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ON \mathcal{C} -STARCOMPACT SPACES

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Abstract. A space X is \mathcal{C} -starcompact if for every open cover \mathcal{U} of X , there exists a countably compact subset C of X such that $\text{St}(C, \mathcal{U}) = X$. In this paper we investigate the relations between \mathcal{C} -starcompact spaces and other related spaces, and also study topological properties of \mathcal{C} -starcompact spaces.

Keywords: compact space, countably compact space, Lindelöf space, \mathcal{K} -starcompact space, \mathcal{C} -starcompact space, \mathcal{L} -starcompact space

MSC 2010: 54D20, 54D55

1. INTRODUCTION

By a space we mean a topological space. Let us recall that a space X is *countably compact* if every countable open cover of X has a finite subcover. Fleischman [4] defined a space X to be *starcompact* if for every open cover \mathcal{U} of X , there exists a finite subset F of X such that $\text{St}(F, \mathcal{U}) = X$, where $\text{St}(F, \mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap F \neq \emptyset\}$ and he proved that every countably compact space is starcompact. Conversely, van Douwen-Reed-Roscoe-Tree [1] proved that every Hausdorff starcompact space is countably compact, but this does not hold for T_1 -spaces (see [7] Example 12). As generalizations of starcompactness, the following classes of spaces were introduced:

Definition 1.1. A space X is \mathcal{C} -starcompact if for every open cover \mathcal{U} of X , there exists a countably compact subset C of X such that $\text{St}(C, \mathcal{U}) = X$.

Definition 1.2 ([3], [7], [10]). A space X is \mathcal{L} -starcompact if for every open cover \mathcal{U} of X , there exists a Lindelöf subset L of X such that $\text{St}(L, \mathcal{U}) = X$.

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Definition 1.3 ([5], [7], [9]). A space X is \mathcal{K} -starcompact if for every open cover \mathcal{U} of X , there exists a compact subset K of X such that $\text{St}(K, \mathcal{U}) = X$.

From the above definitions it is not difficult to see that every \mathcal{K} -starcompact space is \mathcal{C} -starcompact and every \mathcal{K} -starcompact space is \mathcal{L} -starcompact. In the second section, we show the relationships between these spaces by giving some examples.

The cardinality of a set A is denoted by $|A|$. Let ω be the first infinite cardinal, ω_1 the first uncountable cardinal and \mathfrak{c} the cardinality of the set of all real numbers. As usual, a cardinal is the initial ordinal and an ordinal is the set of smaller ordinals. For each ordinals α, β with $\alpha < \beta$ we write $(\alpha, \beta) = \{\gamma: \alpha < \gamma < \beta\}$, $(\alpha, \beta] = \{\gamma: \alpha < \gamma \leq \beta\}$ and $[\alpha, \beta] = \{\gamma: \alpha \leq \gamma \leq \beta\}$. Every cardinal is often viewed as a space with the usual order topology. Other terms and symbols that we do not define follow [2].

2. \mathcal{C} -STARCOMPACT SPACES AND RELATED SPACES

In this section we give some examples showing the relations between \mathcal{C} -starcompact spaces and other related spaces. The symbol $\beta(X)$ means the Čech-Stone compactification of a Tychonoff space X .

Example 2.1. There exists an \mathcal{L} -starcompact Tychonoff space which is not \mathcal{C} -starcompact (hence, not \mathcal{K} -starcompact).

Proof. Let D be a discrete space of cardinality ω . Define

$$X = (\beta(D) \times (\omega + 1)) \setminus ((\beta(D) \setminus D) \times \{\omega\}).$$

Then X is \mathcal{L} -starcompact, since $\beta(D) \times \omega$ is a Lindelöf dense subset of X .

Next, we show that X is not \mathcal{C} -starcompact. Since $|D| = \omega$, we can enumerate D as $\{d_n: n \in \omega\}$. Let us consider the open cover

$$\mathcal{U} = \{\{d_n\} \times [n, \omega]: n \in \omega\} \cup \{(\beta(D) \setminus \{m \in \omega: m \leq n\}) \times \{n\}: n \in \omega\}$$

of X . Let C be a countably compact subset of X . Then $\{n \in \omega: C \cap (\{d_n\} \times [n, \omega]) \neq \emptyset\}$ is finite. Hence, there exists an $n_1 \in \omega$ such that

$$(1) \quad C \cap (\{d_n\} \times [n, \omega]) = \emptyset \quad \text{for each } n > n_1.$$

