Robert Černý; Silvie Mašková

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A SHARP FORM OF AN EMBEDDING INTO MULTIPLE EXPONENTIAL SPACES

Robert Černý, Silvie Mašková, Praha

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Abstract. Let Ω be a bounded open set in $\mathbb{R}^n$, $n \geqslant 2$. In a well-known paper *Indiana Univ. Math. J.*, 20, 1077–1092 (1971) Moser found the smallest value of K such that

$$\sup\Big\{\int_\Omega \exp\Big(\Big(\frac{|f(x)|}{K}\Big)^{n/(n-1)}\Big)\colon f \in W_0^{1,n}(\Omega), \|\nabla f\|_{L^n} \leqslant 1\Big\} < \infty.$$

We extend this result to the situation in which the underlying space L^n is replaced by the generalized Zygmund space $L^n \log^{n-1} L \log^\alpha \log L$ $(\alpha < n-1)$, the corresponding space of exponential growth then being given by a Young function which behaves like $\exp(\exp(t^{n/(n-1-\alpha)}))$ for large t. We also discuss the case of an embedding into triple and other multiple exponential cases.

Keywords: Orlicz spaces, Orlicz-Sobolev spaces, embedding theorems, sharp constants

MSC 2010: 46E35, 46E30

1. Introduction

Throughout the paper Ω denotes an open bounded set in $\mathbb{R}^n$, $n \geqslant 2$, we write $n' = n/(n-1)$ (i.e. $1/n + 1/n' = 1$), and ω_{n-1} stands for the measure of the surface of the unit sphere in $\mathbb{R}^n$.

The classical Sobolev embedding theorem states that $W_0^{1,p}(\Omega)$ is continuously embedded into $L^{p^*}(\Omega)$ if $1 \leqslant p < n$ and $p^* = pn/(n-p)$. If $p > n$ then every function from $W_0^{1,p}(\Omega)$ is bounded (i.e. belongs to $L^\infty(\Omega)$) and in the limiting case $p = n$, it is known that every function from $W_0^{1,n}(\Omega)$ belongs to $L^q(\Omega)$ for every $1 \leqslant q < \infty$ but not neccesarily to $L^\infty(\Omega)$. A famous result of Trudinger (see

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[10], [15], [17] and [18]) implies that the first-order Sobolev space $W_0^{1,n}(\Omega)$ may be continuously embedded into the Orlicz space $L^\Phi(\Omega)$ with the Young function Φ of an exponential type $\Phi(t) = \exp(t^{n'}) - 1$, $t > 0$.

In [12] Moser proved that for $K \geqslant n^{-(n-1)/n}\omega_{n-1}^{-1/n}$ we have

$$\sup\Big\{\int_\Omega \exp\Big(\Big(\frac{|f(x)|}{K}\Big)^{n'}\Big)\,\mathrm{d}x\colon\ f \in W_0^{1,n}(\Omega), \|\nabla f\|_{L^n} \leqslant 1\Big\} < \infty \tag{1}$$

but that for $K < n^{-(n-1)/n}\omega_{n-1}^{-1/n}$ the integral $\int_\Omega \exp\big(\big(|f(x)|/K\big)^{n'}\big)\,\mathrm{d}x$ can be made arbitrarily large by an appropriate choice of $f \in W_0^{1,n}(\Omega)$, $\|\nabla f\|_{L^n} \leqslant 1$. Our aim is to study a similar phenomenon for other embeddings into exponential and multiple exponential spaces.

Here by ∇f we denote the generalized derivative of f while $W_0^{1,n}(\Omega)$ and $L_0^\Phi(\Omega)$ stand for the closure of $C_0^1(\Omega)$ in $W^{1,n}(\Omega)$ and $L^\Phi(\Omega)$, respectively. For the definition of the norm in $L^\Phi(\Omega)$ see Preliminaries. By $WL_0^\Phi(\Omega)$ we denote the set of all functions f such that $|\nabla f| \in L_0^\Phi(\Omega)$.

Let $\alpha < n-1$ and set

$$\gamma = \frac{n}{n-1-\alpha} > 0 \qquad \text{and} \qquad B = 1 - \frac{\alpha}{n-1} = \frac{n'}{\gamma} > 0. \tag{2}$$

Then the following analogue of the embedding result given above is well known. The space $WL^n\log^\alpha L(\Omega)$ of the Sobolev type, modeled on the Zygmund space $L^n\log^\alpha L(\Omega)$, is continuously embedded into the Orlicz space with the Young function that behaves like $\exp(t^\gamma)$ for large t. These results are due to Fusco, Lions, Sbordone [9] for $\alpha < 0$ and Edmunds, Gurka, Opic [3] in general. Moreover, it is shown in [3] (see also [2] and [4]) that in the limiting case $\alpha = n-1$ we have the embedding into a double exponential space, i.e., the space $WL^n\log^{n-1}L\log^\alpha\log L(\Omega)$, $\alpha < n-1$, is continuously embedded into the Orlicz space with the Young function that behaves like $\exp(\exp(t^\gamma))$ for large t. Further, in the limiting case $\alpha = n-1$ we have the embedding into the triple exponential space and so on. For other results concerning these spaces we refer the reader to [4], [5], [6], [7], [8] and [13].

In paper [11], the author studies the analogue of (1) for the case of an embedding into single and double exponential spaces. In the case of the single exponential space, i.e for a Young function Φ satisfying

$$\lim_{t\to\infty}\frac{\Phi(t)}{t^n\log^\alpha(t)} = 1,$$

$\alpha < n-1$, $n \geqslant 2$, it is shown that the critical constant for the uniform boundedness of $\int_\Omega \exp\big(\big(|f(x)|/K\big)^\gamma\big)\,\mathrm{d}x$ is $K = B^{-(n-1)/n}n^{-1/\gamma}\omega_{n-1}^{-1/n}$. In the case of the double exponential space, i.e. when Φ satisfies

$$\lim_{t\to\infty} \frac{\Phi(t)}{t^n \log^{n-1}(t) \log^{\alpha}(\log(t))} = 1,$$

$\alpha < n-1$, and the integral considered is $\int_\Omega \exp\bigl(\exp\bigl(\bigl(|f(x)|/K\bigr)^\gamma\bigr)\bigr)\,\mathrm{d}x$, the critical constant is $K = B^{-(n-1)/n}\omega_{n-1}^{-1/n}$.

