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COUNTING IRREDUCIBLE POLYNOMIALS OVER FINITE FIELDS

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Abstract. In this paper we generalize the method used to prove the Prime Number Theorem to deal with finite fields, and prove the following theorem:

$$\pi(x) = \frac{q}{q-1}\frac{x}{\log_q x} + \frac{q}{(q-1)^2}\frac{x}{\log_q^2 x} + O\Big(\frac{x}{\log_q^3 x}\Big), \quad x = q^n \to \infty$$

where $\pi(x)$ denotes the number of monic irreducible polynomials in $F_q[t]$ with norm $\leqslant x$.

Keywords: finite fields, distribution of irreducible polynomials, residue

MSC 2010: 11T55

1. Introduction

Let F_q be a finite field with character p, and $N(f)$ be the norm of f which is equal to the number of elements in the quotient ring $F_q[t]/(f(t))$. We consider the irreducible polynomials in $F_q[t]$ with norm less than or equal to x.

Let $\pi(x)$ denote the number of monic irreducible polynomials in $F_q[t]$ with norm $\leqslant x$. In 1990, M. Kruse and H. Stichtenoth (see [1]) proved that

$$\pi(x) \sim \frac{q}{q-1}\frac{x}{\log_q x}, \quad x = q^n \to \infty.$$

In this paper we generalize the method used to prove the Prime Number Theorem to deal with finite fields, and prove the following more precise result:

$$\pi(x) = \frac{q}{q-1}\frac{x}{\log_q x} + \frac{q}{(q-1)^2}\frac{x}{\log_q^2 x} + O\Big(\frac{x}{\log_q^3 x}\Big),$$

where $x = q^n \to \infty$.

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2. The prime number theorem for $F_q[t]$

Let $f(t)$ be a polynomial in $F_q[t]$ with degree n. It is easily seen that $N(f) = q^n$. The zeta function of $F_q[t]$ is defined as

$$\zeta(s) = \sum_f N(f)^{-s},$$

where the sum is taken over all monic polynomials in $F_q[t]$. There are q^n monic polynomials in $F_q[t]$ with degree n. Hence

$$\zeta(s) = \sum_{n=0}^{\infty} \frac{q^n}{q^{ns}} = \sum_{n=0}^{\infty} q^{n(1-s)}$$

converges for $\mathrm{Re}(s) > 1$. Whence

$$\zeta(s) = \frac{1}{1 - q^{1-s}}. \tag{2.1}$$

Hence we obtain an analytic continuation of $\zeta(s)$ which has poles at $s = 1 + 2k\pi\mathrm{i}/\log q$, $k \in \mathbb{Z}$ and does not vanish everywhere.

Since every monic polynomial can be factored as a product of monic irreducible polynomials uniquely, we have the Euler product formula:

$$\zeta(s) = \prod_P \Big(1 - \frac{1}{N(P)^s}\Big)^{-1} \quad \text{for } \mathrm{Re}(s) > 1, \tag{2.2}$$

where the product is taken over all monic irreducible polynomials in $F_q[t]$. By applying logarithms to both sides in equation (2.2), and then differentiating, we obtain

$$-\frac{\zeta'}{\zeta}(s) = \sum_P \frac{N(P)^{-s} \log N(P)}{1 - N(P)^{-s}} = \sum_P \sum_{n=1}^{\infty} N(P)^{-ns} \log N(P) = \sum_f \frac{\Lambda(f)}{N(f)^s},$$

where the sum is taken over all monic polynomials in $F_q[t]$ and

$$\Lambda(f) = \begin{cases} \log N(P) & \text{if } f \text{ is a power of some irreducible polynomial } P, \\ 0 & \text{otherwise.} \end{cases}$$

From the equation (2.1), we see that

$$-\frac{\zeta'}{\zeta}(s) = \frac{q^{1-s} \log q}{1 - q^{1-s}}, \tag{2.3}$$

which has simple poles at $s = 1 + 2k\pi\mathrm{i}/\log q$, $k \in \mathbb{Z}$, and with residue 1.

