Aleksander Misiak; Eugeniusz Stasiak

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G-SPACE OF ISOTROPIC DIRECTIONS AND G-SPACES OF φ-SCALARS WITH $G = O(n, 1, \mathbb{R})$

Aleksander Misiak, Eugeniusz Stasiak, Szczecin

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Abstract. There exist exactly four homomorphisms φ from the pseudo-orthogonal group of index one $G = O(n, 1, \mathbb{R})$ into the group of real numbers $\mathbb{R}_0$. Thus we have four G-spaces of φ-scalars $(\mathbb{R}, G, h_\varphi)$ in the geometry of the group G. The group G operates also on the sphere S^{n-2} forming a G-space of isotropic directions $(S^{n-2}, G, *)$. In this note, we have solved the functional equation $F(A*q_1, A*q_2, \ldots, A*q_m) = \varphi(A) \cdot F(q_1, q_2, \ldots, q_m)$ for given independent points $q_1, q_2, \ldots, q_m \in S^{n-2}$ with $1 \leqslant m \leqslant n$ and an arbitrary matrix $A \in G$ considering each of all four homomorphisms. Thereby we have determined all equivariant mappings $F\colon (S^{n-2})^m \to \mathbb{R}$.

Keywords: G-space, equivariant map, pseudo-Euclidean geometry

MSC 2010: 53A55

1. Introduction

For $n \geqslant 2$ consider the matrix $E_1 = \operatorname{diag}(+1, \ldots, +1, -1) \in GL(n, \mathbb{R})$.

Definition 1. A pseudo-orthogonal group of index one is a subgroup of the group $GL(n, \mathbb{R})$ satisfying the condition

$$G = O(n, 1, \mathbb{R}) = \{A\colon\ A \in GL(n, \mathbb{R}) \wedge A^T \cdot E_1 \cdot A = E_1\}.$$

It is known that there exist exactly four homomorphisms φ from the group G into the group $\mathbb{R}_0$. Denoting $A = [A_i^j]_1^n \in G$ we can specify these homomorphisms, namely $1(A) = 1, \varepsilon(A) = \det A = \operatorname{sign}(\det A)$, $\eta(A) = \operatorname{sign}(A_n^n)$ and $\varepsilon(A) \cdot \eta(A)$.

Definition 2. A G-space is the triple (M, G, f), where f is an operation of the group G on the set M.

Definition 3. By a G-space of φ-scalars we understand the triple $(\mathbb{R}, G, h_\varphi)$, where the mappings $\varphi\colon\ G \longrightarrow \mathbb{R}_0$ and $h_\varphi\colon\ \mathbb{R}\times G \longrightarrow \mathbb{R}$ fulfil the conditions

a) $\bigwedge\limits_{A,B\in G} \varphi(A\cdot B) = \varphi(A)\cdot\varphi(B)$,

b) $\bigwedge\limits_{x\in\mathbb{R}} \bigwedge\limits_{A\in G} h_\varphi(x, A) = \varphi(A)\cdot x$.

Let two G-spaces (M_α, G, f_α) and (M_β, G, f_β) be given.

Definition 4. A mapping $F_{\alpha\beta}\colon\ M_\alpha \longrightarrow M_\beta$ is called equivariant if the condition

$$\bigwedge_{x\in M_\alpha} \bigwedge_{A\in G} F_{\alpha\beta}(f_\alpha(x, A)) = f_\beta(F_{\alpha\beta}(x), A) \tag{1}$$

is fulfilled.

The class of G-spaces with equivariant maps as morphisms constitutes a category which is called a pseudo-Euclidean geometry of index one. In particular, there exist in this geometry the G-space of contravariant vectors

$$(\mathbb{R}^n, G, f), \qquad \text{where } \bigwedge_{u\in\mathbb{R}^n} \bigwedge_{A\in G} f(u, A) = A\cdot u, \tag{2}$$

and four G-spaces of objects with one component and linear transformation rule

$$(\mathbb{R}, G, h), \text{ where } \bigwedge_{x\in\mathbb{R}} \bigwedge_{A\in G} h(x, A) = \begin{cases} 1\cdot x & \text{for -scalars,} \\ \varepsilon(A)\cdot x & \text{for } \varepsilon\text{-scalars,} \\ \eta(A)\cdot x & \text{for } \eta\text{-scalars,} \\ \varepsilon(A)\cdot\eta(A)\cdot x & \text{for } \varepsilon\eta\text{-scalars.} \end{cases} \tag{3}$$