On the other hand, $\{n \in \omega: C \cap ((\beta(D) \setminus \{m \in \omega: m \leq n\}) \times \{n\}) \neq \emptyset\}$ is finite, since C is countably compact, Hence, there exists an $n_2 \in \omega$ such that

$$(2) \quad C \cap ((\beta(D) \setminus \{m \in \omega: m \leq n\}) \times \{n\}) = \emptyset \quad \text{for each } n > n_2.$$

Choose $n \in \omega$ such that $n > \max\{n_1, n_2\}$. Then $\{d_n\} \times [n, \omega]$ is the only element of \mathcal{U} containing the point $\langle d_n, \omega \rangle$ and $(\{d_n\} \times [n, \omega]) \cap C = \emptyset$ by (1) and (2). It follows that $\langle d_n, \omega \rangle \notin \text{St}(C, \mathcal{U})$, which shows that X is not \mathcal{C} -starcompact. \square

Example 2.2. There exists a \mathcal{C} -starcompact Tychonoff space which is not \mathcal{L} -starcompact (hence, not \mathcal{K} -starcompact).

Proof. Let $X_1 = \mathfrak{c} + 1$ denote the usual order topology. Then X_1 is compact. Let $X_2 = \mathfrak{c} + 1$. We topologize X_2 as follows: for each $\alpha < \mathfrak{c}$, α is isolated and a set U containing \mathfrak{c} is open if and only if $X \setminus U$ is finite. Then X_2 is compact. Let

$$X = (X_1 \times X_2) \setminus \{\langle \mathfrak{c}, \mathfrak{c} \rangle\}$$

be a subspace of the product of X_1 and X_2 . Then X is \mathcal{C} -starcompact, since $\mathfrak{c} \times X_2$ is a countably compact dense subset of X .

Now, we show that X is not \mathcal{L} -starcompact. For each $\alpha < \mathfrak{c}$, let $U_\alpha = (\alpha, \mathfrak{c}] \times \{\alpha\}$. Then $U_\alpha \cap U_{\alpha'} = \emptyset$ for $\alpha \neq \alpha'$. Let us consider the open cover

$$\mathcal{U} = \{U_\alpha : \alpha < \mathfrak{c}\} \cup \{\mathfrak{c} \times X_2\}$$

of X . Let L be a Lindelöf subset of X . Then $\Lambda = \{\alpha : \langle \mathfrak{c}, \alpha \rangle \in L\}$ is countable, since $\{\langle \mathfrak{c}, \alpha \rangle : \alpha < \mathfrak{c}\}$ is discrete and closed in X . Let $L' = L \setminus \bigcup\{U_\alpha : \alpha \in \Lambda\}$. If $L' = \emptyset$, then there exists an $\alpha_0 < \mathfrak{c}$ such that $L \cap U_{\alpha_0} = \emptyset$, hence $\langle \mathfrak{c}, \alpha_0 \rangle \notin \text{St}(L, \mathcal{U})$, since U_{α_0} is the only element of \mathcal{U} containing the point $\langle \mathfrak{c}, \alpha_0 \rangle$. On the other hand, if $L' \neq \emptyset$, since L' is closed in L , L' is Lindelöf and $L' \subseteq \mathfrak{c} \times X_2$, hence $\pi(L')$ is a Lindelöf subset of a countably compact space \mathfrak{c} , where $\pi: \mathfrak{c} \times X_2 \rightarrow \mathfrak{c}$ is the projection. Hence, there exists $\alpha_1 < \mathfrak{c}$ such that $\pi(L') \cap (\alpha_1, \mathfrak{c}) = \emptyset$. Choose $\alpha < \mathfrak{c}$ such that $\alpha > \alpha_1$ and $\alpha \notin \Lambda$. Then $\langle \mathfrak{c}, \alpha \rangle \notin \text{St}(L, \mathcal{U})$, since U_α is the only element of \mathcal{U} containing the point $\langle \mathfrak{c}, \alpha \rangle$ and $U_\alpha \cap L = \emptyset$, which shows that X is not \mathcal{L} -starcompact. \square

Recall from [7] that a space X is called $1\frac{1}{2}$ -starcompact if for every open cover \mathcal{U} of X there exists a finite subset \mathcal{V} of \mathcal{U} such that $\text{St}(\bigcup\mathcal{V}, \mathcal{U}) = X$. In [1], a $1\frac{1}{2}$ -starcompact space is called 1-starcompact. From the above definitions, it is not difficult to see that every \mathcal{K} -starcompact space is $1\frac{1}{2}$ -starcompact. It is well-known that every countably compact Lindelöf space is compact. The following example shows that the result cannot be generalized to starcompact. For showing the example, we need the following lemma from [7, Theorem 28].

