) If $a \in I$, then $\{a, (b,c)^u\}^l \subseteq a^l \subseteq I$.

(ii) If $a \notin I$, then $b, c \in I$ and hence $(b,c)^{ul} \subseteq I$. Therefore $\{a, (b,c)^u\}^l \subseteq (b,c)^{ul} \subseteq I$. Thus I is semiprime. $\square$

R e m a r k 4. The converse of Lemma 3 does not hold in general. In the poset depicted in Figure 1, the ideal $I = \{0\}$ is semiprime but not prime, as $(a,b)^l \subseteq I$ and neither a nor b is in I.

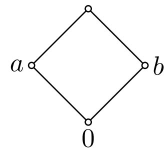

Figure 1

We consider the following definition of meet-irreducible elements.

Definition 5. An element x of a poset P is called *meet-irreducible* if x cannot be obtained as a meet of two elements different from x.

Theorem 6. *Every prime ideal of a poset P is a meet-irreducible element of* $\mathrm{Id}\,(P)$.

Proof. Let I be a prime ideal such that $I = J \cap K$ for $J, K \in \mathrm{Id}\,(P)$. We have to show that either $I = J$ or $I = K$.

Clearly $I \subseteq J$ and $I \subseteq K$. Suppose $I \neq J$ and $I \neq K$; then there exist $x, y \in P$ such that $x \in J \setminus I$ and $y \in K \setminus I$. But J and K are ideals, so we have $(x, y)^l \subseteq J \cap K \subseteq I$. Since I is prime, either $x \in I$ or $y \in I$, a contradiction with the fact that $x, y \notin I$. □

Remark 7. The converse of Theorem 6 is not true in general. Consider the poset P depicted in Figure 2 and its ideal lattice $\mathrm{Id}(P)$, depicted in Figure 3. Observe that $(a]$ is a meet-irreducible element of $\mathrm{Id}(P)$. However, $(a]$ is not prime as $(b, c)^l \subseteq (a]$ and neither b nor c is in $(a]$.

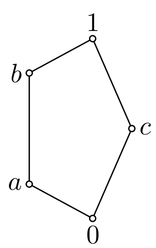

Figure 2

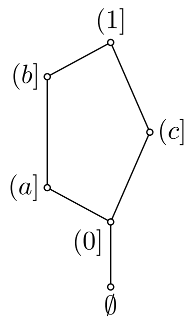

Figure 3

For ideals $I, J \subseteq P$ denote $C_1(I, J) = \bigcup\{(a, b)^{ul};\ a, b \in I \cup J\}$. Inductively, let $C_{n+1}(I, J) = \bigcup\{(a, b)^{ul};\ a, b \in C_n(I, J)\}$ for each $n \in \mathbb{N}$. It is easy to observe that the sets $C_n\,(I, J)$ form a chain, in other words, $C_1 \subseteq \ldots \subseteq C_{n-1} \subseteq C_n \subseteq \ldots$

The following result describes the join of two elements in $\mathrm{Id}(P)$.

Lemma 8 (Halaš [7]). *Let P be a poset and $I, J \in \mathrm{Id}(P)$. Then $I \vee J = \bigcup\{C_n(I,J);\ n \in \mathbb{N}\}$.*

The following statement characterizes semiprime ideals which are prime.

Theorem 9. *Let I be a semiprime ideal in a poset P. Then I is prime if and only if I is a meet-irreducible element of* $\mathrm{Id}(P)$.

Proof. ($\Rightarrow$) Follows by Theorem 6.

($\Leftarrow$) Let $(a,b)^l \subseteq I$. We claim that $I = (I \vee (a]) \cap (I \vee (b])$. Clearly, it is enough to show that $(I \vee (a]) \cap (I \vee (b]) \subseteq I$. In view of Lemma 8, we have to show that $C_n(I,(a]) \cap C_m(I,(b]) \subseteq I$ for all $n, m \in \mathbb{N}$. We proceed by induction on $n+m$.