All equivariant maps from the product of linearly independent contravariant vectors into G-spaces of φ-scalars were determined in [4], [5] and [6]. In particular, the equivariant in the G-space of 1-scalars of a pair of vectors u and v is the invariant $p(u, v) = u^T\cdot E_1\cdot v$. In fact, for an arbitrary matrix $A\in G$ we have $p(Au, Av) = (Au)^T\cdot E_1\cdot(Av) = u^T\cdot(A^T\cdot E_1\cdot A)\cdot v = u^T\cdot E_1\cdot v = p(u, v)$. The invariant p enables us to determine an invariant subset of isotropic vectors, namely the transitive, isotropic cone $\overset{0}{V} = \{u\colon\ u\in\mathbb{R}^n \wedge p(u, u) = 0 \wedge u \neq 0\}$. Let us introduce in addition the sphere S^{n-2} included in the hyperplane $q^n = 1$ and immersed in the space $\mathbb{R}^n$, namely

$$S^{n-2} = \left\{q\colon\ q = [q^1, q^2, \ldots, q^{n-1}, 1]^T, \text{ where } \sum_{i=1}^{n-1}(q^i)^2 = 1 = q^n\right\}.$$

Let $q \in S^{n-2}$ and $A \in G$. For brevity let us denote $W(q,A) = \sum_{i=1}^{n} A_i^n q^i$. Let us recall (see [5]) that

$$\text{(4)} \qquad \bigwedge_{q \in S^{n-2}} \bigwedge_{A \in G} \operatorname{sign} W(q,A) = \operatorname{sign}(A_n^n) = \eta(A).$$

Because of $u^n \neq 0$ we can write every isotropic vector $u \in \overset{0}{V}$ in the form

$$u = [u^1, u^2, \ldots, u^n]^T = u^n \cdot \left[\frac{u^1}{u^n}, \ldots, \frac{u^{n-1}}{u^n}, 1\right]^T = u^n \cdot [q^1, q^2, \ldots, q^{n-1}, 1]^T = u^n \cdot q,$$

where $q \in S^{n-2}$. Let us call $u^n = u^n(u)$ the parameter and $q = q(u)$ the direction of the isotropic vector u. For an arbitrary matrix $A \in G$ we have $A \cdot u \in \overset{0}{V}$ and applying the transformation rule for the vector (2) we get

$$A \cdot u = \left[\sum_{i=1}^{n} A_i^1 u^i, \ldots, \sum_{i=1}^{n} A_i^n u^i\right]^T = \left(\sum_{i=1}^{n} A_i^n u^i\right) \cdot \left[\frac{\sum_{i=1}^{n} A_i^1 u^i}{\sum_{i=1}^{n} A_i^n u^i}, \ldots, \frac{\sum_{i=1}^{n} A_i^{n-1} u^i}{\sum_{i=1}^{n} A_i^n u^i}, 1\right]^T$$
$$= (u^n \cdot W(q,A)) \cdot \left(\frac{1}{W(q,A)} \cdot A \cdot q\right).$$

So, we have obtained the transformation rules for the parameter and the direction of the isotropic vector u:

$$\text{(5)} \qquad u^n(A \cdot u) = u^n(u) \cdot W(q,A) \text{ and } q(A \cdot u) = \frac{1}{W(q,A)} \cdot A \cdot q(u) = A * q.$$

Let us observe that $B * (A * q) = (B \cdot A) * q$ holds for $A, B \in G$ and $E * q = q$ for the unit matrix E. In what follows the group G operates on the sphere S^{n-2}.

Definition 5. The G-space

$$\text{(6)} \qquad (S^{n-2}, G, *), \quad \text{where} \quad \bigwedge_{q \in S^{n-2}} \bigwedge_{A \in G} *(q,A) = A * q = \frac{A \cdot q}{W(q,A)},$$

is called a G-space of isotropic directions.

Definition 6. The system of directions $q_i = q(\underset{i}{u}) \in S^{n-2}$ for $i = 1, 2, \ldots, m$ is called independent if the system of vectors $\underset{1}{u}, \underset{2}{u}, \ldots, \underset{m}{u} \in \overset{0}{V}$ is linearly independent.