Our aim is to study the problem for higher multiple exponential spaces. For $k \in \mathbb{N}$, $k \geqslant 2$, we write

$$\log_{[k]}(t) = \log(\log_{[k-1]}(t)), \qquad \text{where } \log_{[1]}(t) = \log(t)$$

and

$$\exp_{[k]}(t) = \exp(\exp_{[k-1]}(t)), \qquad \text{where } \exp_{[1]}(t) = \exp(t).$$

Let $\alpha < n-1$ and $k \in \mathbb{N}$, $k \geqslant 2$. We consider a Young function Φ satisfying

$$\lim_{t\to\infty} \frac{\Phi(t)}{t^n \left(\prod_{j=1}^{k-1} \log_{[j]}^{n-1}(t)\right) \log_{[k]}^{\alpha}(t)} = 1 \tag{3}$$

and the embedding into the Orlicz space with the Young function that behaves like $\exp_{[k]}(t^\gamma)$ for large t.

Theorem 1.1. *Let $k \in \mathbb{N}$, $k \geqslant 2$, and $\alpha < n-1$. Let Φ be a Young function satisfying* (3) *and let*

$$K > B^{-(n-1)/n}\omega_{n-1}^{-1/n}. \tag{4}$$

Suppose that $f \in WL_0^\Phi(\Omega)$ and $\int_\Omega \Phi(|\nabla f|)\,\mathrm{d}x \leqslant 1$. Then

$$\int_\Omega \exp_{[k]}\Bigl(\Bigl(\frac{|f(x)|}{K}\Bigr)^\gamma\Bigr)\,\mathrm{d}x < c$$

where c depends on n, k, α, $\mathcal{L}_n(\Omega)$, K and Φ only.

Theorem 1.2. *Let $k \in \mathbb{N}$, $k \geqslant 2$, and $\alpha < n-1$. Let Φ be a Young function satisfying* (3)*. Let $R > 0$, $m \in \mathbb{N}$ and*

$$K < B^{-(n-1)/n} \omega_{n-1}^{-1/n}.$$

Then there is a radial function $f\colon B(0,R) \to \mathbb{R}$ such that $f \in WL_0^{\Phi}(B(0,R))$ and $\int_{B(0,R)} \Phi(|\nabla f|) \leqslant 1$, but

$$\int_{B(0,R)} \exp_{[k]}\Big(\Big(\frac{|f(x)|}{K}\Big)^{\gamma}\Big)\, \mathrm{d}x > m.$$

Recall that both the theorems were proved by Hencl in [11] only for $k = 2$, and for $k \geqslant 3$ our results are new. It is also quite surprising that the value of the critical constant is always $K_0 = B^{-(n-1)/n}\omega_{n-1}^{-1/n}$ for any $k \geqslant 2$. Recall that $K_0 = B^{-(n-1)/n} n^{-1/\gamma} \omega_{n-1}^{-1/n}$ for $k = 1$ (see [11]).

The above theorems do not give any information in the borderline case

$$K_0 = B^{-(n-1)/n}\omega_{n-1}^{-1/n}. \tag{5}$$

In the last section we show that the condition (3) is not enough to guarantee anything in general. We prove that if we replace (3) by a suitable growth condition, then the statement of Theorem 1.2 is valid even for K given by (5). On the other hand, if we require an additional condition on the growth of Φ like (68) then we obtain the statement of Theorem 1.1 even in the case (5). These results are new even in the double exponential case $k = 2$.

2. Preliminaries

We denote by $\mathcal{L}_n$ the n-dimensional Lebesgue measure.

By $B(0,R)$ we denote an open Euclidean ball in $\mathbb{R}^n$ with its center at the origin and the radius $R > 0$.

For given functions g, h we say that $g(t) \gg h(t)$ for t big enough if we have $\lim\limits_{t\to\infty} g(t)/h(t) = \infty$. Analogously $g(t) \gg h(t)$ for t small enough if $\lim\limits_{t\to 0_+} g(t)/h(t) = \infty$.

A function $\Phi\colon \mathbb{R}^+ \to \mathbb{R}^+$ is a Young function if Φ is increasing, convex and satisfies $\Phi(0) = 0$.

Denote by $L^{\Phi}(A, \mathrm{d}\mu)$ the Orlicz space corresponding to a Young function Φ on a set A with a measure μ. This space is equipped with the norm

$$\|f\|_{L^{\Phi}} = \inf\Big\{\lambda > 0\colon \int_A \Phi\Big(\frac{|f(x)|}{\lambda}\Big)\, \mathrm{d}\mu(x) \leqslant \Phi(1)\Big\}. \tag{6}$$

Note that this is slightly different from the usual definition where the condition $\int_A \Phi\big(|f(x)|/\lambda\big)\,\mathrm{d}\mu(x) \leqslant \Phi(1)$ is replaced by $\int_A \Phi\big(|f(x)|/\lambda\big)\,\mathrm{d}\mu(x) \leqslant 1$. We use (6) to have the inequality (7) with a sharp constant.

Given a differentiable Young function Φ we can define the generalized inverse to $\varphi(u) = \Phi'(u)$ by

$$\psi(s) = \inf\{u\colon\ \varphi(u) > s\} \qquad \text{for } s > 0$$

and further define its associated Young function Ψ by

$$\Psi(t) = \int_0^t \psi(s)\,\mathrm{d}s \qquad \text{for } t \geqslant 0.$$

The dual space to L^Φ can be identified with the Orlicz space L^Ψ. If in addition we have $\Phi(1) + \Psi(1) = 1$ then the following generalization of the Hölder inequality is valid (see [14] page 58 for a proof):

$$\int_A |f(y)g(y)|\,\mathrm{d}\mu(y) \leqslant \|f\|_{L^\Phi(A,\,\mathrm{d}\mu)}\|g\|_{L^\Psi(A,\,\mathrm{d}\mu)}. \tag{7}$$

We use this inequality for a measurable subset $A \subset \mathbb{R}$ and the measure $\mathrm{d}\mu(y) = \omega_{n-1}y^{n-1}\,\mathrm{d}y$. For an introduction to Orlicz spaces see e.g. [14].

The non-increasing rearrangement f^* of a measurable function f on Ω is defined by

$$f^*(t) = \inf\big\{s > 0\colon\ \mathcal{L}_n(\{x \in \Omega\colon\ |f(x)| > s\}) \leqslant t\big\},\ t > 0.$$

We also define the non-increasing radially symmetric rearrangement $f^\#$ by

$$f^\#(x) = f^*\Big(\frac{\omega_{n-1}}{n}|x|^n\Big) \qquad \text{for } x \in B(0,R),\ \mathcal{L}_n(B(0,R)) = \mathcal{L}_n(\Omega).$$

For an introduction to these rearrangements see e.g. [16]. We need the fact that for every Young function Φ and for every measurable function $f\colon\ \Omega \to \mathbb{R}$ we have

$$\int_\Omega \Phi(|f(x)|)\,\mathrm{d}x = \int_{B(0,R)} \Phi(|f^\#(x)|)\,\mathrm{d}x = \int_0^{\mathcal{L}_n(\Omega)} \Phi(|f^*(y)|)\,\mathrm{d}y.$$

We also use the Polya-Szegö principle (see e.g. Talenti [16] for the proof).