(i) Assume $n+m=2$ and let $z \in C_1(I,(a]) \cap C_1(I,(b])$. We have $z \in (x_1,y_1)^{ul} \cap (x_2,y_2)^{ul}$ for $x_1, y_1 \in I \cup (a]$ and $x_2, y_2 \in I \cup (b]$. We distinguish two cases:

(1) If $x_1, y_1 \in I$ or $x_1, y_1 \in (a]$ or $x_2, y_2 \in I$ or $x_2, y_2 \in (b]$, by Definition 1 we obtain $z \in I$.

(2) Assume $x_1 \in I$, $y_1 \in (a]$, $x_2 \in I$ and $y_2 \in (b]$. Then $(x_1,x_2)^l \subseteq I$ and $(x_1,y_2)^l \subseteq I$. By semiprimeness of I we have $\{x_1,(x_2,y_2)^u\}^l \subseteq I$. Since $z \in (x_2,y_2)^{ul}$, we get $(x_1,z)^l \subseteq I$. Similarly, since $(y_1,x_2)^l \subseteq I$, $(y_1,y_2)^l \subseteq I$ and $z \in (x_2,y_2)^{ul}$, we have $(y_1,z)^l \subseteq I$. Now, $(x_1,z)^l \subseteq I$ and $(y_1,z)^l \subseteq I$ together yield $\{z,(x_1,y_1)^u\}^l \subseteq I$. But $z \in (x_1,y_1)^{ul}$, thus $z \in I$. Therefore the statement is true for $n+m=2$.

(ii) Suppose the statement is true for $n+m=r$; in other words, suppose $C_n(I,(a]) \cap C_m(I,(b]) \subseteq I$ holds for $n+m=r$. We shall show that the statement is true for $n+m=r+1$. Let $z \in C_n(I,(a]) \cap C_{m+1}(I,(b])$. Then $z \in (x_1,y_1)^{ul} \cap (x_2,y_2)^{ul}$ for $x_1,y_1 \in C_{n-1}(I,(a])$ and $x_2,y_2 \in C_m(I,(b])$. Observe that $(x_2,y_1)^l \subseteq C_{n-1}(I,(a]) \cap C_m(I,(b])$. Since the sets $C_n(I,(a])$ form a chain (i.e., $C_1 \subseteq \ldots \subseteq C_{n-1} \subseteq C_n \subseteq \ldots$), we have $(x_2,y_1)^l \subseteq C_n(I,(a]) \cap C_m(I,(b])$. By the induction hypothesis, $(x_2,y_1)^l \subseteq I$. Also, $(x_2,x_1)^l \subseteq C_{n-1}(I,(a]) \cap C_m(I,(b]) \subseteq C_n(I,(a]) \cap C_m(I,(b]) \subseteq I$. By semiprimeness of I, we get $\{x_2,(x_1,y_1)^u\}^l \subseteq I$. But $z \in (x_1,y_1)^{ul}$, thus $(x_2,z)^l \subseteq I$. Similarly, we get $(y_2,z)^l \subseteq I$. Again by semiprimeness of I and the fact that $z \in (x_2,y_2)^{ul}$ we have $z \in \{z,(x_2,y_2)^u\}^l \subseteq I$. Hence the statement is true for each $n, m \in \mathbb{N}$.

Now, since I is a meet-irreducible element of $\mathrm{Id}(P)$ and $I = (I \vee (a]) \cap (I \vee (b])$, we have either $I = (I \vee (a])$ or $I = (I \vee (b])$. Therefore, $a \in I$ or $b \in I$ and thus I is prime. □

Now we present more properties of semiprime ideals in posets by using the following sets. For a semiprime ideal I and a nonempty subset A of a poset P, define

$$I : A = \{z \in P;\ (a,z)^l \subseteq I \text{ for all } a \in A\}.$$

Observe that $I \subseteq I : A$. If $A = \{x\}$, then we write $I : x$ instead of $I : \{x\}$.

Lemma 10. *Let I be a semiprime ideal of a poset P. Then the following statements hold:*
(i) $(a,b)^l \subseteq I : x$ *if and only if* $(x,a,b)^l \subseteq I$.
(ii) $\{x,(a,b)^u\}^l \subseteq I$ *if and only if* $(a,b)^{ul} \subseteq I : x$.
(iii) $I : x = P$ *if and only if* $x \in I$.

Note: The statement (i) does not require semiprimeness.