In this paper we determine all equivariant mappings from the product of isotropic directions into φ-scalars. More accurately, having in mind (1), (3) and (6) we solve the functional equations

$$F(A*q_1, A*q_2, \dots, A*q_m) = 1 \cdot F(q_1, q_2, \dots, q_m), \tag{7}$$

$$F(A*q_1, A*q_2, \dots, A*q_m) = \varepsilon(A) \cdot F(q_1, q_2, \dots, q_m), \tag{8}$$

$$F(A*q_1, A*q_2, \dots, A*q_m) = \eta(A) \cdot F(q_1, q_2, \dots, q_m), \tag{9}$$

$$F(A*q_1, A*q_2, \dots, A*q_m) = \varepsilon(A) \cdot \eta(A) \cdot F(q_1, q_2, \dots, q_m) \tag{10}$$

for an arbitrary matrix $A \in G$ and the given system of independent points $q_1, q_2, \dots, q_m \in S^{n-2}$ with $1 \leqslant m \leqslant n$.

2. Certain particular solutions

For the pair of points $q_i, q_j \in S^{n-2}$ let us denote $1 - \sum\limits_{k=1}^{n-1} q_i^k q_j^k = Q(q_i, q_j) = Q_{ij}$ for brevity. The Euclidean distance between these points

$$\|q_i, q_j\| = \sqrt{\sum_{k=1}^{n-1} (q_j^k - q_i^k)^2} = \sqrt{2 \cdot \left(1 - \sum_{k=1}^{n-1} q_i^k q_j^k\right)} = \sqrt{2 \cdot Q(q_i, q_j)} = \sqrt{2 \cdot Q_{ij}}$$

is not an invariant under the operation of the group G. Let the isotropic vectors $\underset{i}{u}, \underset{j}{u}$ correspond to the directions q_i, q_j, respectively. Since we have $p(A\underset{i}{u}, A\underset{j}{u}) = p(\underset{i}{u}, \underset{j}{u})$ for an arbitrary matrix $A \in G$, according to (5) we get

$$Q(A*q_i, A*q_j) = \frac{Q(q_i, q_j)}{W(q_i, A) \cdot W(q_j, A)}, \tag{11}$$

which means

$$\|A*q_i, A*q_j\| = \frac{\|q_i, q_j\|}{\sqrt{W(q_i, A) \cdot W(q_j, A)}}.$$

For different points $q_1, q_2, q_3, q_4 \in S^{n-2}$, which is possible if $n > 2$, we can construct easily two simple but nontrivial invariants

$$\frac{Q_{13}Q_{24}}{Q_{12}Q_{34}}, \frac{Q_{14}Q_{23}}{Q_{12}Q_{34}} \quad \text{or equivalently} \quad \frac{\|q_1, q_3\| \cdot \|q_2, q_4\|}{\|q_1, q_2\| \cdot \|q_3, q_4\|}, \frac{\|q_1, q_4\| \cdot \|q_2, q_3\|}{\|q_1, q_2\| \cdot \|q_3, q_4\|}$$

which can be interpreted in a quadrilateral or tetrahedron with vertices q_1, q_2, q_3, q_4. In addition we have

$$\det(A\underset{1}{u}, A\underset{2}{u}, \dots, A\underset{n}{u}) = \varepsilon(A) \cdot \det(\underset{1}{u}, \underset{2}{u}, \dots, \underset{n}{u}),$$

so, in particular, for isotropic vectors $\underset{1}{u}, \underset{2}{u}, \ldots, \underset{n}{u}$ in view of (5) we get

$$\text{(12)} \qquad \det(A * q_1, A * q_2, \ldots, A * q_n) = \frac{\varepsilon(A) \cdot \det(q_1, q_2, \ldots, q_n)}{W(q_1, A) \cdot W(q_2, A) \cdot \ldots \cdot W(q_n, A)}.$$

Now (12) together with (4) yields

$$\text{(13)} \quad \operatorname{sign}\det(A * q_1, \ldots, A * q_n) = \begin{cases} \varepsilon(A) \cdot \operatorname{sign}\det(q_1, \ldots, q_n) \text{ for even } n, \\ \varepsilon(A) \cdot \eta(A) \cdot \operatorname{sign}\det(q_1, \ldots, q_n) \text{ for odd } n. \end{cases}$$

Lemma 1. *For arbitrary possible $m = 1, 2, \ldots$ and an arbitrary matrix $A \in G$ the functional equation*

$$F(A * q_1, \ldots, A * q_m) = \begin{cases} \eta(A) \cdot F(q_1, \ldots, q_m) & \text{if } n = 2, 3, 4, \ldots, \\ \varepsilon(A) \cdot F(q_1, \ldots, q_m) & \text{if } n = 3, 5, 7, \ldots, \\ \varepsilon(A) \cdot \eta(A) \cdot F(q_1, \ldots, q_m) & \text{if } n = 2, 4, 6, \ldots \end{cases}$$

has only the trivial solution $F(q_1, q_2, \ldots, q_m) = 0$.

