Mathematica Bohemica

E. M. E. Zayed; M. A. El-Moneam

On the rational recursive sequence $x_{n+1} = \left(A + \sum_{i=0}^{k} \alpha_i x_{n-i}\right) \Big/ \sum_{i=0}^{k} \beta_i x_{n-i}$

Mathematica Bohemica, Vol. 133 (2008), No. 3, 225–239

Persistent URL: `http://dml.cz/dmlcz/140612`

Terms of use:

© Institute of Mathematics AS CR, 2008

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.

This document has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* `http://dml.cz`

ON THE RATIONAL RECURSIVE SEQUENCE $x_{n+1} = \left(A + \sum\limits_{i=0}^{k} \alpha_i x_{n-i}\right) \Big/ \sum\limits_{i=0}^{k} \beta_i x_{n-i}$

E. M. E. Zayed, El-Taif, M. A. El-Moneam, Zagazig

(Received December 17, 2006)

Abstract. The main objective of this paper is to study the boundedness character, the periodic character, the convergence and the global stability of positive solutions of the difference equation

$$x_{n+1} = \left(A + \sum_{i=0}^{k} \alpha_i x_{n-i}\right) \Big/ \sum_{i=0}^{k} \beta_i x_{n-i}, \quad n = 0, 1, 2, \ldots$$

where the coefficients A, α_i, β_i and the initial conditions $x_{-k}, x_{-k+1}, \ldots, x_{-1}, x_0$ are positive real numbers, while k is a positive integer number.

Keywords: difference equations, boundedness character, period two solution, convergence, global stability

MSC 2010: 39A10, 39A11, 39A99, 34C99

1. Introduction

Our goal in this paper is to investigate the boundedness character, the periodic character, the convergence and the global stability of positive solutions of the difference equation

$$x_{n+1} = \left(A + \sum_{i=0}^{k} \alpha_i x_{n-i}\right) \Big/ \sum_{i=0}^{k} \beta_i x_{n-i}, \qquad n = 0, 1, 2, \ldots \tag{1}$$

where the coefficients A, α_i, β_i and the initial conditions $x_{-k}, x_{-k+1}, \ldots, x_{-1}, x_0$ are positive real numbers, while k is a positive integer number. The case when any of A, α_i, β_i is allowed to be zero gives different special cases of the equation (1) which have been studied by many authors (see for example [1]–[14]). For related work see [15]–[27]. The study of these equations is challenging and rewarding and is still in its infancy. We believe that nonlinear rational difference equations are of paramount importance in their own right. Furthermore, the results about such equations offer prototypes for the development of the basic theory of the global behavior of nonlinear difference equations.

Definition 1. A difference equation of order $(k+1)$ is of the form

$$x_{n+1} = F(x_n, x_{n-1}, \ldots, x_{n-k}), \qquad n = 0, 1, 2, \ldots$$

where F is a continuous function which maps some set J^{k+1} into J and J is a set of real numbers. An equilibrium point $\tilde{x}$ of this equation is a point that satisfies the condition $\tilde{x} = F(\tilde{x}, \tilde{x}, \ldots, \tilde{x})$. That is, the constant sequence $\{x_n\}_{n=-k}^{\infty}$ with $x_n = \tilde{x}$ for all $n \geqslant -k$ is a solution of that equation.

Definition 2. Let $\tilde{x} \in (0, \infty)$ be an equilibrium point of the difference equation (1). Then

(i) An equilibrium point $\tilde{x}$ of the difference equation (1) is called locally stable if for every $\varepsilon > 0$ there exists $\delta > 0$ such that, if $x_{-k}, \ldots, x_{-1}$, $x_0 \in (0, \infty)$ with $|x_{-k} - \tilde{x}| + \ldots + |x_{-1} - \tilde{x}| + |x_0 - \tilde{x}| < \delta$, then $|x_n - \tilde{x}| < \varepsilon$ for all $n \geqslant -k$.

(ii) An equilibrium point $\tilde{x}$ of the difference equation (1) is called locally asymptotically stable if it is locally stable and there exists $\gamma > 0$ such that, if $x_{-k}, \ldots, x_{-1}, x_0 \in (0, \infty)$ with $|x_{-k} - \tilde{x}| + \ldots + |x_{-1} - \tilde{x}| + |x_0 - \tilde{x}| < \gamma$, then

$$\lim_{n \to \infty} x_n = \tilde{x}.$$

(iii) An equilibrium point $\tilde{x}$ of the difference equation (1) is called a global attractor if for every $x_{-k}, \ldots, x_{-1}, x_0 \in (0, \infty)$ we have

$$\lim_{n \to \infty} x_n = \tilde{x}.$$

(iv) An equilibrium point $\tilde{x}$ of the equation (1) is called globally asymptotically stable if it is locally stable and a global attractor.

(v) An equilibrium point $\tilde{x}$ of the difference equation (1) is called unstable if it is not locally stable.

Definition 3. We say that a sequence $\{x_n\}_{n=-k}^{\infty}$ is bounded and persists if there exist positive constants m and M such that

$$m \leqslant x_n \leqslant M \quad \text{for all } n \geqslant -k.$$

Definition 4. A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with period p if $x_{n+p} = x_n$ for all $n \geqslant -k$. A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with prime period p if p is the smallest positive integer having this property.