Let $\psi(x) = \sum_{N(f) \leqslant x} \Lambda(f)$, where f are monic polynomials in $F_q[t]$. Beginning with the fundamental line integral

$$\frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} y^s \frac{\mathrm{d}s}{s} = \begin{cases} 1 & \text{if } y > 1, \\ \frac{1}{2} & \text{if } y = 1, \\ 0 & \text{if } y < 1, \end{cases}$$

for any $c > 1$ we have

$$\psi_0(x) = \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} -\frac{\zeta'}{\zeta}(s) \frac{x^s}{s} \, \mathrm{d}s,$$

where

$$\psi_0(x) = \begin{cases} \psi(x) - \dfrac{1}{2} \sum_{N(f)=x} \Lambda(f) & \text{if } x = q^n, \ \ n \in \mathbb{N}, \\ \psi(x) & \text{otherwise.} \end{cases}$$

Then by (2.3) we get

$$\psi_0(x) = \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{q^{1-s} \log q}{1 - q^{1-s}} \frac{x^s}{s} \, \mathrm{d}s.$$

As a consequence of this calculus we get

Lemma 2.1. *Let $f(s)$ be continuous on $\Gamma_R\colon\ s = c + R\mathrm{e}^{\mathrm{i}\theta}$ ($\frac{1}{2}\pi \leqslant \theta \leqslant \frac{3}{2}\pi$), and $f(s) \to 0$ as $R \to +\infty$, then $\int_{\Gamma_R} f(s) x^s \, \mathrm{d}s \to 0$ as $R \to +\infty$, for any $x > 1$.*

Let $f(s) = (q^{1-s} \log q)/(1 - q^{1-s})s$. We have

$$(2.4) \qquad \frac{1}{2\pi \mathrm{i}} \int_{c-iR}^{c+iR} f(s) x^s \, \mathrm{d}s \to \psi_0(x), \quad \text{as } R \to +\infty, \text{ where } c > 1.$$

If $R = R_0 = \sqrt{(c-1)^2 + ((2k+1)^2 \pi^2)/\log^2 q}$, (2.4) holds also for $k \to +\infty$.

If $\Gamma_R\colon\ s = c + R_0 \mathrm{e}^{\mathrm{i}\theta}$ ($\frac{1}{2}\pi \leqslant \theta \leqslant \frac{3}{2}\pi$), it is easily seen that we can apply Lemma 2.1 to $f(s)$. Hence we deduce the following proposition:

Proposition 2.1.

$$\psi_0(x) = \frac{q\log q}{1-q} + x\sum_{k=-\infty}^{\infty} \frac{\cos(ky)(\log q)^2 + 2k\pi\log q\sin(ky)}{(\log q)^2 + 4k^2\pi^2},$$

for any $x > 1$, *where* $y = 2\pi\log x/\log q$.

Proof. By Lemma 2.1 we get $\int_{\Gamma_{R_0}} f(s)x^s\,\mathrm{d}s \to 0$, as $R_0 \to \infty$ for any $x > 1$. Hence by contour integration we have

$$\psi_0(x) = \frac{q\log q}{1-q} + \sum_{k=-\infty}^{\infty} \frac{x^{1+2k\pi\mathrm{i}/\log q}}{1+2k\pi\mathrm{i}/\log q}. \tag{2.5}$$

Indeed, this is obtained by the integral on the line $\mathrm{Re}(s) = c$ and by moving it to Γ_{R_0}. The simple poles at $s = 0$, $s = 1 + 2k\pi\mathrm{i}/\log q$ produce the corresponding terms in (2.5). Since $\psi_0(x)$ is a real valued function, imaginary part of it must be zero, and the result follows. □

Corollary 2.1.

$$\sum_{k=-\infty}^{\infty} \frac{\log^2 q}{\log^2 q + 4k^2\pi^2} = \frac{q+1}{2(q-1)}\log q.$$