Lemma 2.3. *If a regular space X contains a discrete closed subspace Y such that $|X| = |Y| \geq \omega$, then X is not $1\frac{1}{2}$ -starcompact (hence, not \mathcal{K} -starcompact).*

Example 2.4. There exists a \mathcal{C} -starcompact and \mathcal{L} -starcompact Tychonoff space which is not \mathcal{K} -starcompact.

Proof. Let $S_1 = (X_1 \times X_2) \setminus \{\langle \mathfrak{c}, \mathfrak{c} \rangle\}$ be the same space X as in Example 2.2. Then S_1 is not \mathcal{K} -starcompact by Lemma 2.3.

Let $S_2 = \omega \cup \mathcal{R}$ be the Isbell-Mrówka space [9], where \mathcal{R} is a maximal almost disjoint family of infinite subsets of ω with $|\mathcal{R}| = \mathfrak{c}$. Then S_2 is \mathcal{L} -starcompact because it is separable. However, S_2 is not \mathcal{K} -starcompact by Lemma 2.3.

Assume $S_1 \cap S_2 = \emptyset$. Let $\varphi: \{\mathfrak{c}\} \times \mathfrak{c} \rightarrow \mathcal{R}$ be a bijection. Let X be the quotient space obtained from the discrete sum $S_1 \oplus S_2$ by identifying the $\langle \mathfrak{c}, \alpha \rangle$ and $\varphi(\langle \mathfrak{c}, \alpha \rangle)$ for each $\alpha < \mathfrak{c}$. Let $\pi: S_1 \oplus S_2 \rightarrow X$ be the quotient map. Then X is not \mathcal{K} -starcompact by Lemma 2.3.

Now, we show that X is \mathcal{C} -starcompact. Let \mathcal{U} be an open cover of X . Since $\pi(\mathfrak{c} \times X_2)$ is a countably compact subset of $\pi(S_1)$, we have

$$\pi(S_1) \subseteq \text{St}(\pi(\mathfrak{c} \times X_2), \mathcal{U}).$$

On the other hand, since $\pi(S_2)$ is homeomorphic to S_2 , every infinite subset of $\pi(\omega)$ has an accumulation point in $\pi(S_2)$. Hence, there exists a finite subset F_1 of $\pi(\omega)$ such that $\pi(\omega) \subseteq \text{St}(F_1, \mathcal{U})$. Indeed, if $\pi(\omega) \not\subseteq \text{St}(B, \mathcal{U})$ for any finite subset $B \subseteq \pi(\omega)$, then, by induction, we can define a sequence $\{x_n: n \in \omega\}$ in $\pi(\omega)$ such that $x_n \notin \text{St}(\{x_i: i < n\}, \mathcal{U})$ for each $n \in \omega$. By the property $\pi(\omega)$ mentioned above, the sequence $\{x_n: n \in \omega\}$ has a limit point x_0 in $\pi(S_2)$. Pick $U \in \mathcal{U}$ such that $x_0 \in U$. Choose $n < m < \omega$ such that $x \in U$ and $x_m \in U$. Then $x_m \in \text{St}(\{x_i: i < n\}, \mathcal{U})$, which contradicts the definition of the sequence $\{x_n: n \in \omega\}$. Let $F = F_1 \cup \pi(\beta(D) \times \mathfrak{c})$. Then F is countably compact and $X = \text{St}(F, \mathcal{U})$. Hence, X is \mathcal{C} -starcompact.

Next, we show that X is \mathcal{L} -starcompact. Since $\pi(\omega)$ is a countable dense subset of $\pi(S_2)$, we have $\pi(S_2) \subseteq \text{St}(\pi(\omega), \mathcal{U})$. On the other hand, since $\pi(\mathfrak{c} \times X_2)$ is countably compact, there exists a finite subset F_1 of $\pi(\mathfrak{c} \times X_2)$ such that $\pi(\mathfrak{c} \times X_2) \subseteq \text{St}(F_1, \mathcal{U})$. If we put $L = \pi(\omega) \cup F_1$, then L is a countable subset of X and $X = \text{St}(L, \mathcal{U})$, which shows that X is \mathcal{L} -starcompact. \square

Remark 1. The author does not know if there exists a normal \mathcal{C} -starcompact and \mathcal{L} -starcompact space which is not \mathcal{K} -starcompact.

3. PROPERTIES OF \mathcal{C} -STARCOMPACT SPACES

In Example 2.2, the closed subset $\{c\} \times \mathfrak{c}$ of a Tychonoff \mathcal{C} -starcompact space X is not \mathcal{C} -starcompact, which shows that a closed subset of a \mathcal{C} -starcompact space need not be \mathcal{C} -starcompact. In the following, we show that a regular closed subspace of a \mathcal{C} -starcompact space need not be \mathcal{C} -starcompact by using Example 2.4.