Proof. If $A \in G$ then obviously $(-A) \in G$ and $A * q = (-A) * q$. Inserting A and then $(-A)$ into the first equation and having in mind $\eta(-A) = -\eta(A)$ we get simultaneously

$$F(q_1, \ldots, q_m) = \eta(A) \cdot F(A * q_1, \ldots, A * q_m) = -\eta(A) \cdot F(A * q_1, \ldots, A * q_m).$$

An analogous result is obtained for the two remaining equations using $\varepsilon(-A) = -\varepsilon(A)$ in the case of n odd and $\varepsilon(-A) \cdot \eta(-A) = -\varepsilon(A) \cdot \eta(A)$ in the case of n even. □

We have to consider the cases $n = 2$ and $n = 3$. If $n = 2$ then the sphere S^0 has only two different points $q_1 = [q_1^1, 1]^T$ and $q_2 = [q_2^1, 1]^T = [-q_1^1, 1]^T$ where $(q_1^1)^2 = 1$. An arbitrary pseudo-orthogonal matrix is of the form

$$A(\varepsilon, \eta, x) = \begin{bmatrix} \varepsilon \cdot \eta \cdot \cosh x & \varepsilon \cdot \eta \cdot \sinh x \\ \eta \cdot \sinh x & \eta \cdot \cosh x \end{bmatrix},$$

where $\varepsilon^2 = 1$, $\eta^2 = 1$, $x \in \mathbb{R}$. Since we have $A * q_1 = [\varepsilon q_1^1, 1]^T$, so putting the matrix $A(q_1^1, \eta, x)$ into functional equations (7) and (8) we get solutions

1-scalars) $F(q_1) = c$ and $F(q_1, q_2) = c$,

ε-scalars) $F(q_1) = c \cdot q_1^1$ and $F(q_1, q_2) = 2c \cdot q_1^1 = -2c \cdot q_2^1 = c \cdot \begin{vmatrix} q_1^1 & 1 \\ q_2^1 & 1 \end{vmatrix}$,

where c denotes a constant.

In the case $n = 3$ the circle S^1 is an uncountable set. For the given different points $q_1, q_2, q_3 \in S^1$ there exists a matrix $\underset{0}{A} \in G$ such that $\varepsilon(\underset{0}{A}) = 1$, $\eta(\underset{0}{A}) = \operatorname{sign}\det(q_1, q_2, q_3)$ and $\underset{0}{A} * q_1 = [0, 1, 1]^T$, $\underset{0}{A} * q_2 = [0, -1, 1]^T$ and $\underset{0}{A} * q_3 = [1, 0, 1]^T$. Inserting this matrix into equations (7) and (10) we get solutions

1-scalars) $F(q_1) = c$ and $F(q_1, q_2) = c$ and $F(q_1, q_2, q_3) = c$,

$\varepsilon\eta$-scalars) $F(q_1) = 0$ and $F(q_1, q_2) = 0$ and $F(q_1, q_2, q_3) = c \cdot \operatorname{sign}[\det(q_1, q_2, q_3)]$, where c again denotes an arbitrary constant.

Just in the case $m = 4$ and $q_4 \notin \{q_1, q_2, q_3\}$ we get two non-trivial invariants and general solutions of the equations:

1-scalars) $F(q_1, q_2, q_3, q_4) = \Theta(Q_{13}Q_{24}/Q_{12}Q_{34}, Q_{14}Q_{23}/Q_{12}Q_{34}) = \Theta(x_4, y_4)$,

$\varepsilon\eta$-scalars) $F(q_1, q_2, q_3, q_4) = \Theta(x_4, y_4) \cdot \operatorname{sign}\det(q_1, q_2, q_3)$, where Θ is an arbitrary function of two variables.