Assume that $\tilde{a} = \sum\limits_{i=0}^{k} \alpha_i$, $\bar{a} = \sum\limits_{i=0}^{k} (-1)^i \alpha_i$, $\tilde{b} = \sum\limits_{i=0}^{k} \beta_i$ and $\bar{b} = \sum\limits_{i=0}^{k} (-1)^i \beta_i$. Since the coefficients A, α_i, β_i are positive, a positive equilibrium point $\tilde{x}$ of Eq. (1) is a solution of the equation

$$\tilde{x} = \frac{A + \tilde{a}\tilde{x}}{\tilde{b}\tilde{x}}. \tag{2}$$

Consequently, the positive equilibrium point $\tilde{x}$ of the difference equation (1) is given by

$$\tilde{x} = \tilde{x}_{1,2} = \frac{\tilde{a} \pm \sqrt{\tilde{a}^2 + 4A\tilde{b}}}{2\tilde{b}}.$$

Let $F\colon\ (0, \infty)^{k+1} \longrightarrow (0, \infty)$ be a continuous function defined by

$$F(u_0, u_1, \ldots, u_k) = \left(A + \sum_{i=0}^{k} \alpha_i u_i\right) \Big/ \sum_{i=0}^{k} \beta_i u_i. \tag{3}$$

We have

$$y_{n+1} = \sum_{j=0}^{k} \frac{\partial F(\tilde{x}, \ldots, \tilde{x})}{\partial u_j}\, y_{n-j},$$

and then the linearized equation is

$$y_{n+1} = \sum_{j=0}^{k} b_j y_{n-j}, \tag{4}$$

where

$$b_j = (\alpha_j - \beta_j \tilde{x})/\tilde{b}\,\tilde{x}. \tag{5}$$

The characteristic equation of the linearized equation (4) is given by

$$\lambda^{n+1} = \sum_{j=0}^{k} b_j \lambda^{n-j}, \tag{6}$$

which can be rewritten in the form

$$\sum_{j=0}^{k} b_j \lambda^{-j-1} = 1. \tag{7}$$

2. Main results

In this section we establish some results which show that the positive equilibrium point $\tilde{x}$ of the difference equation (1) is globally asymptotically stable and every positive solution of the difference equation (1) is bounded and has prime period two.

Theorem 1 ([13] The linearized stability theorem)**.**

Suppose F is a continuously differentiable function defined on an open neighbourhood of the equilibrium $\tilde{x}$. Then the following statements are true.

(i) *If all roots of the characteristic equation* (6) *of the linearized equation* (4) *have absolute value less than one, then the equilibrium point $\tilde{x}$ is locally asymptotically stable.*

(ii) *If at least one root of Eq.* (6) *has absolute value greater than one, then the equilibrium point $\tilde{x}$ is unstable.*

(iii) *If all roots of Eq.* (6) *have absolute value greater than one, then the equilibrium point $\tilde{x}$ is a source.*

Theorem 2 (See [4], [10], [13], [17])**.** *Assume that $a, b \in \mathbb{R}$ and $k \in \{0, 1, 2, \ldots\}$. Then*

$$|a| + |b| < 1 \tag{8}$$

is a sufficient condition for the asymptotic stability of the difference equation

$$x_{n+1} + ax_n + bx_{n-k} = 0, \quad n = 0, 1, \ldots \tag{9}$$

Remark 1 (See [13]). Theorem 1 can be easily extended to a general linear difference equation of the form

$$x_{n+k} + p_1 x_{n+k-1} + \ldots + p_k x_n = 0, \quad n = 0, 1, 2, \ldots \tag{10}$$

where $p_1, p_2, \ldots, p_k \in \mathbb{R}$ and $k \in \{1, 2, \ldots\}$. We can see that the equation (10) is asymptotically stable provided that

$$\sum_{i=1}^{k} |p_i| < 1. \tag{11}$$

Theorem 3 (See [13])**.** *Consider the difference equation*

$$x_{n+1} = F(x_n, x_{n-1}, \ldots, x_{n-k})$$

where $F \in C(I^{k+1}, \mathbb{R})$ where I is an open interval of real numbers and $\mathbb{R}$ is the set of real numbers. Let $\tilde{x} \in I$ be an equilibrium of this equation. Suppose also that

(i) *F is a nondecreasing function in each of its arguments.*

(ii) *F satisfies the negative feedback property*

$$(x-\tilde{x})[F(x,x,\ldots,x)-x]<0 \quad \textit{for all } x\in I-\{\tilde{x}\}.$$

Then the equilibrium point $\tilde{x}$ is a global attractor.

The following lemma is an extension of that obtained in [16], [22] which is needed here.

Lemma 4. *Suppose that b_j $(j=0,1,\ldots,k)$ are real numbers such that $\sum\limits_{j=0}^{k}|b_j|\neq 0$ and σ_j $(j=0,1,\ldots,k)$ are positive integers. Then the equation $\sum\limits_{j=0}^{k}|b_j|x^{-\sigma_j}=1$ has a unique solution in $x\in(0,\infty)$.*

Theorem 5. *If all roots of the polynomial equation* (6) *lie in the open unit disk $|\lambda|<1$, then*

$$\sum_{j=0}^{k}|b_j|<1. \tag{12}$$

P r o o f. Assume that μ is a nonzero root of the equation (6) satisfying $|\mu|<1$. Let us write $\mu=r\exp(\mathrm{i}\theta)$, $\mathrm{i}=\sqrt{-1}$ and then write (7) in the form

$$\sum_{j=0}^{k}b_j r^{-j-1}\cos(j+1)\theta=1 \tag{13}$$

and

$$\sum_{j=0}^{k}b_j r^{-j-1}\sin(j+1)\theta=0. \tag{14}$$

Let us now discuss the following cases:

C a s e 1. If $b_j>0$ $(j=0,1,\ldots,k)$, then by virtue of Lemma 4 we see that the equation $\sum\limits_{j=0}^{k}b_j\varrho_1^{-j-1}=1$ has a unique solution $\varrho_1\in(0,\infty)$. Thus, $(r,\theta)=(\varrho_1,n\pi)$ where $n=0,2,4,\ldots$ is a solution of the equations (13), (14). This implies that $\varrho_1=r=|\mu|<1$. But then we get

$$1=\sum_{j=0}^{k}b_j\varrho_1^{-j-1}>\sum_{j=0}^{k}|b_j|.$$

C a s e 2. If $b_j < 0$ $(j = 0, 2, 4, \ldots)$ and $b_j > 0$ $(j = 1, 3, 5, \ldots)$, then by virtue of Lemma 4 we see that the equation $\sum\limits_{j=0}^{k} |b_j| \varrho_2^{-j-1} = 1$ has a unique solution $\varrho_2 \in (0, \infty)$. Thus, $(r, \theta) = (\varrho_2, n\pi)$ where $n = 1, 3, 5, \ldots$ is a solution of the equations (13), (14). This implies that $\varrho_2 = r = |\mu| < 1$. But then we get

$$1 = \sum_{j=0}^{k} |b_j| \varrho_2^{-j-1} > \sum_{j=0}^{k} |b_j|.$$

Thus, the proof of Theorem 5 is completed. $\square$

Theorem 6. *Let $\{x_n\}_{n=-k}^{\infty}$ be a positive solution of the difference equation* (1) *such that for some $n_0 \geqslant 0$,*

$$\text{(15)} \qquad \textit{either} \quad x_n \geqslant \tilde{x}_1 \quad \textit{for all } n \geqslant n_0$$

$$\text{(16)} \qquad \textit{or} \quad x_n \leqslant \tilde{x}_1 \quad \textit{for all } n \geqslant n_0.$$

Then $\{x_n\}$ converges to the equilibrium point $\tilde{x}_1$ as $n \to \infty$.

P r o o f. Assume that (15) holds. The case when (16) holds is similar and will be omitted. Then for $n \geqslant n_0 + k$ we deduce that

$$\begin{aligned} x_{n+1} &= \Big(A + \sum_{i=0}^{k} \alpha_i x_{n-i}\Big) \Big/ \sum_{i=0}^{k} \beta_i x_{n-i} \\ &= \Big[\sum_{i=0}^{k} \alpha_i x_{n-i}\Big] \Big[\Big(1 + \frac{A}{\sum\limits_{i=0}^{k} \alpha_i x_{n-i}}\Big) \Big/ \sum_{i=0}^{k} \beta_i x_{n-i}\Big] \\ &\leqslant \Big[\sum_{i=0}^{k} \alpha_i x_{n-i}\Big] \frac{[1 + (A/\tilde{a}\tilde{x}_1)]}{\tilde{b}\tilde{x}_1} = \Big[\sum_{i=0}^{k} \alpha_i x_{n-i}\Big] \frac{(A + \tilde{a}\tilde{x}_1)}{\tilde{a}\tilde{b}\tilde{x}_1^2}. \end{aligned}$$

With the aid of (2) the last inequality becomes

$$x_{n+1} \leqslant \Big[\sum_{i=0}^{k} \alpha_i x_{n-i}\Big] \frac{(A + \tilde{a}\tilde{x}_1)}{\tilde{b}\tilde{x}_1} \Big(\frac{1}{\tilde{a}\tilde{x}_1}\Big) \leqslant \sum_{i=0}^{k} \alpha_i x_{n-i} / \tilde{a},$$

and so

$$\text{(17)} \qquad x_{n+1} \leqslant \max_{0 \leqslant i \leqslant k} \{x_{n-i}\} \quad \text{for } n \geqslant n_0 + k.$$

Set

$$y_n = \max_{0\leqslant i\leqslant k}\{x_{n-i}\} \quad \text{for } n \geqslant n_0 + k. \tag{18}$$

Then clearly

$$y_n \geqslant x_{n+1} \geqslant \tilde{x}_1 \quad \text{for } n \geqslant n_0 + k. \tag{19}$$

Next we claim that

$$y_{n+1} \leqslant y_n \quad \text{for } n \geqslant n_0 + k. \tag{20}$$

We have

$$y_{n+1} = \max_{0\leqslant i\leqslant k}\{x_{n+1-i}\} = \max\{x_{n+1}, \max_{0\leqslant i\leqslant k-1}\{x_{n-i}\}\} \leqslant \max\{x_{n+1},\ y_n\} = y_n.$$

From (19) and (20) it follows that the sequence $\{y_n\}$ is convergent and that

$$y = \lim_{n\to\infty} y_n \geqslant \tilde{x}_1. \tag{21}$$

To complete the proof, it suffices to prove that $y \leqslant \tilde{x}_1$. To this end, we note that

$$x_{n+1} \leqslant \left(A + \sum_{i=0}^{k} \alpha_i x_{n-i}\right)/\tilde{b}\tilde{x}_1 \leqslant (A + \tilde{a}\ y_n)/\tilde{b}\tilde{x}_1.$$

From this and by using (20) we obtain

$$x_{n+i} \leqslant (A + \tilde{a}y_{n+i-1})/\tilde{b}\tilde{x}_1 \leqslant (A + \tilde{a}y_n)/\tilde{b}\tilde{x}_1 \quad \text{for } i = 1, \ldots, k+1.$$

Then

$$y_{n+k+1} = \max_{1\leqslant i\leqslant k+1}\{x_{n+i}\} \leqslant (A + \tilde{a}y_n)/\tilde{b}\tilde{x}_1, \tag{22}$$

and letting $n \longrightarrow \infty$, we have

$$y \leqslant \frac{A + \tilde{a}y}{\tilde{b}\tilde{x}_1}.$$

Consequently, we obtain

$$y\left(1 - \frac{\tilde{a}}{\tilde{b}\tilde{x}_1}\right) \leqslant \frac{A}{\tilde{b}\tilde{x}_1}. \tag{23}$$