Example 3.1. There exists a \mathcal{C} -starcompact Tychonoff space having a regular-closed subset which is not \mathcal{C} -starcompact.

Proof. Let X be the same space as the space X in the proof of Example 2.4. As we proved above, X is \mathcal{C} -starcompact. Since $S_2 = \omega \cup \mathcal{R}$ is the Isbell-Mrówka space, hence S_2 is not \mathcal{K} -starcompact by Lemma 2.3. By the construction of the topology of S_2 , it is clear that every countably compact subset K of S_2 is countable and $K \cap \mathcal{R}$ is finite, hence it is compact. Thus, S_2 is not \mathcal{C} -starcompact. So, $\varphi[S_2]$ is a regular-closed subspace of X which is not \mathcal{C} -starcompact, which completes the proof. \square

Since a continuous image of a countably compact space is countably compact, it is not difficult to show the following result.

Theorem 3.2. *A continuous image of an \mathcal{C} -starcompact space is \mathcal{C} -starcompact*

Next, we turn to considering preimages. To show that the preimage of a \mathcal{C} -starcompact space under a closed 2-to-1 continuous map need not be \mathcal{C} -starcompact, we use the Alexandroff duplicate $A(X)$ of a space X . The underlying set of $A(X)$ is $X \times \{0, 1\}$; each point of $X \times \{1\}$ is isolated and a basic neighborhood of a point $\langle x, 0 \rangle \in X \times \{0\}$ is of the form $(U \times \{0\}) \cup ((U \times \{1\}) \setminus \{\langle x, 1 \rangle\})$, where U is a neighborhood of x in X .

Example 3.3. There exists a closed 2-to-1 continuous map $f: X \rightarrow Y$ such that Y is a \mathcal{C} -starcompact space, but X is not \mathcal{C} -starcompact.

Proof. Let Y be the space X from the proof of Example 2.2. Then Y is \mathcal{C} -starcompact and has the infinite discrete closed subset $\{\mathfrak{c}^+\} \times \mathfrak{c}$. Let X be the Alexandroff duplicate $A(Y)$ of Y . Then X is not \mathcal{C} -starcompact, since $F \times \{1\}$ is an infinite discrete, open and closed set in X . Let $f: X \rightarrow Y$ be the natural map. Then f is a closed 2-to-1 continuous map, which completes the proof. \square

Now, we give a positive result:

Theorem 3.4. *Let f be an open perfect map from a space X to a \mathcal{C} -starcompact space Y . Then X is \mathcal{C} -starcompact.*

Proof. Since $f(X)$ is open and closed in Y , we may assume that $f(X) = Y$. Let \mathcal{U} be an open cover of X and let $y \in Y$. Since $f^{-1}(y)$ is compact, there exists a finite subcollection \mathcal{U}_y of \mathcal{U} such that $f^{-1}(y) \subseteq \bigcup \mathcal{U}_y$ and $U \cap f^{-1}(y) \neq \emptyset$ for each $U \in \mathcal{U}_y$. Pick an open neighborhood V_y of y in Y such that $f^{-1}(V_y) \subseteq \bigcup \{U : U \in \mathcal{U}_y\}$; we can assume that

$$(1) \quad V_y \subseteq \bigcap \{f(U) : U \in \mathcal{U}_y\},$$

because f is open. Taking such an open set V_y for each $y \in Y$, we have an open cover $\mathcal{V} = \{V_y : y \in Y\}$ of Y . Let C be a countably compact subset of the \mathcal{C} -starcompact space Y such that $\text{St}(C, \mathcal{V}) = Y$. Since f is perfect, the set $f^{-1}(C)$ is a countably compact subset of X . To show that $\text{St}(f^{-1}(C), \mathcal{U}) = X$, let $x \in X$. Then there exists $y \in Y$ such that $f(x) \in V_y$ and $V_y \cap C \neq \emptyset$. Since

$$x \in f^{-1}(V_y) \subseteq \bigcup \{U : U \in \mathcal{U}_y\},$$

we can choose $U \in \mathcal{U}_y$ with $x \in U$. Then $V_y \subseteq f(U)$ by (1), and hence $U \cap f^{-1}(C) \neq \emptyset$. Therefore, $x \in \text{St}(f^{-1}(C), \mathcal{U})$. Consequently, we have that $\text{St}(f^{-1}(C), \mathcal{U}) = X$. \square

By Theorem 3.4 we have the following corollary.