Proof. (i) Suppose that $(x,a,b)^l \subseteq I$ and $z \in (a,b)^l$. It is clear that $(x,z)^l \subseteq (x,a,b)^l \subseteq I$. Thus $z \in I : x$.

Conversely, suppose that $(a,b)^l \subseteq I : x$ and $z \in (x,a,b)^l$. Since $z \in (x,a,b)^l \subseteq I : x$, we get $(x,z)^l \subseteq I$. But $z \leqslant x$, therefore $z \in I$.

(ii) Let $\{x,(a,b)^u\}^l \subseteq I$ and $z \in (a,b)^{ul}$. We have $(x,z)^l \subseteq \{x,(a,b)^u\}^l \subseteq I$, hence $z \in I : x$.

Conversely, suppose that $(a,b)^{ul} \subseteq I : x$. Then clearly $a, b \in I : x$. Hence $(x,a)^l \subseteq I$ and $(x,b)^l \subseteq I$. By semiprimeness we get $\{x,(a,b)^u\}^l \subseteq I$.

(iii) Observe that $x \in I$ iff $(x,z)^l \subseteq I$ for all $z \in P$ iff $I : x = P$. □

A nonempty subset Q of a poset P is called *up directed*, if $Q \cap (x,y)^u \neq \emptyset$ for any $x, y \in Q$. Dually, we have the concept of a *down directed* subset. If an ideal I (filter F) is an up (down) directed set of a poset P, then it is called a u-ideal (l-filter).

Definition 11. Let P be a poset and $Q \subseteq P$. Denote the set of all maximal or minimal elements of Q by $\mathrm{Max}(Q)$ or $\mathrm{Min}(Q)$, respectively.

In the following we use a statement the proof of which is trivial:

Lemma 12. *Every l-filter of a finite poset P is principal.*

We prove some characterizations of primeness and semiprimeness in the case of finite posets. To this end we introduce the concept of an I-atom in posets. Beran [3] defined the concept of an I-atom in lattices and has shown that this concept plays a crucial role in the study of prime ideals.

Definition 13. Let I be an ideal of a poset P. An element $i \in P$ is called an I-atom if
(i) $i \notin I$, and
(ii) for $x \in P$, if $x < i$, then $x \in I$.

Dually, we define an F-coatom for a given filter F of P.

R e m a r k 14. Note that for every finite poset P and I a proper ideal of P, an I-atom exists. However, an I-atom need not exist for P infinite. For example, in the poset P depicted in Figure 4, the ideal $I = \{0\}$ does not have any I-atom.

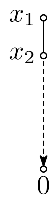

Figure 4

Note that two distinct I-atoms of a poset P are either equal or incomparable.

The following theorem is a characterization of semiprimeness in terms of $I : x$, $x \in P$.

Theorem 15. *Let I be an ideal of a poset P. Then I is semiprime if and only if $I : x$ is a semiprime ideal for all $x \in P$. Moreover, if P is finite, then I is semiprime iff $I : i$ is a principal prime ideal for every I-atom $i \in P$.*

P r o o f. First we show that $I : x$ is an ideal for all $x \in P$. Assume that I is semiprime and $a, b \in I : x$. We have to show that $(a, b)^{ul} \subseteq I : x$. Since $a, b \in I : x$, we obtain $(x, a)^l \subseteq I$ and $(x, b)^l \subseteq I$. By semiprimeness we get $\{x, (a, b)^u\}^l \subseteq I$. Due to Lemma 10(ii) we conclude $(a, b)^{ul} \subseteq I : x$.

To show that $I : x$ is semiprime, suppose that $(a, b)^l \subseteq I : x$ and $(a, c)^l \subseteq I : x$. We obtain $(x, a, b)^l \subseteq I$ and $(x, a, c)^l \subseteq I$, thus by Lemma 10(i) we have $(x, a)^l \subseteq I : b$ and $(x, a)^l \subseteq I : c$. We claim that $\{x, a, (b, c)^u\}^l \subseteq I$. Indeed, let $z \in \{x, a, (b, c)^u\}^l$. Since $z \in (x, a)^l \subseteq I : b$ and $z \in (b, c)^{ul}$, we have $z \in I : b$ and $z \in I : c$, i.e., $b, c \in I : z$. Since $I : z$ is an ideal, $(b, c)^{ul} \subseteq I : z$ and $z \in (b, c)^{ul}$, which yields $z \in I : z$. Thus $z \in I$.