Theorem 2.1. *Let Ω be an open bounded set and let $R > 0$ satisfy $\mathcal{L}_n(B(0,R)) = \mathcal{L}_n(\Omega)$. Let Φ be a Young function. Suppose that a function $f\colon \Omega \to \mathbb{R}$ is Lipschitz continuous, $\int_\Omega \Phi(|\nabla f|) < \infty$ and $f \in WL_0^\Phi(\Omega)$. Then f^* is locally absolutely continuous and*

$$\int_\Omega \Phi(|\nabla f(x)|)\,\mathrm{d}x \geqslant \int_{B(0,R)} \Phi(|\nabla f^{\#}(x)|)\,\mathrm{d}x.$$

We denote by C a generic positive constant which may depend on n, k, α, $\mathcal{L}_n(\Omega)$, K and Φ. This constant may vary from expression to expression. Some lemmata state that for every $\varepsilon > 0$ something is true. Then the constants C in the proof of such a lemma may depend also on a fixed $\varepsilon > 0$.

In the following lemma we show that the function $\log_{[k]}$ has similar asymptotical behaviour similar to log.

Lemma 2.2. *Let $t_1, p, q, \delta, E, L > 0$ and $k \in \mathbb{N}$ and let functions $f, h\colon \mathbb{R} \mapsto (0,\infty)$ and $g\colon \mathbb{R} \mapsto \mathbb{R}$ satisfy*

$$g(t) + Ef(t) > \exp_{[k]}(0) \qquad \text{and} \qquad Eh^q(t)f^p(t) > \exp_{[k]}(0) \qquad \text{on } (t_1,\infty),$$

$$\lim_{t\to\infty} f(t) = \infty, \quad \frac{g(t)}{f(t)} \in [-E+\delta, L] \quad \text{and} \quad \frac{\log(h(t))}{\log(f(t))} \in \Big[-\frac{p}{q} + \delta, L\Big] \quad \text{on } (t_1,\infty).$$

Then there is $t_0 > t_1$ such that if $t > t_0$ then

$$(8) \quad 1 - \frac{C}{\log_{[k]}\big(f(t)\big)} < \frac{\log_{[j]}\big(g(t) + Ef(t)\big)}{\log_{[j]}\big(f(t)\big)} < 1 + \frac{C}{\log_{[k]}\big(f(t)\big)} \quad \text{for } j \in \{1,\dots,k\}$$

and

$$(9) \quad 1 - \frac{C}{\log_{[k]}\big(f(t)\big)} < \frac{\log_{[j]}\big(Eh^q(t)f^p(t)\big)}{\log_{[j]}\big(f(t)\big)} < 1 + \frac{C}{\log_{[k]}\big(f(t)\big)} \quad \text{for } j \in \{2,\dots,k\}.$$

Proof. The proof is based on induction with respect to j. Let us prove (8). For $j = 1$ we have

$$\frac{\log\big(g(t) + Ef(t)\big)}{\log\big(f(t)\big)} = 1 + \frac{\log\big(g(t)/f(t) + E\big)}{\log\big(f(t)\big)}.$$

Since $g(t)/f(t) + E$ is bounded and bounded away from zero on (t_1,∞), the estimate follows.

Suppose that $j \in \{2, \ldots, k\}$ and that we have proved (8) for $(j-1)$. Increasing $t_0 > t_1$ eventually we obtain

$$\begin{aligned}
\frac{\log_{[j]}\bigl(g(t) + Ef(t)\bigr)}{\log_{[j]}\bigl(f(t)\bigr)} &= \frac{\log\bigl(\log_{[j-1]}(g(t) + Ef(t))\bigr)}{\log_{[j]}\bigl(f(t)\bigr)} \\
&\leqslant \frac{\log\bigl((1 + C/\log_{[k]}(f(t)))\log_{[j-1]}\bigl(f(t)\bigr)\bigr)}{\log_{[j]}\bigl(f(t)\bigr)} \\
&\leqslant 1 + \frac{\log\bigl(1 + C/\log_{[k]}(f(t))\bigr)}{\log_{[j]}\bigl(f(t)\bigr)} \\
&\leqslant 1 + \frac{C}{\log_{[k]}\bigl(f(t)\bigr)}.
\end{aligned}$$

The estimate from below is obtained in the same way.

Let us prove (9). For $j = 2$ the estimate follows from the assumptions of the lemma and

$$\begin{aligned}
\frac{\log_{[2]}\bigl(Eh^q(t)f^p(t)\bigr)}{\log_{[2]}\bigl(f(t)\bigr)} &= \frac{\log\bigl(\log(E) + q\log(h(t)) + p\log(f(t))\bigr)}{\log_{[2]}\bigl(f(t)\bigr)} \\
&= 1 + \frac{\log\bigl(\log(E)/\log(f(t)) + q\log(h(t))/\log(f(t)) + p\bigr)}{\log_{[2]}\bigl(f(t)\bigr)}.
\end{aligned}$$

Let $j \in \{3, \ldots, k\}$ and suppose we have proved (9) for $(j-1)$. We obtain

$$\begin{aligned}
\frac{\log_{[j]}\bigl(Eh^q(t)f^p(t)\bigr)}{\log_{[j]}\bigl(f(t)\bigr)} &= \frac{\log\bigl(\log_{[j-1]}\bigl(Eh^q(t)f^p(t)\bigr)\bigr)}{\log_{[j]}\bigl(f(t)\bigr)} \\
&\leqslant \frac{\log\bigl((1 + C/\log_{[k]} f(t))\log_{[j-1]} f(t)\bigr)}{\log_{[j]}\bigl(f(t)\bigr)} \\
&\leqslant 1 + \log\Bigl(1 + \frac{C}{\log_{[k]}(t)}\Bigr)/\log_{[j]}\bigl(f(t)\bigr) \\
&\leqslant 1 + \frac{C}{\log_{[k]}\bigl(f(t)\bigr)}.
\end{aligned}$$