Proof. By Proposition 1 let $x = q$. We have

$$\psi_0(q) = \frac{q\log q}{1-q} + q\sum_{k=-\infty}^{\infty} \frac{\log^2 q}{\log^2 q + 4k^2\pi^2} = \frac{q}{2}\log q,$$

and the result follows. □

Let $x = q^n$. We get

$$\psi_0(q^n) = \frac{q\log q}{1-q} + q^n\sum_{k=-\infty}^{\infty} \frac{\log^2 q}{\log^2 q + 4k^2\pi^2} = \frac{(q^{n+1} + q^n - 2q)\log q}{2(q-1)} \tag{2.6}$$

and

$$\psi(q^n) = \psi_0(q^n) + \frac{1}{2}\sum_{N(f)=q^n} \Lambda(f) = 2\psi_0(q^n) - \psi(q^{n-1}), \tag{2.7}$$

$$\psi(q) = q\log q. \tag{2.8}$$

Then by (2.6), (2.7) and (2.8) we deduce that

$$\psi(q^n) = \frac{q^{n+1} - q}{q-1}\log q. \tag{2.9}$$

□

Lemma 2.2.

$$\sum_{i=1}^{n} \frac{q^i}{i} = \frac{q^{n+1}}{n(q-1)} + \frac{q^{n+1}}{n^2(q-1)^2} + O\Big(\frac{q^n}{n^2}\Big), \quad \text{as } n \to \infty.$$

Proof. We have

$$\sum_{i=1}^{n} \frac{q^i}{i} = \frac{q^n}{n} \sum_{i=0}^{n-1} \frac{nq^{-i}}{n-i} = \frac{q^n}{n}\Big(\sum_{i=0}^{n-1}\Big(1+\frac{i}{n-i}\Big)q^{-i}\Big),$$

and

$$\sum_{i=0}^{n-1} q^{-i} = \frac{q}{q-1} + O(q^{-n}),$$

$$\sum_{i=0}^{n-1} \frac{i}{n-i} q^{-i} = q^{-n} \sum_{i=1}^{n-1} \frac{n-i}{i} q^i.$$

By Poisson's summation formula we get

$$\sum_{i=1}^{n-1} \frac{n-i}{i} q^i = \frac{qn}{n-1} \sum_{i=1}^{n-2} \frac{q^i}{i(i+1)} + \frac{q}{n-1} \frac{1-q^{n-1}}{1-q} + O(n),$$

and

$$\sum_{i=1}^{n-2} \frac{q^i}{i(i+1)} = \frac{q}{q-1} \frac{q^{n-2}-1}{(n-2)(n-1)} + O\Big(\frac{q^n}{n^3}\Big).$$

Therefore

$$\sum_{i=1}^{n-1} \frac{n-i}{i} q^i = \frac{q^{n+1}}{n(q-1)^2} + O\Big(\frac{q^n}{n^2}\Big),$$

and the result follows. □

Theorem 2.1.

$$\pi(x) = \frac{q}{q-1} \frac{x}{\log_q x} + \frac{q}{(q-1)^2} \frac{x}{\log_q^2 x} + O\Big(\frac{x}{\log_q^3 x}\Big), \quad \text{where } x = q^n \to \infty.$$

Proof. Let

$$\pi_1(x) = \sum_{N(f) \leqslant x} \frac{\Lambda(f)}{\log N(f)}.$$

We have

$$\pi_1(x) = \sum_{N(P^m) \leqslant x} \frac{\log N(P)}{m \log N(P)} = \pi(x) + \frac{1}{2}\pi(x^{1/2}) + \frac{1}{3}\pi(x^{1/3}) + \dots \tag{2.10}$$

$$= \pi(x) + o(x^{1/2}) \quad \text{(see [2])}.$$

□

By (2.9) and Lemma 2.2 we have

$$\begin{aligned}\pi_1(x) &= \sum_{i=1}^{n} \frac{\psi(q^i)-\psi(q^{i-1})}{i\log q} \\ &= \sum_{i=1}^{n} \frac{q^i}{i} = \frac{q^{n+1}}{n(q-1)} + \frac{q^{n+1}}{n^2(q-1)^2} + O\Big(\frac{q^n}{n^2}\Big),\end{aligned} \tag{2.11}$$

as $x = q^n \to \infty$. By (2.10) and (2.11) we deduce the theorem. $\square$

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