Now we prove that $\{a, (b, c)^u\}^l \subseteq I : x$. Let $t \in \{a, (b, c)^u\}^l$. Clearly we have $(x, t)^l \subseteq \{x, a, (b, c)^u\}^l \subseteq I$. Hence $t \in I : x$, so $\{a, (b, c)^u\}^l \subseteq I : x$. Therefore $I : x$ is semiprime.

Conversely, suppose $I : x$ is an ideal for all $x \in P$. We shall show that I is semiprime. Let $(x, y)^l \subseteq I$ and $(x, z)^l \subseteq I$. Since $y, z \in I : x$ and $I : x$ is an ideal, we have $(y, z)^{ul} \subseteq I : x$. By using Lemma 10(ii), we get $\{x, (y, z)^u\}^l \subseteq I$ as required.

Further, let P be finite, let I be a semiprime ideal of P and i an I-atom of P. First we show that $I : i$ is prime. To this end assume $(x, y)^l \subseteq I : i$ and $x \notin I : i$. As $x \notin I : i$, we have, $(x, i)^l \nsubseteq I$. Therefore there exists an element $k \in (x, i)^l$ such that $k \notin I$. Clearly, $k \leqslant i$ and we claim $k = i$. Indeed, $k < i$ yields $k \in I$, a contradiction.

Consequently, we have $(x,i)^l = i^l$, i.e., $i \leqslant x$. Since $(x,y)^l \subseteq I : i$, by Lemma 10(i) we obtain $(x,y,i)^l \subseteq I$. Therefore $(y,i)^l \subseteq I$, $y \in I : i$ and thus $I : i$ is prime.

We show that $I : i$ is principal. In view of the statement dual to Lemma 12, it suffices to show that $I : i$ is an u-ideal. Suppose on the contrary that $I : i$ is not a u-ideal. Then there exist $b, c \in I : i$ such that there is no $x \in (b,c)^u$ for which $(i,x)^l \subseteq I$. Denote $(b,c)^u = \{x_1, x_2, \ldots, x_n\}$. Then for all x_j, $j = 1, 2, \ldots, n$ we have $(i, x_j)^l \nsubseteq I$. Therefore $(i,x_1)^l = (i,x_2)^l = \ldots = (i,x_n)^l = i^l$ and hence $i \leqslant x_j$ for all $x_j \in (b,c)^u$. Since I is semiprime, we have $\{i, (b,c)^u\}^l \subseteq I$. Therefore $i^l = \{i, (b,c)^u\}^l \subseteq I$, which is a contradiction to the fact that $i \notin I$. Thus $I : i$ is a u-ideal. $\square$

Let I be a proper ideal of a poset P. Then I is said to be a *maximal* ideal of P if the only ideal properly containing I is P.

The following statement gives another condition for semiprime ideals to be prime.

Theorem 16. *Every maximal semiprime ideal of a poset P is a prime ideal.*

Proof. Let I be a maximal semiprime ideal of P and $(x,y)^l \subseteq I$. We have $x \in I : y$, where $I : y$ is a semiprime ideal (see Theorem 15) with $I \subseteq I : y$. So we have two cases:

(i) If $I = I : y$, then $x \in I$.

(ii) If $I \subset I : y$, then by maximality of I we have $I : y = P$. By Lemma 10(iii), $y \in I$. Thus I is prime. $\square$

As a consequence, we have

Corollary 17. *Let I be a maximal ideal of a poset P. Then I is semiprime if and only if I is prime.*

By using Theorem 16 and Theorem 6, we obtain

Corollary 18. *Every maximal semiprime ideal of a poset P is a meet-irreducible element of* $\mathrm{Id}(P)$.

Theorem 19. *Let I be an ideal of a finite poset P. Then I is prime if and only if P has exactly one I-atom.*

Proof. Suppose I is a prime ideal. If I has two different I-atoms i_1 and i_2, then $(i_1, i_2)^l \subseteq I$ but $i_1, i_2 \notin I$, a contradiction.