From (2) and (23) we deduce that

$$\frac{y}{\tilde{x}_1}\left(\frac{\tilde{b}\tilde{x}_1 - \tilde{a}}{\tilde{b}}\right) \leqslant \left(\frac{\tilde{b}\tilde{x}_1 - \tilde{a}}{\tilde{b}}\right).$$

Since $\tilde{x}_1 > \tilde{a}/\tilde{b}$, the term in the two brackets is positive. Thus, we have $y \leqslant \tilde{x}_1$. Therefore, we have $\lim_{n\to\infty} y_n = \tilde{x}_1$ and with help of (19) we obtain $\lim_{n\to\infty} x_n = \tilde{x}_1$. The proof of Theorem 6 is completed. □

Theorem 7. *If* $\{x_n\}_{n=-k}^{\infty}$ *is a positive solution of Eq.* (1) *which is monotonic increasing, then it is bounded and persists.*

Proof. Let $\{x_n\}_{n=-k}^{\infty}$ be a positive solution of the difference equation (1). It follows from Eq. (1) that

$$x_{n+1} = (A + \alpha_0 x_n + \alpha_1 x_{n-1} + \ldots + \alpha_k x_{n-k})/(\beta_0 x_n + \beta_1 x_{n-1} + \ldots + \beta_k x_{n-k}).$$

Since $\beta_0 x_n < \beta_0 x_n + \beta_1 x_{n-1} + \ldots + \beta_k x_{n-k}$, we have

$$A/(\beta_0 x_n + \beta_1 x_{n-1} + \ldots + \beta_k x_{n-k}) < A/(\beta_0 x_n),$$

and also we note that

$$(\alpha_0 x_n)/(\beta_0 x_n + \beta_1 x_{n-1} + \ldots + \beta_k x_{n-k}) < \frac{\alpha_0}{\beta_0}.$$

Similarly, we can show that

$$(\alpha_1 x_{n-1})/(\beta_0 x_n + \beta_1 x_{n-1} + \ldots + \beta_k x_{n-k}) < \frac{\alpha_1}{\beta_1},$$

and so on. Now, we deduce that

$$x_{n+1} \leqslant \frac{A}{\beta_0 x_n} + \sum_{i=0}^{k} \frac{\alpha_i}{\beta_i}, \quad n \geqslant 0. \tag{24}$$

Since the sequence $\{x_n\}_{n=-k}^{\infty}$ is positive and monotonic increasing, we have $x_{n+1} \geqslant x_n$ and hence (24) can be rewritten in the form

$$x_n^2 - x_n \sum_{i=0}^{k} \frac{\alpha_i}{\beta_i} \leqslant \frac{A}{\beta_0}.$$

Consequently, we have

$$\left(x_n - \frac{1}{2}\sum_{i=0}^{k} \frac{\alpha_i}{\beta_i}\right)^2 \leqslant \frac{1}{4}\left(\sum_{i=0}^{k} \frac{\alpha_i}{\beta_i}\right)^2 + \frac{A}{\beta_0}.$$

From this we deduce that

$$x_n \leqslant \frac{1}{2}\left[\left(\sum_{i=0}^{k} \frac{\alpha_i}{\beta_i}\right) + \sqrt{\left(\sum_{i=0}^{k} \frac{\alpha_i}{\beta_i}\right)^2 + \frac{4A}{\beta_0}}\right] = M, \tag{25}$$

where M is a positive constant. On the other hand, the change of variables $x_n = 1/z_n$ transforms the equation (1) to

$$\frac{1}{z_{n+1}} = \left(A + \sum_{i=0}^{k} \frac{\alpha_i}{z_{n-i}}\right) \Big/ \sum_{i=0}^{k} \frac{\beta_i}{z_{n-i}}. \tag{26}$$

Consequently, we get

$$\begin{aligned} z_{n+1} = {} & \beta_0 z_{n-1} \dots z_{n-k} + \beta_1 z_n z_{n-2} \dots z_{n-k} + \dots + \beta_k z_n z_{n-1} \dots z_{n-k+1} \\ & \times (A z_n z_{n-1} \dots z_{n-k} + \alpha_0 z_{n-1} \dots z_{n-k} \\ & \quad + \alpha_1 z_n z_{n-2} \dots z_{n-k} + \dots + \alpha_k z_n z_{n-1} \dots z_{n-k+1})^{-1}, \end{aligned}$$

from which we deduce that

$$\alpha_0 z_{n-1} \dots z_{n-k} < A z_n z_{n-1} \dots z_{n-k} + \alpha_0 z_{n-1} \dots z_{n-k} + \dots + \alpha_k z_n z_{n-1} \dots z_{n-k+1},$$

and hence

$$\begin{aligned} & \beta_0 z_{n-1} \dots z_{n-k} \\ & \quad \times (A z_n z_{n-1} \dots z_{n-k} + \alpha_0 z_{n-1} \dots z_{n-k} + \dots + \alpha_k z_n z_{n-1} \dots z_{n-k+1})^{-1} < \frac{\beta_0}{\alpha_0}. \end{aligned}$$