The estimate from below is obtained in the same way. $\square$

3. Embedding into multiple exponential spaces

3.1. Lower estimate.

Proof of Theorem 1.2. As $K < B^{-(n-1)/n}\omega_{n-1}^{-1/n}$ there is $\varepsilon > 0$ such that $A > K(1+\varepsilon)$, where

$$A = \Big(\frac{\omega_{n-1}^{-1} - 2\varepsilon}{(1+\varepsilon)^3}\frac{1}{B^{n-1}}\Big)^{1/n}. \tag{10}$$

Fix $T > \exp_{[k]}(1)$. For $s > T$ set $f_s(x) = g_s(|x|)$ where

$$g_s(y) = \begin{cases} (-2R^{-1}y+2)A\log_{[k]}^{B}(T+2)s^{1/\gamma-B} & \text{for } y \in [\frac12 R, R],\\ A\log_{[k]}^{B}(T+R/y)s^{1/\gamma-B} & \text{for } y \in [R\exp_{[k]}^{-1/n}(s), \frac12 R],\\ A\log_{[k]}^{B}(T+\exp_{[k]}^{1/n}(s))s^{1/\gamma-B} & \text{for } y \in [0, R\exp_{[k]}^{-1/n}(s)]. \end{cases}$$

As $|(\log(T+R/y))'| = (T+R/y)^{-1}R/y^2 = (R+Ty)^{-1}R/y$, for $y \in [R\exp_{[k]}^{-1/n}(s), \frac12 R]$ we have

$$|g_s'(y)| = AB\log_{[k]}^{B-1}\Big(T+\frac{R}{y}\Big)\Big(\prod_{j=1}^{k-1}\log_{[j]}^{-1}\Big(T+\frac{R}{y}\Big)\Big)\frac{1}{R+Ty}\frac{R}{y}s^{1/\gamma-B}. \tag{11}$$

Put $M = M(s) = 1/s^{\log(s)}$. Plainly there is $s_1 > T$ such that for $s > s_1$ we have

$$R\exp_{[k]}^{-1/n}(s) < M < \frac{R}{T}. \tag{12}$$

Therefore

$$\int_0^R \Phi(|g_s'(y)|)y^{n-1}\,\mathrm{d}y = \int_{R\exp_{[k]}^{-\frac1n}(s)}^{M} + \int_M^{\frac{R}{T}} + \int_{\frac{R}{T}}^{R} = I_1 + I_2 + I_3. \tag{13}$$

Using (3) we can see that there is $E > 0$ large enough such that

$$\Phi(t) \leqslant (1+\varepsilon)t^n\Big(\prod_{j=1}^{k-1}\log_{[j]}^{n-1}(t)\Big)\log_{[k]}^{\alpha}(t) \qquad \text{for } t \in [E,\infty). \tag{14}$$

Further, as Φ is a Young function, hence increasing, convex and satisfying $\Phi(0) = 0$, we also have

$$\Phi(t) \leqslant t\frac{\Phi(E)}{E} = Ct \qquad \text{for } t \in [0, E]. \tag{15}$$

Estimates (14) and (15) give

$$\Phi(t) \leqslant Ct + Ct^n |\log(t)|^n \qquad \text{for } t \in [0, \infty). \tag{16}$$

Clearly $|g_s'(y)| = Cs^{1/\gamma - B}$ for $y \in (R/2, R)$. Similarly, $|g_s'(y)| \leqslant Cs^{1/\gamma-B}$ for $y \in (R/T, R/2)$ by (11). As $1/\gamma - B < 0$ for every $\alpha < n-1$ by (2), for s large enough we can apply (15) to obtain

$$I_3 \leqslant C \int_{\frac{R}{T}}^{R} |g_s'(y)| y^{n-1}\,\mathrm{d}y \leqslant Cs^{1/\gamma - B} \underset{s\to\infty}{\longrightarrow} 0.$$

Thus there is $s_2 > s_1$ such that for all $s > s_2$ we have

$$I_3 < \varepsilon. \tag{17}$$

Plainly, (11) for $y \in (M, R/T)$ implies

$$\begin{aligned} |\log(|g_s'(y)|)|^n &\leqslant C\Big(\log^n(s) + \log^n\Big(\frac{R}{y}\Big)\Big) \\ &\leqslant C\Big(\log^n(s) + \log^n\Big(\frac{R}{M}\Big)\Big) \leqslant C\log^{2n}(s). \end{aligned} \tag{18}$$

Further, for $y \in (M, R/T)$ we have $R/y \leqslant T + R/y \leqslant 2R/y$ and thus the following inequalities are satisfied:

$$\begin{aligned} &\log_{[j]}^{-1}\Big(T + \frac{R}{y}\Big) \leqslant \log_{[j]}^{-1}\Big(\frac{R}{y}\Big) \qquad \text{for } j \in \{1, \dots, k-1\}, \\ &\log_{[k]}^{B-1}\Big(T + \frac{R}{y}\Big) \leqslant C\log_{[k]}^{B-1}\Big(\frac{R}{y}\Big). \end{aligned} \tag{19}$$

Therefore from (11), (16), (18) and (19) we have

$$\begin{aligned} I_2 &\leqslant C \int_M^{\frac{R}{T}} \Big(|g_s'(y)| + |g_s'(y)|^n \big|\log(|g_s'(y)|)\big|^n\Big) y^{n-1}\,\mathrm{d}y \\ &\leqslant Cs^{1/\gamma - B} \int_M^{\frac{R}{T}} \log_{[k]}^{B-1}\Big(\frac{R}{y}\Big)\Big(\prod_{j=1}^{k-1}\log_{[j]}^{-1}\Big(\frac{R}{y}\Big)\Big)\frac{\mathrm{d}y}{y} \\ &\quad + Cs^{(\frac{1}{\gamma}-B)n} \int_M^{\frac{R}{T}} \log_{[k]}^{(B-1)n}\Big(\frac{R}{y}\Big)\Big(\prod_{j=1}^{k-1}\log_{[j]}^{-n}\Big(\frac{R}{y}\Big)\Big)\log^{2n}(s)\frac{\mathrm{d}y}{y} \\ &= J_0 + J_1, \end{aligned}$$

where (recall that $B > 0$ by (2))

$$\begin{aligned} J_0 &= Cs^{1/\gamma - B} \int_M^{\frac{R}{T}} \log_{[k]}^{B-1}\Big(\frac{R}{y}\Big)\Big(\prod_{j=1}^{k-1} \log_{[j]}^{-1}\Big(\frac{R}{y}\Big)\Big)\frac{\mathrm{d}y}{y} \\ &= Cs^{1/\gamma - B}\frac{1}{B}\Big[-\log_{[k]}^{B}\Big(\frac{R}{y}\Big)\Big]_M^{\frac{R}{T}} \leqslant Cs^{1/\gamma - B}\Big(1 + \log_{[k]}^{B}\Big(\frac{R}{M}\Big)\Big) \end{aligned}$$

and

$$\begin{aligned} J_1 &= Cs^{(1/\gamma - B)n}\log^{2n}(s) \int_M^{\frac{R}{T}} \log_{[k]}^{(B-1)n}\Big(\frac{R}{y}\Big)\Big(\prod_{j=1}^{k-1} \log_{[j]}^{-n}\Big(\frac{R}{y}\Big)\Big)\frac{\mathrm{d}y}{y} \\ &\leqslant Cs^{(\frac{1}{\gamma} - B)n}\log^{2n}(s) \int_M^{\frac{R}{T}} \log_{[k]}^{(B-1)n}\Big(\frac{R}{y}\Big)\Big(\prod_{j=1}^{k-1} \log_{[j]}^{-1}\Big(\frac{R}{y}\Big)\Big)\frac{\mathrm{d}y}{y}. \end{aligned}$$