3. General solution of equation (7)

For $n = 4, 5, 6, \ldots$ let n independent points $q_i = [q_i^1, q_i^2, \ldots, q_i^{n-1}, 1]^T \in S^{n-2}$ be given, where $i = 1, 2, \ldots, n$, and let $Q(s) = \det[Q_{ij}]_1^s$ for $s = 2, 3, \ldots, n$. Let us remark that $[\det(q_1, q_2, \ldots, q_n)]^2 = (-1)^{n+1}Q(n)$ and $(-1)^{s+1}Q(s) > 0$. We are going to construct a matrix $C = C(q_1, q_2, \ldots, q_n) = [C_i^j]_1^n \in G$ which will enable us to solve equation (7). We start with the last three rows. For $i = 1, 2, \ldots, n-1$ let

$$C_i^{n-2} = \frac{Q_{23}q_1^i + Q_{13}q_2^i - Q_{12}q_3^i}{(-1)^n\sqrt{Q(3)}} \quad \text{and} \quad C_n^{n-2} = \frac{Q_{12} - Q_{13} - Q_{23}}{(-1)^n\sqrt{Q(3)}},$$

$$C_i^{n-1} = \frac{Q_{13}q_2^i - Q_{23}q_1^i}{(-1)^n\sqrt{Q(3)}} \quad \text{and} \quad C_n^{n-1} = \frac{Q_{23} - Q_{13}}{(-1)^n\sqrt{Q(3)}},$$

$$C_i^{n} = \frac{Q_{23}q_1^i + Q_{13}q_2^i}{(-1)^n\sqrt{Q(3)}} \quad \text{and} \quad C_n^{n} = \frac{-Q_{13} - Q_{23}}{(-1)^n\sqrt{Q(3)}}.$$

We have formulas for the $(n-2)$-nd and $(n-1)$-st components of an arbitrary point $C * q_r$, namely

$$\text{(14)} \qquad \begin{cases} (C * q_r)^{n-2} = \dfrac{Q_{13}Q_{2r} + Q_{23}Q_{1r} - Q_{12}Q_{3r}}{Q_{13}Q_{2r} + Q_{23}Q_{1r}}, \\[2ex] (C * q_r)^{n-1} = \dfrac{Q_{13}Q_{2r} - Q_{23}Q_{1r}}{Q_{13}Q_{2r} + Q_{23}Q_{1r}}. \end{cases}$$

These components in accordance with (11) are 1-scalars. In particular, for $r = 1, 2, 3$ we get

$$\text{(15)} \quad C * q_1 = [0, \ldots, 0, 1, 1]^T, C * q_2 = [0, \ldots, 0, -1, 1]^T, C * q_3 = [0, \ldots, 0, 1, 0, 1]^T.$$

Let the elements of the first row C_i^1 of the matrix C be coefficients of z_i in the Laplace expansion in terms of elements of the last row of the determinant

$$C^1 = \frac{\operatorname{sign}\det(q_1,\dots,q_n)}{\sqrt{(-1)^n Q(n-1)}} \begin{vmatrix} q_1^1 & q_1^2 & \dots & q_1^{n-1} & 1 \\ q_2^1 & q_2^2 & \dots & q_2^{n-1} & 1 \\ \dots & \dots & \dots & \dots & \dots \\ q_{n-1}^1 & q_{n-1}^2 & \dots & q_{n-1}^{n-1} & 1 \\ z_1 & z_2 & \dots & z_{n-1} & z_n \end{vmatrix}.$$

Then we have $(C * q_r)^1 = 0$ for $r = 1, 2, \dots, n-1$. Analogously, the coefficients of z_i in the Laplace expansion in terms of elements of the last row of the determinant

$$C^2 = \frac{1}{\sqrt{(-1)^{n-1} Q(n-2)}} \begin{vmatrix} q_1^1 & q_1^2 & \dots & q_1^{n-1} & 1 \\ \dots & \dots & \dots & \dots & \dots \\ q_{n-2}^1 & q_{n-2}^2 & \dots & q_{n-2}^{n-1} & 1 \\ C_1^1 & C_2^1 & \dots & C_{n-1}^1 & -C_n^1 \\ z_1 & z_2 & \dots & z_{n-1} & z_n \end{vmatrix}$$

are the elements C_i^2 of the second row of the matrix C. Now, $(C * q_r)^2 = 0$ for $r = 1, 2, \dots, n-2$. Proceeding in the same way we can determine $(k-1)$ rows of the matrix C and then the k-th row using the determinant

$$C^k = \frac{1}{\sqrt{(-1)^{n-k+1} Q(n-k)}} \begin{vmatrix} q_1^1 & q_1^2 & \dots & q_1^{n-1} & 1 \\ \dots & \dots & \dots & \dots & \dots \\ q_{n-k}^1 & q_{n-k}^2 & \dots & q_{n-k}^{n-1} & 1 \\ C_1^1 & C_2^1 & \dots & C_{n-1}^1 & -C_n^1 \\ \dots & \dots & \dots & \dots & \dots \\ C_1^{k-1} & C_2^{k-1} & \dots & C_{n-1}^{k-1} & -C_n^{k-1} \\ z_1 & z_2 & \dots & z_{n-1} & z_n \end{vmatrix}.$$