Conversely, suppose P has exactly one I-atom and I is not a prime ideal. Then there exist $a, b \in P$ such that $(a,b)^l \subseteq I$ and neither a nor b is in I. However, there exist two I-atoms of P, say i_1 and i_2, for which $i_1 \leqslant a$, $i_2 \leqslant b$, $i_1 \neq i_2$, a contradiction. $\square$

Theorem 20. *Let I be a proper ideal of a poset P. Then I is prime if and only if $I : x = I$ for all $x \in P - I$.*

P r o o f. Suppose that I is prime, $z \in I : x$ and $x \in P - I$. Since I is prime, $(z, x)^l \subseteq I$ and $x \notin I$, we have $z \in I$.

Conversely, assume $I : x = I$ for all $x \in P - I$ and let $(x, y)^l \subseteq I$. If $x \notin I$, then $y \in I : x = I$. Thus I is prime. $\square$

Lemma 21. *Let I be a semiprime ideal of a finite poset P. Then $I = \bigcap_i I : i$ for all I-atoms i of P.*

P r o o f. We shall show that $\bigcap_i I : i \subseteq I$, as the converse inclusion always holds. Suppose on the contrary that $z \in \bigcap_i I : i$ and $z \notin I$. Then there exists an I-atom $j \in P$ such that $j \leqslant z$ and $j \notin I$. Since $z \in \bigcap_i I : i$, we have $z \in I : j$, which gives $(j, z)^l = j^l \subseteq I$, thus $j \in I$, a contradiction. $\square$

Lemma 22. *The intersection of any nonempty family of prime ideals of a poset P is a semiprime ideal.*

P r o o f. Suppose $I = \bigcap_n J_n$, $n \in \Gamma$, where each J_n is a prime ideal, and let $(x, y)^l \subseteq I$, $(x, z)^l \subseteq I$. We have to show that $\{x, (y, z)^u\}^l \subseteq I$. Since $(x, y)^l \subseteq J_n$ and $(x, z)^l \subseteq J_n$ for all n, by primeness of J_n's we have $x \in J_n$ or $y, z \in J_n$ for each n. In either case, we have $\{x, (y, z)^u\}^l \subseteq J_n$ for each n. Hence $\{x, (y, z)^u\}^l \subseteq \bigcap_n J_n = I$. Thus I is semiprime. $\square$

As an immediate consequence of Theorem 15, Lemma 21 and Lemma 22 in the case of finite posets we obtain

Theorem 23. *Let I be a proper ideal of a finite poset. Then I is semiprime if and only if I is representable as an intersection of prime ideals.*

We consider the following definition of a distributive poset which is essentialy due to Larmerová and Rachůnek [9].

Definition 24. A poset P is called *distributive* if $\{a, (b, c)^u\}^l = \{(a, b)^l, (a, c)^l\}^{ul}$ for all $a, b, c \in P$.

Note that in a distributive poset not every ideal is semiprime. Indeed, for the distributive poset P depicted in Figure 5, the ideal $I = \{0, a, b, c\}$ is not semiprime. Nonetheless, for principal ideals of P we have the following result.

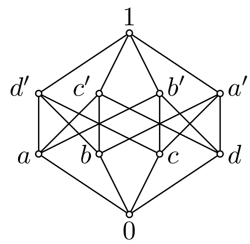

Figure 5

Theorem 25. *Let P be a poset. Then P is distributive if and only if $(x]$ is a semiprime ideal for all $x \in P$.*

Proof. ($\Rightarrow$) Suppose that $(a,b)^l \subseteq (x]$, $(a,c)^l \subseteq (x]$ and $z \in \{a,(b,c)^u\}^l$. We have $z^l = \{z,(b,c)^u\}^l$ and applying distributivity, we get $z^l = \{(z,b)^l,(z,c)^l\}^{ul}$. Since $z \leqslant a$, we obtain $(z,b)^l \subseteq (a,b)^l \subseteq (x]$ and $(z,c)^l \subseteq (a,c)^l \subseteq (x]$. Therefore $(z,b)^l \cup (z,c)^l \subseteq (x]$, i.e., $\{(z,b)^l,(z,c)^l\}^{ul} \subseteq (x]$. Thus $z^l \subseteq (x]$ which gives $z \in (x]$.