Similarly, we see that

$$\begin{aligned} & \beta_1 z_n z_{n-2} \dots z_{n-k} \\ & \quad \times (A z_n z_{n-1} \dots z_{n-k} + \alpha_0 z_{n-1} \dots z_{n-k} + \dots + \alpha_k z_n z_{n-1} \dots z_{n-k+1})^{-1} < \frac{\beta_1}{\alpha_1}, \end{aligned}$$

and so on. Now, we deduce that

$$z_{n+1} \leqslant \sum_{i=0}^{k} \frac{\beta_i}{\alpha_i} = H, \qquad \text{for all } n \geqslant 0.$$

Thus, we obtain

$$x_n = \frac{1}{z_n} \geqslant \frac{1}{H} = m, \tag{27}$$

where H and m are positive constants. From (25) and (27) we get

$$m \leqslant x_n \leqslant M.$$

Therefore, the solution of the difference equation (1) is bounded and persists. The proof of Theorem 7 is completed. □

Theorem 8. *The positive equilibrium points $\tilde{x}_{1,2}$ of the difference equation* (1) *are globally asymptotically stable.*

P r o o f. The linearized equation (4) with the equation (5) can be written in the form

$$y_{n+1} + \sum_{j=0}^{k} \frac{\beta_j \tilde{x}_i - \alpha_j}{\tilde{b}\tilde{x}_i} y_{n-j} = 0 \quad (i = 1, 2),$$

and its characteristic equation is

$$\lambda^{n+1} + \sum_{j=0}^{k} \frac{\beta_j \tilde{x}_i - \alpha_j}{\tilde{b}\tilde{x}_i} \lambda^{n-j} = 0 \quad (i = 1, 2).$$

Now, we discuss the following cases:

C a s e 1. Since $\tilde{x}_1 > \tilde{a}/\tilde{b}$, we have

$$\sum_{j=0}^{\infty} \Big| \frac{\beta_j \tilde{x}_1 - \alpha_j}{\tilde{b}\tilde{x}_1} \Big| = \sum_{j=0}^{\infty} \frac{\beta_j \tilde{x}_1 - \alpha_j}{\tilde{b}\tilde{x}_1} = \frac{\tilde{b}\tilde{x}_1 - \tilde{a}}{\tilde{b}\tilde{x}_1} = \frac{\sqrt{\tilde{a}^2 + 4A\tilde{b}} - \tilde{a}}{\sqrt{\tilde{a}^2 + 4A\tilde{b}} + \tilde{a}} < 1.$$

C a s e 2. Since $\tilde{x}_2 < \tilde{a}/\tilde{b}$, we have

$$\sum_{j=0}^{\infty} \Big| \frac{\beta_j \tilde{x}_2 - \alpha_j}{\tilde{b}\tilde{x}_2} \Big| = \sum_{j=0}^{\infty} \frac{\alpha_j - \beta_j \tilde{x}_2}{\tilde{b}\tilde{x}_2} = \frac{\tilde{a} - \tilde{b}\tilde{x}_2}{\tilde{b}\tilde{x}_2} = \frac{\tilde{a} - \sqrt{\tilde{a}^2 + 4A\tilde{b}}}{\tilde{a} + \sqrt{\tilde{a}^2 + 4A\tilde{b}}} < 1.$$

Applying Theorem 1 we deduce that the equilibrium points $\tilde{x}_{1,2}$ are locally asymptotically stable. It remains to prove that $\tilde{x}_{1,2}$ are global attractors. To this end, we apply Theorem 3 to the function $F(u_0, u_1, \ldots, u_k)$ given by the formula (3) as follows: The function $F\colon (0,\infty)^{k+1} \longrightarrow (0,\infty)$ given by (3) is continuous and nondecreasing in each of its arguments. In addition, we deduce for $x \in (0,\infty)$ that

$$\begin{aligned}[F(x, x, \ldots, x) - x](x - \tilde{x}_1) &= \Big[\frac{A + \tilde{a}x}{\tilde{b}x} - x\Big](x - \tilde{x}_1) \\ &= -\Big(\frac{\tilde{b}x^2 - \tilde{a}x - A}{\tilde{b}x}\Big)(x - \tilde{x}_1) = -\frac{(x - \tilde{x}_1)^2 (x - \tilde{x}_2)}{x} < 0\end{aligned}$$

for all $x > \tilde{x}_2$. Thus, the conditions of Theorem 3 are satisfied. This proves that the equilibrium point $\tilde{x}_1$ is a global attractor. Similarly, we can show that

$$[F(x, x, \ldots, x) - x](x - \tilde{x}_2) = -\frac{(x - \tilde{x}_1)(x - \tilde{x}_2)^2}{x} < 0$$

for all $x > \tilde{x}_1$. This proves that the equilibrium point $\tilde{x}_2$ is a global attractor. Now, we have shown that the equilibrium points $\tilde{x} = \tilde{x}_{1,2}$ are global attractors. The proof of Theorem 8 is completed. □

Theorem 9. *A necessary and sufficient condition for the difference equation* (1) *to have a positive prime period two solution is that the inequality*

$$A(\tilde{b}-\bar{b})^2-\bar{a}(\tilde{a}+\bar{a})(\tilde{b}-\bar{b})<\bar{b}\bar{a}^2 \tag{28}$$

is valid, provided $\bar{a}<0$ *and* $\bar{b}>0$.