Now, if $(B-1)n \neq -1$, then

$$\begin{aligned} \int_M^{\frac{R}{T}} \log_{[k]}^{(B-1)n}\Big(\frac{R}{y}\Big)\Big(\prod_{j=1}^{k-1} \log_{[j]}^{-1}\Big(\frac{R}{y}\Big)\Big)\frac{\mathrm{d}y}{y} &= \frac{1}{(B-1)n+1}\Big[-\log_{[k]}^{(B-1)n+1}\Big(\frac{R}{y}\Big)\Big]_M^{\frac{R}{T}} \\ &\leqslant C + C\log_{[k]}^{(B-1)n+1}\Big(\frac{R}{M}\Big) \end{aligned}$$

and if $(B-1)n = -1$, then

$$\begin{aligned} \int_M^{\frac{R}{T}} \log_{[k]}^{(B-1)n}\Big(\frac{R}{y}\Big)\Big(\prod_{j=1}^{k-1} \log_{[j]}^{-1}\Big(\frac{R}{y}\Big)\Big)\frac{\mathrm{d}y}{y} &= \Big[-\log_{[k+1]}\Big(\frac{R}{y}\Big)\Big]_M^{\frac{R}{T}} \\ &\leqslant C + C\log_{[k+1]}\Big(\frac{R}{M}\Big). \end{aligned}$$

Consequently, as $M = 1/s^{\log(s)}$ and $(1/\gamma - B)n = -B < 0$ by (2) in both cases we obtain

$$J_0 + J_1 \underset{s\to\infty}{\rightarrow} 0.$$

Hence there is $s_3 > s_2$ such that for all $s > s_3$ we have

$$I_2 < \varepsilon. \tag{20}$$

As $M \ll s^{1/\gamma - B} \ll 1$, by (11) there is $s_4 > s_3$ such that for all $s > s_4$ we have

$$y \in [R\exp_{[k]}^{-1/n}(s), M] \Rightarrow |g_s'(y)| > E.$$

Hence we obtain from (14)

$$(21)\quad I_1 \leqslant (1+\varepsilon)\int_{R\exp_{[k]}^{-1/n}(s)}^{M} |g_s'(y)|^n \Biggl(\prod_{j=1}^{k-1} \log_{[j]}^{n-1}(|g_s'(y)|)\Biggr) \log_{[j]}^{\alpha}(|g_s'(y)|) y^{n-1}\,\mathrm{d}y.$$

Further, for $y \in [R\exp_{[k]}^{-1/n}(s), M]$ we trivially have

$$(22)\qquad \frac{1}{R+Ty}\frac{R}{y} \leqslant \frac{1}{y},$$

$$(23)\qquad \log_{[j]}^{-n}\Bigl(T+\frac{R}{y}\Bigr) \leqslant \log_{[j]}^{-n}\Bigl(\frac{R}{y}\Bigr) \qquad \text{for } j \in \{1,\dots,k-1\}.$$

We can find $s_5 > s_4$ such that for all $s > s_5$ estimate (8) from Lemma 2.2 implies

$$(24)\quad \log_{[k]}^{(B-1)n}\Bigl(T+\frac{R}{y}\Bigr) \leqslant (1+\varepsilon)\log_{[k]}^{(B-1)n}\Bigl(\frac{R}{y}\Bigr) \qquad \text{for } y \in [R\exp_{[k]}^{-1/n}(s), M].$$

Since $1/\gamma - B < 0$ we can find $s_6 > s_5$ such that for all $s > s_6$ we have from (11)

$$y \in [R\exp_{[k]}^{-1/n}(s), M] \Rightarrow |g_s'(y)| < \frac{R}{y}$$

and thus for $y \in [R\exp_{[k]}^{-1/n}(s), M]$ we have

$$(25)\qquad \log_{[j]}^{n-1}(|g_s'(y)|) \leqslant \log_{[j]}^{n-1}\Bigl(\frac{R}{y}\Bigr) \qquad \text{for } j \in \{1,\dots,k-1\}.$$

By estimate (9) in Lemma 2.2 and (11) there is $s_7 > s_6$ such that for every $s > s_7$ and for every $y \in [R\exp_{[k]}^{-1/n}(s), M]$ we obtain

$$(26)\qquad \log_{[k]}^{\alpha}(|g_s'(y)|) \leqslant (1+\varepsilon)\log_{[k]}^{\alpha}\Bigl(\frac{R}{y}\Bigr).$$

Therefore we go on estimating I_1 and from (11), (21), (22), (23), (24), (25), (26) and from $(B-1)n+\alpha = B-1 \neq -1$ we obtain

$$\begin{aligned} I_1 &\leqslant s^{(1/\gamma-B)n}(1+\varepsilon)^3 A^n B^n \int_{R\exp_{[k]}^{-1/n}(s)}^{M} \log_{[k]}^{(B-1)n+\alpha}\Bigl(\frac{R}{y}\Bigr)\Biggl(\prod_{j=1}^{k-1}\log_{[j]}^{-1}\Bigl(\frac{R}{y}\Bigr)\Biggr)\frac{\mathrm{d}y}{y} \\ &\leqslant (1+\varepsilon)^3 A^n B^n \frac{1}{(B-1)n+\alpha+1} s^{(1/\gamma-B)n}\Bigl[-\log_{[k]}^{(B-1)n+\alpha+1}\Bigl(\frac{R}{y}\Bigr)\Bigr]_{R\exp_{[k]}^{-1/n}(s)}^{M}. \end{aligned}$$