We get $(C * q_r)^k = 0$ only for $r = 1, 2, \dots, n-k$. In this way we construct the rows number $k = 2, 3, \dots n-3$ and $(n-2)$ again. We describe the k-th coordinate of the point $C * q_r$ by the formula

$$(16) \qquad (C * q_r)^k = \frac{\sqrt{Q(3)} \cdot W_r^k}{(Q_{13}Q_{2r} + Q_{23}Q_{1r})\sqrt{-Q(n-k)Q(n-k+1)}}$$

where

$$W_r^k = \begin{vmatrix} 0 & Q_{12} & Q_{13} & \dots & Q_{1,n-k-1} & Q_{1,n-k} & Q_{1,n-k+1} \\ Q_{21} & 0 & Q_{23} & \dots & Q_{2,n-k-1} & Q_{2,n-k} & Q_{2,n-k+1} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ Q_{n-k,1} & Q_{n-k,2} & Q_{n-k,3} & \dots & Q_{n-k,n-k-1} & 0 & Q_{n-k,n-k+1} \\ Q_{r1} & Q_{r2} & Q_{r3} & \dots & Q_{r,n-k-1} & Q_{r,n-k} & Q_{r,n-k+1} \end{vmatrix},$$

which holds true for $k = 1, 2, \ldots, n-2$ and arbitrary r. Considering the formulas (14) and (16) we see that $C * q_r$ depends on $q_1, q_2, \ldots, q_r$ only, in spite of $C = C(q_1, q_2, \ldots, q_n)$. It allows us to select the lacking points of the sphere and construct the matrix C in the case $m < n$. Formula (11) implies that $(C * q_r)^k$ is an invariant. Considering the case when $n = 2$, $n = 3$ and (15) we have

Lemma 2. *In the case* $1 \leqslant m < 4$, *equation* (7) *has only the trivial solution* $F(q_1) = c$ *for* $n \geqslant 2$, $F(q_1, q_2) = c$ *for* $n \geqslant 2$ *and* $F(q_1, q_2, q_3) = c$ *for* $n > 2$, *where* c *is an arbitrary constant.*

Considering the case $n = 3$ and formulas (14) and (15) and using for $m = n = 4$ simply formula (16) we obtain

Lemma 3. *The general solution of equation* (7) *in the case* $n > 2$ *and* $m = 4$ *is of the form*

$$F(q_1, q_2, q_3, q_4) = \Theta\Big(\frac{\|q_1, q_3\| \cdot \|q_2, q_4\|}{\|q_1, q_2\| \cdot \|q_3, q_4\|}, \frac{\|q_1, q_4\| \cdot \|q_2, q_3\|}{\|q_1, q_2\| \cdot \|q_3, q_4\|}\Big)$$

where Θ *is an arbitrary function of two variables.*

We can conclude with

Lemma 4. *The general solution of equation* (7) *for arbitrary* $4 \leqslant m \leqslant n$ *is of the form*

$$F(q_1, q_2, \ldots, q_m) = \Theta((C * q_r)^k)$$

where r *runs from* 4 *to* m *and for every fixed* r *the index* k *changes from* $(n+1-r)$ *to* $(n-1)$ *and* Θ *is an arbitrary function of* $\frac{1}{2}(m-3)(m+2)$ *variables.*

Despite omitting in Lemma 4 the trivial 1-scalars $-1, 0, +1$, we have relations $C * q_r \in S^{n-2}$ as a result of the fact that $(m-3)$ arguments of the function Θ are dependent on the others. Analysing formula (16) one can suppose that other kinds of invariants exist, in addition to the arguments of the function Θ in Lemma 3. Because it is easy to find the correct number $\frac{1}{2}m(m-3)$ of simple and independent 1-scalars, we have