P r o o f. First, suppose that there exists a positive prime period two solution

$$\ldots, P, Q, P, Q, \ldots$$

of the difference equation (1). We shall prove that the condition (28) holds. It follows from the difference equation (1) that if k is even, then $x_n = x_{n-k}$ and we have

$$P=\frac{A+\alpha_0 Q+\alpha_1 P+\alpha_2 Q+\alpha_3 P+\ldots+\alpha_k Q}{\beta_0 Q+\beta_1 P+\beta_2 Q+\beta_3 P+\ldots+\beta_k Q}$$

and

$$Q=\frac{A+\alpha_0 P+\alpha_1 Q+\alpha_2 P+\alpha_3 Q+\ldots+\alpha_k P}{\beta_0 P+\beta_1 Q+\beta_2 P+\beta_3 Q+\ldots+\beta_k P},$$

while if k is odd, then $x_{n+1}=x_{n-k}$ and we have

$$P=\frac{A+\alpha_0 Q+\alpha_1 P+\alpha_2 Q+\alpha_3 P+\ldots+\alpha_k P}{\beta_0 Q+\beta_1 P+\beta_2 Q+\beta_3 P+\ldots+\beta_k P}$$

and

$$Q=\frac{A+\alpha_0 P+\alpha_1 Q+\alpha_2 P+\alpha_3 Q+\ldots+\alpha_k Q}{\beta_0 P+\beta_1 Q+\beta_2 P+\beta_3 Q+\ldots+\beta_k Q}.$$

Now, we discuss the case when k is even (and in a similar way we can discuss the case when k is odd which is omitted here). Consequently, we obtain

$$A+\alpha_0 Q+\alpha_1 P+\alpha_2 Q+\ldots+\alpha_k Q=\beta_0 PQ+\beta_1 P^2+\beta_2 PQ+\ldots+\beta_k PQ \tag{29}$$

and

$$A+\alpha_0 P+\alpha_1 Q+\alpha_2 P+\ldots+\alpha_k P=\beta_0 PQ+\beta_1 Q^2+\beta_2 PQ+\ldots+\beta_k PQ. \tag{30}$$

By subtracting, we deduce after some reduction that

$$P+Q=\frac{-\bar{a}}{\beta_1+\beta_3+\ldots+\beta_{k-1}}, \tag{31}$$

while by adding we obtain

$$PQ=\frac{A(\beta_1+\beta_3+\ldots+\beta_{k-1})-\bar{a}(\alpha_0+\alpha_2+\ldots+\alpha_k)}{\bar{b}(\beta_1+\beta_3+\ldots+\beta_{k-1})}, \tag{32}$$

where $\beta_i > 0$, $\bar{a} < 0$ and $\bar{b} > 0$. Assume that P and Q are two positive distinct real roots of the quadratic equation

$$t^2 - (P+Q)t + PQ = 0. \tag{33}$$

We deduce that

$$\Big(\frac{-\bar{a}}{\beta_1+\beta_3+\ldots+\beta_{k-1}}\Big)^2 > 4\,\frac{A(\beta_1+\beta_3+\ldots+\beta_{k-1}) - \bar{a}(\alpha_0+\alpha_2+\ldots+\alpha_k)}{\bar{b}(\beta_1+\beta_3+\ldots+\beta_{k-1})}. \tag{34}$$

From (34), we obtain

$$A(\tilde{b}-\bar{b})^2 - (\tilde{a}+\bar{a})(\tilde{b}-\bar{b})\bar{a} < \bar{b}\bar{a}^2,$$

and hence the condition (28) is valid. Conversely, suppose that the condition (28) is valid. Then we deduce immediately from (28) that the inequality (34) holds. Consequently, there exist two positive distinct real numbers P and Q such that

$$P = \frac{-\bar{a}}{2(\beta_1+\beta_3+\ldots+\beta_{k-1})} - \frac{1}{2}\sqrt{T_1} \tag{35}$$

and

$$Q = \frac{-\bar{a}}{2(\beta_1+\beta_3+\ldots+\beta_{k-1})} + \frac{1}{2}\sqrt{T_1}, \tag{36}$$

where $T_1 > 0$ is given by the formula

$$T_1 = \Big(\frac{-\bar{a}}{\beta_1+\beta_3+\ldots+\beta_{k-1}}\Big)^2 - 4\,\frac{A(\beta_1+\beta_3+\ldots+\beta_{k-1}) - \bar{a}(\alpha_0+\alpha_2+\ldots+\alpha_k)}{\bar{b}(\beta_1+\beta_3+\ldots+\beta_{k-1})}. \tag{37}$$

Thus, P and Q represent two positive distinct real roots of the quadratic equation (33). Now, we are going to prove that P and Q are positive solutions of prime period two for the difference equation (1). To this end, we assume that

$$x_{-k} = P,\ x_{-k+1} = Q,\ \ldots,\ x_{-1} = Q, \quad \text{and} \quad x_0 = P.$$

We wish to show that

$$x_1 = Q \quad \text{and} \quad x_2 = P.$$

To this end, we deduce from the difference equation (1) that

$$
(38)\qquad \begin{aligned} x_1 &= \frac{A+\alpha_0 x_0+\alpha_1 x_{-1}+\ldots+\alpha_k x_{-k}}{\beta_0 x_0+\beta_1 x_{-1}+\ldots+\beta_k x_{-k}} \\ &= \frac{A+P(\alpha_0+\alpha_2+\ldots+\alpha_k)+Q(\alpha_1+\alpha_3+\ldots+\alpha_{k-1})}{P(\beta_0+\beta_2+\ldots+\beta_k)+Q(\beta_1+\beta_3+\ldots+\beta_{k-1})}. \end{aligned}
$$