An easy computation gives $(B-1)n+\alpha+1=B=-(1/\gamma-B)n$. Further, there is $s_8>s_7$ such that $\log_{[k]}(R/M)>0$ for all $s>s_8$. Hence for all $s>s_8$ we obtain, thanks to (10), that

$$I_1 \leqslant (1+\varepsilon)^3 A^n B^{n-1} s^{-B} \log_{[k]}^B(\exp_{[k]}^{1/n}(s)) \leqslant (1+\varepsilon)^3 A^n B^{n-1} = \frac{1}{\omega_{n-1}} - 2\varepsilon. \tag{27}$$

From (13), (17), (20) and (27) it follows that for $s>s_8$ we have

$$\int_{B(0,R)} \Phi(|\nabla f_s|) = \omega_{n-1}\int_0^R \Phi(|g_s'(y)|)y^{n-1}\,\mathrm{d}y = \omega_{n-1}(I_1+I_2+I_3) \leqslant 1.$$

By estimate (9) from Lemma 2.2 there is $s_9>s_8$ such that if $s>s_9$ we observe

$$y\in[0,R\exp_{[k]}^{-1/n}(s)] \Rightarrow g_s(y) = A\log_{[k]}^B(T+\exp_{[k]}^{1/n}(s))s^{1/\gamma-B} \geqslant \frac{As^{1/\gamma}}{1+\varepsilon}.$$

This and $A>K(1+\varepsilon)$ give

$$\begin{aligned}\int_{B(0,R)} \exp_{[k]}\Bigl(\Bigl(\frac{|f_s(x)|}{K}\Bigr)^{\gamma}\Bigr) &\geqslant \int_{B\left(0,R\exp_{[k]}^{-1/n}(s)\right)} \exp_{[k]}\Bigl(\Bigl(\frac{|f_s(x)|}{K}\Bigr)^{\gamma}\Bigr)\,\mathrm{d}x\\ &\geqslant C\exp_{[k]}^{-1}(s)\exp_{[k]}\Bigl(\Bigl(\frac{A}{(1+\varepsilon)K}\Bigr)^{\gamma}s\Bigr) \underset{s\to\infty}{\to} \infty.\end{aligned}$$

□

3.2. Upper estimate.

Suppose that the function $\Phi\colon \mathbb{R}^+\to\mathbb{R}^+$ satisfies (3). It is not difficult to show that there is a function $\Phi_1\colon \mathbb{R}^+\to\mathbb{R}^+$ such that

$$\begin{gathered}\Phi_1 \text{ is a Young function,}\\ \Phi_1' \text{ is continuous and increasing on } (0,\infty),\\ \Phi_1(t)=\frac{1}{n}t^n \text{ for } t\in[0,1],\\ \lim_{t\to\infty}\frac{\Phi_1(t)}{\frac{1}{n}t^n\Bigl(\prod_{j=1}^{k-1}\log_{[j]}^{n-1}(t)\Bigr)\log_{[k]}^{\alpha}(t)}=1,\\ \lim_{t\to\infty}\frac{\Phi_1'(t)}{t^{n-1}\Bigl(\prod_{j=1}^{k-1}\log_{[j]}^{n-1}(t)\Bigr)\log_{[k]}^{\alpha}(t)}=1\\ \text{and there is } G>\exp_{[k]}(1) \text{ such that } \Phi_1(t)\leqslant\frac{1}{n}\Phi(t) \text{ for } t\in[G,\infty).\end{gathered} \tag{28}$$

Denote by Ψ the Young function associated with the function Φ_1. Easy computation gives $\Psi(t) = t^{n'}/n'$ for $t \in [0,1]$. Thus $\Phi_1(1) + \Psi(1) = 1$ and therefore (Φ_1, Ψ) is a normalized complementary Young pair and we can use inequality (7).

We first estimate the growth of Ψ.

Lemma 3.1. *There is $E > 0$ such that for every $t \in \mathbb{R}$ we have*

$$\Psi(t) < \hat{\Psi}(t) := Et^{n/(n-1)}\big(1 + |\log(t)|^E\big). \tag{29}$$

Moreover, for every $\varepsilon > 0$ there is $A > 0$ such that if $t \in [A, \infty)$ then

$$\Psi(t) \leqslant \tilde{\Psi}(t) := \Big(\frac{(n-1)^2}{n} + \varepsilon\Big) t^{n/(n-1)} \left(\prod_{j=1}^{k-1} \log_{[j]}^{-1}(t)\right) \log_{[k]}^{-\alpha/(n-1)}(t). \tag{30}$$

P r o o f. Set $\varphi = \Phi_1'$ and $\psi = \varphi^{-1}$, hence $\Psi(t) = \int_0^t \psi$. Given $\varepsilon > 0$ we can find $\delta \in (0, \frac{1}{2})$ such that

$$\frac{(n-1)^2}{n}(1+\delta)^2 \frac{1}{(1-\delta)^{(k+1)/(n-1)}} < \frac{(n-1)^2}{n} + \varepsilon. \tag{31}$$

Assumptions (28) give that there is $A_1 > \exp_{[k]}(1)$ such that if $t > A_1$ we have

$$\varphi(t) \geqslant \tilde{\varphi}(t) := (1-\delta) t^{n-1} \left(\prod_{j=1}^{k-1} \log_{[j]}^{n-1}(t)\right) \log_{[k]}^{\alpha}(t). \tag{32}$$

Set $P = n - 1/(1-\delta)^{(k+1)/(n-1)}$ and

$$\tilde{\psi}(t) = P t^{1/(n-1)} \left(\prod_{j=1}^{k-1} \log_{[j]}^{-1}(t)\right) \log_{[k]}^{-\alpha/(n-1)}(t). \tag{33}$$

Plainly there is $A_2 > A_1$ such that for $t > A_2$ we have

$$\begin{aligned} &\frac{\log^{n-1}\big(\tilde{\psi}(t)\big)}{\log^{n-1}(t)} \\ &= \Big(\frac{1}{n-1}\Big)^{n-1} \left(1 + \frac{\log\Big(P^{n-1}\Big(\prod_{j=1}^{k-1} \log_{[j]}^{-(n-1)}(t)\Big) \log_{[k]}^{-\alpha}(t)\Big)}{\log t}\right)^{n-1} \\ &\geqslant \Big(\frac{1}{n-1}\Big)^{n-1} (1-\delta). \end{aligned} \tag{34}$$

Further, by (9) from Lemma 2.2 and (33), there is $A_3 > A_2$ such that for every $t > A_3$ we have

$$\log_{[j]}^{n-1}\big(\tilde{\psi}(t)\big) \geqslant (1-\delta)\log_{[j]}^{n-1}(t) \qquad \text{provided } j \in \{2,\dots,k-1\} \tag{35}$$

and

$$\log_{[k]}^{\alpha}\big(\tilde{\psi}(t)\big) \geqslant (1-\delta)\log_{[k]}^{\alpha}(t). \tag{36}$$