($\Leftarrow$) It is enough to prove $\{a,(b,c)^u\}^l \subseteq \{(a,b)^l,(a,c)^l\}^{ul}$, as the converse inclusion is always true. Let $x \in \{a,(b,c)^u\}^l$ and $y \in \{(a,b)^l,(a,c)^l\}^u$. We claim that $x \leqslant y$. Indeed, since $\{(a,b)^l,(a,c)^l\}^{ul} \subseteq y^l$, we have $(a,b)^l \subseteq y^l$ and $(a,c)^l \subseteq y^l$. By semiprimeness of $(y]$ we conclude $x \in \{a,(b,c)^u\}^l \subseteq y^l$. Thus $x \leqslant y$ as required. $\square$

An immediate consequence of Theorem 23 and Theorem 25 is

Corollary 26. *Let P be a finite poset. Then P is distributive if and only if every proper principal ideal is representable as an intersection of prime ideals.*

For an ideal I of a poset P, consider the set

$$F_I = \{z \in P;\ I : z = I\}.$$

In the following, we establish some properties of F_I and its connections with I.

Lemma 27. *Let I be a proper semiprime ideal of a finite poset P. Then $I : x \cap F_I = \emptyset$ for all $x \in P - I$.*

Proof. Let $x \in P - I$ and $z \in I : x \cap F_I$. We have $(z,x)^l \subseteq I$ and $I : z = I$. Hence $x \in I : z = I$, a contradiction to the fact that $x \notin I$. $\square$

Theorem 28. *Let I be an ideal of a poset P. Then F_I is a filter. Moreover, if I is a semiprime ideal in a finite poset P, then F_I is semiprime.*

Proof. Suppose $x, y \in F_I$ and $z \in (x,y)^{lu}$. To show that $z \in F_I$, it is enough to verify that $I : z \subseteq I$. Let $a \in I : z$. Then $(z,a)^l \subseteq I$. Since $(x,y)^l \subseteq z^l$, we obtain $(x,y,a)^l \subseteq (z,a)^l \subseteq I$. This yields $(y,a)^l \subseteq I : x$ by Lemma 10(i). Since $x \in F_I$, we get $(y,a)^l \subseteq I = I : x$. Hence $a \in I : y = I$, as $y \in F_I$. Thus $a \in I$.

Now, suppose that I is a semiprime ideal of a finite poset P. We have to show that F_I is semiprime. Suppose that $(x,y)^u \subseteq F_I$ and $(x,z)^u \subseteq F_I$. Let $a \in \{x,(y,z)^l\}^u$ and $a \notin F_I$. Then $I \subset I : a$, therefore there exists an element $b \in P$ such that $b \in I : a$ and $b \notin I$. Consequently, there exists an I-atom i of P such that $i \leqslant b$. Clearly $i \in I : a$ and so $a \in I : i$. Further, since $a \in \{x,(y,z)^l\}^u$, we have $x \in I : i$ and $(y,z)^l \subseteq I : i$. By Theorem 15, $I : i$ is a prime ideal, hence either $y \in I : i$ or $z \in I : i$. Suppose $y \in I : i$; then by Theorem 15, $I : i$ is a principal ideal and since $x, y \in I : i$, we have $I : i \cap (x,y)^u \neq \emptyset$. But $(x,y)^u \subseteq F_I$. Therefore $I : i \cap F_I \neq \emptyset$, a contradiction with Lemma 27. $\square$

Remark 29. Consider the infinite poset P depicted in Figure 6, and $I = \{0,a,b,c\}$ which is a semiprime ideal of P. Observe that $F_I = \{y_j;\ j \in \mathbb{N}\}$. F_I is a filter which is not semiprime as $(a,b)^u \subseteq F_I$ and $(a,c)^u \subseteq F_I$ but $a^u = \{a,(b,c)^l\}^u \nsubseteq F_I$. Thus, Theorem 28 is not true if we drop the condition of finiteness.