Multiplying the denominator and numerator of (38) by $\gamma=-(\beta_1+\beta_3+\ldots+\beta_{k-1})/\bar{a}$ and using (35)–(37) we obtain

$$
(39)\quad \begin{aligned} x_1 &= \frac{2A\gamma+[1+\sqrt{K_1}](\alpha_0+\alpha_2+\ldots+\alpha_k)}{[1+\sqrt{K_1}](\beta_0+\beta_2+\ldots+\beta_k)+[1-\sqrt{K_1}](\beta_1+\beta_3+\ldots+\beta_{k-1})} \\ &\quad+\frac{[1-\sqrt{K_1}](\alpha_1+\alpha_3+\ldots+\alpha_{k-1})}{[1+\sqrt{K_1}](\beta_0+\beta_2+\ldots+\beta_k)+[1-\sqrt{K_1}](\beta_1+\beta_3+\ldots+\beta_{k-1})} \\ &= \frac{[(\alpha_0+\alpha_2+\ldots+\alpha_k)+(\alpha_1+\alpha_3+\ldots+\alpha_{k-1})+2A\gamma}{[(\beta_0+\beta_2+\ldots+\beta_k)+(\beta_1+\beta_3+\ldots+\beta_{k-1})]+\bar{b}\sqrt{K_1}} \\ &\quad+\frac{[(\alpha_0+\alpha_2+\ldots+\alpha_k)-(\alpha_1+\alpha_3+\ldots+\alpha_{k-1})]\sqrt{K_1}}{[(\beta_0+\beta_2+\ldots+\beta_k)+(\beta_1+\beta_3+\ldots+\beta_{k-1})]+\bar{b}\sqrt{K_1}} \\ &= \frac{[\tilde{a}+2A\gamma]+\bar{a}\sqrt{K_1}}{\tilde{b}+\bar{b}\sqrt{K_1}}, \end{aligned}
$$

where

$$
(40)\qquad K_1 = 1-\Big[\frac{A(\tilde{b}-\bar{b})^2-\bar{a}(\tilde{a}+\bar{a})(\tilde{b}-\bar{b})}{\bar{b}\bar{a}^2}\Big],
$$

and from the condition (28) we deduce that $K_1>0$. Multiplying the denominator and numerator of (39) by

$$
\tilde{b}-\bar{b}\sqrt{K_1},
$$

we have

$$
x_1=\frac{\tilde{b}[\tilde{a}+2A\gamma]-\bar{b}\bar{a}K_1}{\tilde{b}^2-\bar{b}^2K_1}+\frac{[\tilde{b}\bar{a}-\tilde{a}\bar{b}+-2A\bar{b}\gamma]\sqrt{K_1}}{\tilde{b}^2-\bar{b}^2K_1}.
$$

After some reduction, we deduce that

$$
\begin{aligned} x_1 &= \frac{(1+\sqrt{K_1})T_2}{2\gamma T_2} = \frac{-\bar{a}(1+\sqrt{K_1})}{2(\beta_1+\beta_3+\ldots+\beta_{k-1})} \\ &= \frac{-\bar{a}}{2(\beta_1+\beta_3+\ldots+\beta_{k-1})}+\frac{1}{2}\sqrt{T_1}=Q, \end{aligned}
$$

where

$$
\begin{aligned} T_2 &= 2(\alpha_1+\alpha_3+\ldots+\alpha_{k-1})(\beta_0+\beta_2+\ldots+\beta_k) \\ &\quad -2(\alpha_0+\alpha_2+\ldots+\alpha_k)(\beta_1+\beta_3+\ldots+\beta_{k-1})-\frac{2A\bar{b}(\beta_1+\beta_3+\ldots+\beta_{k-1})}{B+\bar{a}}. \end{aligned}
$$

Similarly, we can show that

$$
\begin{aligned}
x_2 &= \frac{A+\alpha_0 x_1+\alpha_1 x_0+\ldots+\alpha_k x_{-(k-1)}}{\beta_0 x_1+\beta_1 x_0+\ldots+\beta_k x_{-(k-1)}} \\
&= \frac{A+Q(\alpha_0+\alpha_2+\ldots+\alpha_k)+P(\alpha_1+\alpha_3+\ldots+\alpha_{k-1})}{Q(\beta_0+\beta_2+\ldots+\beta_k)+P(\beta_1+\beta_3+\ldots+\beta_{k-1})}=P.
\end{aligned}
$$

By using the mathematical induction, we have

$$x_n = P \quad \text{and} \quad x_{n+1}=Q \qquad \text{for all } n \geqslant -k.$$

Thus the difference eqution (1) has a positive prime period two solution

$$\ldots,\ P,\ Q,\ P,\ Q,\ \ldots$$

The proof of Theorem 9 is completed. □

A c k n o w l e d g e m e n t. The authors would like to express their deep thanks to the referee for his interesting suggestions and comments on this paper.

References

[1] *M. T. Aboutaleb, M. A. El-Sayed, A. E. Hamza*: Stability of the recursive sequence $x_{n+1}=(\alpha-\beta x_n)/(\gamma+x_{n-1})$. J. Math. Anal. Appl. *261* (2001), 126–133.

[2] *R. Agarwal*: Difference Equations and Inequalities. Theory, Methods and Applications, Marcel Dekker, New York, 1992.

[3] *A. M. Amleh, E. A. Grove, G. Ladas, D. A. Georgiou*: On the recursive sequence $x_{n+1}=\alpha+(x_{n-1}/x_n)$. J. Math. Anal. Appl. *233* (1999), 790–798.

[4] *R. De Vault, W. Kosmala, G. Ladas, S. W. Schultz*: Global behavior of $y_{n+1}=(p+y_{n-k})/(qy_n+y_{n-k})$. Nonlinear Analysis *47* (2001), 4743–4751.

[5] *R. De Vault, G. Ladas, S. W. Schultz*: On the recursive sequence $x_{n+1}=A/x_n+1/x_{n-2}$. Proc. Amer. Math. Soc. *126* (1998), 3257–3261.