Corollary 3.5. *Let X be a \mathcal{C} -starcompact space and Y a compact space. Then $X \times Y$ is \mathcal{C} -starcompact.*

In [1, Example 3.3.3], Song gave an example showing that the product of a countably compact space X and a Lindelöf space Y is not \mathcal{L} -starcompact. Now, we show that the product $X \times Y$ is not \mathcal{C} -starcompact:

Example 3.6. There exist a countably compact space X and a Lindelöf space Y such that $X \times Y$ is not \mathcal{C} -starcompact.

Proof. Let $X = \omega_1$ with the usual order topology. Let $Y = \omega_1 + 1$ with the following topology: each point α with $\alpha < \omega_1$ is isolated and a set U containing ω_1 is open if and only if $Y \setminus U$ is countable. Then X is countably compact and Y is Lindelöf. Now, we show that $X \times Y$ is not \mathcal{C} -starcompact. For each $\alpha < \omega_1$, let $U_\alpha = [0, \alpha] \times [\alpha, \omega_1]$ and $V_\alpha = (\alpha, \omega_1) \times \{\alpha\}$. Consider the open cover

$$\mathcal{U} = \{U_\alpha : \alpha < \omega_1\} \cup \{V_\alpha : \alpha < \omega_1\}$$

of $X \times Y$ and let C be a countably compact subset of $X \times Y$. Then $\pi_Y(C \cap (\omega_1 \times \omega_1))$ is a countably compact subset of Y , where $\pi_Y : X \times Y \rightarrow Y$ is the projection. By

the definition of the topology of Y , we have that $\pi_Y(C \cap (\omega_1 \times \omega_1))$ is finite, thus there exists $\beta < \omega_1$ such that

$$\pi(C \cap (\omega_1 \times \omega_1)) \cap (\beta, \omega_1) = \emptyset.$$

Pick $\alpha_0 > \beta$. Then $\langle \alpha_0, \beta \rangle \notin \text{St}(C, \mathcal{U})$ since V_{α_0} is the only element of \mathcal{U} containing $\langle \alpha_0, \beta_0 \rangle$ and $V_{\alpha_0} \cap C = \emptyset$. Hence, $X \times Y$ is not \mathcal{C} -starcompact, which completes the proof. \square

In [10], Song gave an example showing that the product of two countably compact spaces is not \mathcal{L} -starcompact. In the following, we show that the product of two countably compact spaces is not \mathcal{C} -starcompact by using his example. Here we give the proof roughly for the sake of completeness (see 10, Example 3.10.19).

Example 3.7. There exist countably compact spaces X and Y such that $X \times Y$ is not \mathcal{C} -starcompact.

Proof. Consider ω with the discrete topology. We can define $X = \bigcup_{\alpha < \omega_1} E_\alpha$, $Y = \bigcup_{\alpha < \omega_1} F_\alpha$, where E_α and F_α are the subsets of $\beta(\omega)$ which are defined inductively so as to satisfy the following conditions (1), (2) and (3):

- (1) $E_\alpha \cap F_\beta = D$ if $\alpha \neq \beta$;
- (2) $|E_\alpha| \leq \mathfrak{c}$ and $|F_\alpha| \leq \mathfrak{c}$;
- (3) every infinite subset of E_α or F_α has an accumulation point in $E_{\alpha+1}$ or $F_{\alpha+1}$, respectively.

The sets E_α and F_α are well-defined since every infinite closed set in $\beta(\omega)$ has cardinality $2^{\mathfrak{c}}$ (see [2]). Then $X \times Y$ is not \mathcal{C} -starcompact, because the diagonal $\{(n, n) : n \in \omega\}$ is a discrete open and closed subset of $X \times Y$ with cardinality ω and \mathcal{C} -starcompactness is preserved by open and closed subsets. \square

Theorem 3.8. Every Tychonoff space can be embedded in a \mathcal{C} -starcompact Tychonoff space as a closed subspace.

Proof. Let X be a Tychonoff space. If we put

$$Z = (\beta(X) \times (\omega_1 + 1)) \setminus ((\beta(X) \times \{\omega_1\}),$$

then $\overline{X} = X \times \{\omega_1\}$ is a closed subset of Z which is homeomorphic to X . Since $\beta(X) \times \omega_1$ is a countably compact dense subset of Z , we conclude that Z is \mathcal{C} -starcompact, which completes the proof. \square

Remark. The author does not know if a Tychonoff space can be embedded in a \mathcal{C} -starcompact Tychonoff space as a G_δ -closed subspace.

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