Theorem 1. *The general solution of the functional equation*

$$F(A * q_1, A * q_2, \ldots, A * q_m) = F(q_1, q_2, \ldots, q_m)$$

for given independent points $q_1, q_2, \ldots, q_m \in S^{n-2}$ *and an arbitrary matrix* $A \in G$ *is of the form*

$$F(q_1, q_2, \ldots, q_m) = \begin{cases} c & \text{if } m = 1, 2, 3, \\ \Theta\left(\dfrac{Q_{13}Q_{24}}{Q_{12}Q_{34}}, \dfrac{Q_{14}Q_{23}}{Q_{12}Q_{34}}\right) & \text{if } m = 4, \\ \Theta\left(\dfrac{Q_{13}Q_{2i}}{Q_{12}Q_{3i}}, \dfrac{Q_{23}Q_{1i}}{Q_{12}Q_{3i}}, \dfrac{Q_{1i}Q_{2j}}{Q_{12}Q_{ij}}\right) & \text{if } 4 < m \leqslant n, \end{cases}$$

where $4 \leqslant j < i = 4, 5, \ldots, m$, c *is an arbitrary constant and* Θ *is an arbitrary function of* $\frac{1}{2}m(m-3)$ *variables.*

4. General solutions to equations (8) and (10)

Theorem 2. *The general solution of the functional equation*

$$F(A * q_1, A * q_2, \ldots, A * q_m) = \varepsilon(A) \cdot F(q_1, q_2, \ldots, q_m)$$

for given independent points $q_1, q_2, \ldots, q_m \in S^{n-2}$ *and an arbitrary matrix* $A \in G$ *is of the form*

$$F(q_1, q_2, \ldots, q_m) = \begin{cases} c \cdot q_1^1 & \text{if } n = 2 \text{ and } m = 1, \\ 0 & \text{if } n \text{ is odd}, \\ 0 & \text{if } n > 2 \text{ and } m < n, \\ \Psi \cdot \operatorname{sign}\det(q_1, q_2, \ldots, q_n) & \text{if } n \text{ is even and } m = n, \end{cases}$$

where c *is an arbitrary constant and* Ψ *is the general solution of equation* (7).

P r o o f. We have already proved the first two cases. Now, let $m < n$ and $n > 2$. Then the matrix C in the case $n \geqslant 4$ (or $\underset{0}{A}$ in the case $n = 3$) satisfies $(C * q_r)^1 = 0$ for $r = 1, 2, \ldots, m$. Let $\overline{C}$ denote a matrix obtained from the matrix C by multiplying its elements of the first row by -1. From the relations $\varepsilon(\overline{C}) = -\varepsilon(C)$ and $(C * q_r) = (\overline{C} * q_r)$ we get simultaneously

$$\begin{aligned} F(q_1, q_2, \ldots, q_m) &= \varepsilon(C)F(C * q_1, C * q_2, \ldots, C * q_m) \\ &= \varepsilon(\overline{C})F(\overline{C} * q_1, \overline{C} * q_2, \ldots, \overline{C} * q_m) \\ &= -\varepsilon(C)F(C * q_1, C * q_2, \ldots, C * q_m). \end{aligned}$$

Let $F(q_1, q_2, \ldots, q_n)$ be the general solution of equation (8) in the case $m = n$ and n even. Then the quotient $F(q_1, q_2, \ldots, q_n) : \operatorname{sign}\det(q_1, q_2, \ldots, q_n)$ is the general solution of equation (7), which proves the assertion of the theorem in the last case. □

Analogously we can prove

Theorem 3. *The general solution of the functional equation*

$$F(A * q_1, A * q_2, \dots, A * q_m) = \varepsilon(A) \cdot \eta(A) \cdot F(q_1, q_2, \dots, q_m)$$

for given independent points $q_1, q_2, \dots, q_m \in S^{n-2}$ and an arbitrary matrix $A \in G$ is of the form

$$F(q_1, q_2, \dots, q_m) = \begin{cases} 0 & \text{if } n \text{ is even or } m < n, \\ \Psi \cdot \operatorname{sign}\det(q_1, q_2, \dots, q_n) & \text{if } n \text{ is odd and } m = n, \end{cases}$$

where Ψ is the general solution of equation (7).

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Authors' address: *Aleksander Misiak*, *Eugeniusz Stasiak*, Instytut Matematyki, Politechnika Szczecińska, Al. Piastów 17, 70-310 Szczecin, Poland, e-mail: `misiak@ps.pl`.