[6] *R. De Vault, S. W. Schultz*: On the dynamics of $x_{n+1}=(\beta x_n+\gamma x_{n-1})/(Bx_n+Dx_{n-2})$. Comm. Appl. Nonlinear Analysis *12* (2005), 35–39.

[7] *H. El-Metwally, E. A. Grove, G. Ladas*: A global convergence result with applications to periodic solutions. J. Math. Anal. Appl. *245* (2000), 161–170.

[8] *H. El-Metwally, G. Ladas, E. A. Grove, H. D. Voulov*: On the global attractivity and the periodic character of some difference equations. J. Difference Equ. Appl. *7* (2001), 837–850.

[9] *H. M. EL-Owaidy, A. M. Ahmed, M. S. Mousa*: On asymptotic behavior of the difference equation $x_{n+1}=\alpha+(x^p_{n-1}/x^p_n)$. J. Appl. Math. & Comput. *12* (2003), 31–37.

[10] *H. M. EL-Owaidy, A. M. Ahmed, Z. Elsady*: Global attractivity of the recursive sequence $x_{n+1}=(\alpha-\beta x_{n-k})/(\gamma+x_n)$. J. Appl. Math. & Comput. *16* (2004), 243–249.

[11] *G. Karakostas*: Convergence of a difference equation via the full limiting sequences method. Diff. Equations and Dynamical. System *1* (1993), 289–294.

[12] *G. Karakostas, S. Stević*: On the recursive sequences $x_{n+1}=A+f(x_n,\ldots,x_{n-k+1})/x_{n-1}$. Commun. Appl. Nonlin. Anal. *11* (2004), 87–99.

[13] *V. L. Kocic, G. Ladas*: Global Behavior of Nonlinear Difference Equations of Higher Order with Applications. Kluwer Academic Publishers, Dordrecht, 1993.
[14] *M. R. S. Kulenovic, G. Ladas*: Dynamics of Second Order Rational Difference Equations with Open Problems and Conjectures. Chapman & Hall/CRC Press, 2002.
[15] *M. R. S. Kulenovic, G. Ladas, W. S. Sizer*: On the recursive sequence $x_{n+1} = (\alpha x_n + \beta x_{n-1})/(\gamma x_n + \delta x_{n-1})$. Math. Sci. Res. Hot-Line *2* (1998), 1–16.
[16] *S. A. Kuruklis*: The asymptotic stability of $x_{n+1} - ax_n + bx_{n-k} = 0$. J. Math. Anal. Appl. *188* (1994), 719–731.
[17] *G. Ladas, C. H. Gibbons, M. R. S. Kulenovic, H. D. Voulov*: On the trichotomy character of $x_{n+1} = (\alpha + \beta x_n + \gamma x_{n-1})/(A + x_n)$. J. Difference Equations and Appl. *8* (2002), 75–92.
[18] *G. Ladas, C. H. Gibbons, M. R. S. Kulenovic*: On the dynamics of $x_{n+1} = (\alpha + \beta x_n + \gamma x_{n-1})/(A + Bx_n)$. Proceeding of the Fifth International Conference on Difference Equations and Applications, Temuco, Chile, Jan. 3–7, 2000, Taylor and Francis, London (2002), 141–158.
[19] *G. Ladas, E. Camouzis, H. D. Voulov*: On the dynamic of $x_{n+1} = (\alpha + \gamma x_{n-1} + \delta x_{n-2})/(A + x_{n-2})$. J. Difference Equ. Appl. *9* (2003), 731–738.
[20] *G. Ladas*: On the recursive sequence $x_{n+1} = (\alpha + \beta x_n + \gamma x_{n-1})/(A + Bx_n + Cx_{n-1})$. J. Difference Equ. Appl. *1* (1995), 317–321.
[21] *W. T. Li, H. R. Sun*: Global attractivity in a rational recursive sequence. Dyn. Syst. Appl. *11* (2002), 339–346.
[22] *Yi-Zhong Lin*: Common domain of asymptotic stability of a family of difference equations. Appl. Math. E-Notes *1* (2001), 31–33.
[23] *S. Stević*: On the recursive sequences $x_{n+1} = x_{n-1}/g(x_n)$. Taiwanese J. Math. *6* (2002), 405–414.
[24] *S. Stević*: On the recursive sequences $x_{n+1} = g(x_n, x_{n-1})/(A + x_n)$. Appl. Math. Letter *15* (2002), 305–308.
[25] *S. Stević*: On the recursive sequences $x_{n+1} = \alpha + (x_{n-1}^p/x_n^p)$. J. Appl. Math. Comput. *18* (2005), 229–234.
[26] *E. M. E. Zayed, M. A. El-Moneam*: On the rational recursive sequence $x_{n+1} = (D + \alpha x_n + \beta x_{n-1} + \gamma x_{n-2})/(Ax_n + Bx_{n-1} + Cx_{n-2})$. Commun. Appl. Nonlin. Anal. *12* (2005), 15–28.
[27] *E. M. E. Zayed, M. A. El-Moneam*: On the rational recursive sequence $x_{n+1} = (\alpha x_n + \beta x_{n-1} + \gamma x_{n-2} + \delta x_{n-3})/(Ax_n + Bx_{n-1} + Cx_{n-2} + Dx_{n-3})$. J. Appl. Math. Comput. *22* (2006), 247–262.

Authors' addresses: *E. M. E. Zayed*, Mathematics Department, Faculty of Science, Taif University, El-Taif, Hawai, P.O. Box 888, Kingdom of Saudi Arabia, e-mail: `emezayed@hotmail.com`; *M. A. El-Moneam*, Mathematics Department, Faculty of Science, Zagazig University, Zagazig, Egypt, e-mail: `mabdelmeneam2004@yahoo.com`.