Hence if $t > A_3$, then (32), (33), (34), (35) and (36) imply

$$\begin{aligned}
\tilde{\varphi}(\tilde{\psi}(t)) &\geqslant (1-\delta)\tilde{\psi}^{n-1}(t)\Big(\prod_{j=1}^{k-1}\log_{[j]}^{n-1}(\tilde{\psi}(t))\Big)\log_{[k]}^{\alpha}(\tilde{\psi}(t)) \\
&\geqslant (1-\delta)^{k+1}\tilde{\psi}^{n-1}(t)\Big(\frac{1}{n-1}\Big)^{n-1}\Big(\prod_{j=1}^{k-1}\log_{[j]}^{n-1}(t)\Big)\log_{[k]}^{\alpha}(t) \\
&\geqslant (1-\delta)^{k+1}P^{n-1}\Big(\frac{1}{n-1}\Big)^{n-1} t = t.
\end{aligned}$$

This estimate and (32) give for $t > A_3$

$$\tilde{\psi}(t) \geqslant \tilde{\varphi}^{-1}(t) \geqslant \varphi^{-1}(t) = \psi(t). \tag{37}$$

Denote

$$\tilde{\Psi}_1(t) = (1+\delta)^2 P\frac{t^{n/(n-1)}}{n/(n-1)}\Big(\prod_{j=1}^{k-1}\log_{[j]}^{-1}(t)\Big)\log_{[k]}^{-\alpha/(n-1)}(t).$$

Hence

$$\tilde{\Psi}_1'(t) = (1+\delta)^2 P t^{1/(n-1)}\Big(\prod_{j=1}^{k-1}\log_{[j]}^{-1}(t)\Big)\log_{[k]}^{-\alpha/(n-1)}(t)\Theta(t),$$

where

$$\Theta(t) = 1 - \sum_{j=1}^{k-1}\frac{n-1}{n}\frac{1}{\prod\limits_{i=1}^{j}\log_{[i]}(t)} - \frac{\alpha}{n}\frac{1}{\prod\limits_{i=1}^{k}\log_{[i]}(t)}.$$

Therefore from (33) and (37) we obtain that there is $A_4 > A_3$ such that for every $t > A_4$ we have

$$\tilde{\Psi}_1'(t) > (1+\delta)\tilde{\psi}(t) \geqslant (1+\delta)\psi(t).$$

Thus there is $A > A_4$ such that for $t > A$ we have $\Psi(t) < \tilde{\Psi}_1(t)$ and (31) gives

$$\Psi(t) < \tilde{\Psi}_1(t) < \tilde{\Psi}(t) := \Big(\frac{(n-1)^2}{n} + \varepsilon\Big)t^{n/(n-1)}\Big(\prod_{j=1}^{k-1}\log_{[j]}^{-1}(t)\Big)\log_{[k]}^{-\alpha/(n-1)}(t).$$

As Ψ is increasing and $\Psi(t) = t^{n'}/n'$ for $t \in [0,1]$, estimate (30) obviously implies (29). □

In the proof of Theorem 1.1 we use the generalized Hölder inequality (7) and thus we need to estimate the term $\|1/y^{n-1}\|_{L^{\Psi}((t,R),\omega_{n-1}y^{n-1}\,\mathrm{d}y)}$.

Lemma 3.2. *For every $\varepsilon_1 > 0$ there is $t_0 \in (0,1)$ such that if $0 < t < t_0$ then*

$$\Big\|\frac{1}{y^{n-1}}\Big\|_{L^{\Psi}((t,R),\omega_{n-1}y^{n-1}\,\mathrm{d}y)} \leqslant D\log_{[k]}^{1/\gamma}\Big(\frac{1}{t}\Big), \tag{38}$$

$$\text{where } \; D^{n/(n-1)} = \frac{\omega_{n-1}}{B} + \varepsilon_1 = \frac{\omega_{n-1}}{1-\frac{\alpha}{n-1}} + \varepsilon_1.$$

Proof. For $t \in \big(0, \exp_{[k]}^{-1}(1)\big)$ we set $\lambda = D\log_{[k]}^{1/\gamma}(1/t)$. Given $\varepsilon_1 > 0$ we can find $\varepsilon > 0$ such that

$$D^{n/(n-1)} = \frac{\omega_{n-1}}{1-\frac{\alpha}{n-1}} + \varepsilon_1 > \frac{(1+\varepsilon)^k\big(\frac{(n-1)^2}{n}+\varepsilon\big)}{(n-1)(1-\frac{\alpha}{n-1})(\frac{n-1}{\omega_{n-1}n}-\varepsilon)}. \tag{39}$$

For this ε we apply Lemma 3.1. From now on ε and A are fixed. Put

$$M = M(t) = \exp\Big(-\log_{[k]}^{\frac{1}{4(E+1)\gamma}}\Big(\frac{1}{t}\Big)\Big).$$

Since $1/t \gg 1/M^{n-1} \gg \lambda$ there is $t_1 \in \big(0, \exp_{[k]}^{-1}(1)\big)$ such that for $0 < t < t_1$ we have

$$t < M < R \quad \text{and} \quad \frac{1}{\lambda M^{n-1}} > A \tag{40}$$

where A comes from condition (30). Therefore from Lemma 3.1 we have

$$\begin{aligned} &\int_t^R \Psi\Big(\frac{1}{\lambda y^{n-1}}\Big)y^{n-1}\,\mathrm{d}y \\ &\leqslant \int_t^M \tilde{\Psi}\Big(\frac{1}{\lambda y^{n-1}}\Big)y^{n-1}\,\mathrm{d}y + \int_M^R \hat{\Psi}\Big(\frac{1}{\lambda y^{n-1}}\Big)y^{n-1}\,\mathrm{d}y = I_1 + I_2. \end{aligned} \tag{41}$$