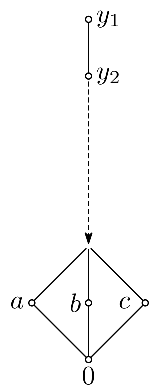

Figure 6

However, if P is a *join-semilattice*, then we have

Lemma 30. *Let I be a semiprime ideal of a join-semilattice P and let $x, y \in P$. Then $I : (x \vee y) = I : x \cap I : y$.*

Proof. We have $a \in I : (x \vee y)$ if and only if $(a, x \vee y)^l \subseteq I$ if and only if $\{(a,x)^l \subseteq I$ and $(a,y)^l \subseteq I\}$ if and only if $a \in I : x \cap I : y$. $\square$

Theorem 31. *Let I be a proper semiprime ideal of a join-semilattice P. Then F_I is a semiprime filter.*

Proof. Let $x \vee y$, $x \vee z \in F_I$ and $a \in \{x, (y,z)^l\}^u$. If $a \notin F_I$, then there exists an element $b \in P$ such that $b \in I : a$ and $b \notin I$, i.e., $(a,b)^l \subseteq I$ and $b \notin I$. Since $x \leqslant a$ and $(y,z)^l \subseteq a^l$, we have

$$(\star) \qquad b \in I : x \quad \text{and} \quad (y,b)^l \subseteq I : z.$$

Because $b \in I : x$, we have $(y,b)^l \subseteq I : x$. By Lemma 30 and $(\star)$, we obtain $(y,b)^l \subseteq I : x \cap I : z = I : (x \vee z) = I$. Thus $b \in I : y$ and $(\star)$ implies $b \in I : x$, which yields $b \in I : (x \vee y) = I$, a contradiction. □

Lemma 32. *Let I be a proper semiprime ideal of a poset P. Then $I \cap F_I = \emptyset$.*

Proof. Suppose $x \in I \cap F_I$. We have $I : x = I$. Since $x \in I$, by Lemma 10(iii) we have $I : x = P$ and consequently $I = I : x = P$, which is a contradiction to the fact that I is proper. □

The following theorem characterizes prime ideals in a poset:

Theorem 33. *Let I be a proper ideal of a poset P. Then I is prime if and only if $I \cup F_I = P$.*

Proof. Suppose I is prime, $x \in P$ and $x \notin F_I$. Since $x \notin F_I$, we have that $I \subset I : x$, i.e., there exists an element $y \in I : x$ such that $y \notin I$. In other words, $(y,x)^l \subseteq I$ and $y \notin I$. By primeness of I, we get $x \in I$ as required.

Conversely, suppose $I \cup F_I = P$, $(y,x)^l \subseteq I$ and $x \notin I$. Clearly, $x \in F_I$ and hence $I : x = I$. Since $y \in I : x$, we have $y \in I$. Thus, I is prime. □

Observe that Lemma 32 and Theorem 33 show that a prime ideal I and the corresponding filter F_I separate elements of a poset P.

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References

[1] *L. Beran*: On semiprime ideals in lattices. J. Pure Appl. Algebra *64* (1990), 223–227.
[2] *L. Beran*: Remarks on special ideals in lattices. Comment. Math. Univ. Carol. *35* (1994), 607–615.
[3] *L. Beran*: Length of ideals in lattices. Collect. Math. *46* (1995), 21–33.
[4] *L. Beran*: Some types of implicative ideals. Comment. Math. Univ. Carol. *39* (1998), 219–225.
[5] *G. Grätzer*: General Lattice Theory. Birkhäuser, second edition, 1998.
[6] *R. Halaš*: Characterization of distributive sets by generalized annihilators. Arch. Math. (Brno) *30* (1994), 25–27.
[7] *R. Halaš*: Decompositions of directed sets with zero. Math. Slovaca *45* (1995), 9–17.
[8] *R. Halaš, J. Rachůnek*: Polars and prime ideals in ordered sets. Discuss. Math., Algebra Stoch. Methods *15* (1995), 43–50.
[9] *J. Larmerová, J. Rachůnek*: Translations of distributive and modular ordered sets. Acta Univ. Palack. Olomouc, Fac. Rer. Nat. 91, Math. *27* (1988), 13–23.
[10] *Y. Rav*: Semiprime ideals in general lattices. J. Pure Appl. Algebra *56* (1989), 105–118.

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