Clearly

$$\begin{aligned} I_2 &\leqslant E\int_M^R \frac{1}{\lambda^{n/(n-1)}}\Big(1+\Big|\log\Big(\frac{1}{\lambda y^{n-1}}\Big)\Big|^E\Big)\frac{\mathrm{d}y}{y} \\ &\leqslant \frac{C}{\lambda^{n/(n-1)}}\int_M^R\Big(1+|\log(\lambda)|^E+|\log(y)|^E\Big)\frac{\mathrm{d}y}{y} = J_1 + J_2, \end{aligned}$$

where

$$\begin{aligned} J_1 &= \frac{C}{\lambda^{n/(n-1)}}\int_M^R\Big(1+|\log(\lambda)|^E\Big)\frac{\mathrm{d}y}{y} \\ &\leqslant C\frac{1}{\log_{[k]}^{n/(n-1)\gamma}(\frac{1}{t})}\Big(1+\log^E\Big(\log_{[k]}\Big(\frac{1}{t}\Big)\Big)\Big)\Big(1+\log\Big(\frac{1}{M}\Big)\Big) \\ &\leqslant C\frac{\log_{[k+1]}^E(\frac{1}{t})\log_{[k]}^{1/(4(E+1)\gamma)}(\frac{1}{t})}{\log_{[k]}^{n/(n-1)\gamma}(\frac{1}{t})} \underset{t\to0_+}{\to} 0 \end{aligned}$$

and

$$J_2 = \frac{C}{\lambda^{n/(n-1)}} \int_M^R |\log(y)|^E \frac{\mathrm{d}y}{y} \leqslant \frac{C}{\log_{[k]}^{n/(n-1)\gamma}(\frac{1}{t})} \Big(1 + \log^{E+1}\Big(\frac{1}{M}\Big)\Big)$$
$$\leqslant \frac{C + C \log_{[k]}^{1/(4\gamma)}(\frac{1}{t})}{\log_{[k]}^{n/(n-1)\gamma}(\frac{1}{t})} \underset{t\to 0_+}{\longrightarrow} 0.$$

Hence there is $t_2 \in (0, t_1)$ such that if $0 < t < t_2$ then

$$I_2 < \varepsilon. \tag{42}$$

Since $\log(1/M^{n-1}) \gg \log(\lambda) > 1$ for small t, we can find $t_3 \in (0, t_2)$ such that for all $0 < t < t_3$ and for $y \in [t, M]$ we have

$$\log^{-1}\Big(\frac{1}{\lambda y^{n-1}}\Big) < (1+\varepsilon)\log^{-1}\Big(\frac{1}{y^{n-1}}\Big) = \frac{1+\varepsilon}{n-1}\log^{-1}\Big(\frac{1}{y}\Big). \tag{43}$$

Moreover, by (9) from Lemma 2.2 we can find $t_4 \in (0, t_3)$ such that for every $0 < t < t_4$ and every $y \in [t, M]$ we obtain

$$\log_{[j]}^{-1}\Big(\frac{1}{\lambda y^{n-1}}\Big) < (1+\varepsilon)\log_{[j]}^{-1}\Big(\frac{1}{y}\Big) \qquad \text{for } j \in \{2, \ldots, k-1\} \tag{44}$$

and

$$\log_{[k]}^{-\alpha/(n-1)}\Big(\frac{1}{\lambda y^{n-1}}\Big) < (1+\varepsilon)\log_{[k]}^{-\alpha/(n-1)}\Big(\frac{1}{y}\Big). \tag{45}$$

Therefore (30), (43), (44) and (45) imply that for $0 < t < t_4$ we have

$$I_1 \leqslant \int_t^M \Big(\frac{(n-1)^2}{n} + \varepsilon\Big)\Big(\frac{1}{\lambda y^{n-1}}\Big)^{n/(n-1)}$$
$$\Big(\prod_{j=1}^{k-1} \log_{[j]}^{-1}\Big(\frac{1}{\lambda y^{n-1}}\Big)\Big) \log_{[k]}^{-\alpha/(n-1)}\Big(\frac{1}{\lambda y^{n-1}}\Big) y^{n-1}\,\mathrm{d}y$$
$$\leqslant \frac{(1+\varepsilon)^k}{n-1}\Big(\frac{(n-1)^2}{n} + \varepsilon\Big)\frac{1}{\lambda^{n/(n-1)}}$$
$$\int_t^M \log^{-1}\Big(\frac{1}{y}\Big)\Big(\prod_{j=2}^{k-1}\log_{[j]}^{-1}\Big(\frac{1}{y}\Big)\Big)\log_{[k]}^{-\alpha/(n-1)}\Big(\frac{1}{y}\Big)\frac{\mathrm{d}y}{y}$$
$$\leqslant \frac{(1+\varepsilon)^k\big(\frac{(n-1)^2}{n} + \varepsilon\big)}{(n-1)D^{n/(n-1)}\log_{[k]}^{n/\gamma(n-1)}(\frac{1}{t})}\Bigg[-\frac{\log_{[k]}^{1-\alpha/(n-1)}(\frac{1}{y})}{1 - \frac{\alpha}{n-1}}\Bigg]_t^M.$$

As $\log_{[k]}^{1-\alpha/(n-1)}(1/M) > 0$, using (2) and (39) we obtain

(46)

$$I_1 \leqslant \frac{(1+\varepsilon)^k\big(\frac{(n-1)^2}{n}+\varepsilon\big)}{(n-1)D^{n/(n-1)}\log_{[k]}^{n/\gamma(n-1)}(\frac{1}{t})}\frac{\log_{[k]}^{1-\alpha/(n-1)}(\frac{1}{t})}{1-\frac{\alpha}{n-1}} \leqslant \Big(\frac{n-1}{\omega_{n-1}n}-\varepsilon\Big)\frac{\log_{[k]}^{B}(\frac{1}{t})}{\log_{[k]}^{B}(\frac{1}{t})}$$
$$\leqslant \frac{n-1}{\omega_{n-1}n}-\varepsilon.$$

From (41), (42) and (46) we obtain that for $0<t<t_0=t_4$ we have

$$\int_t^R \Psi\Big(\frac{1}{\lambda y^{n-1}}\Big)y^{n-1}\,\mathrm{d}y \leqslant I_1+I_2 \leqslant \frac{n-1}{\omega_{n-1}n} = \frac{1}{\omega_{n-1}}\Psi(1).$$

□

P r o o f of Theorem 1.1. Since $C_0^\infty(\Omega)$ functions are dense in $WL_0^\Phi(\Omega)$ (see [6]), we can suppose without loss of generality that f is Lipschitz continuous. Find $R>0$ such that $\mathcal{L}_n(\Omega)=\mathcal{L}_n(B(0,R))$. From the basic properties of radially symmetric rearrangements we obtain

$$\int_\Omega \exp_{[k]}\Big(\Big(\frac{|f(x)|}{K}\Big)^\gamma\Big)\,\mathrm{d}x = \int_{B(0,R)} \exp_{[k]}\Big(\Big(\frac{f^\#(x)}{K}\Big)^\gamma\Big)\,\mathrm{d}x$$

and the Polya-Szegö principle (Theorem 2.1) gives

$$\int_{B(0,R)} \Phi\big(|\nabla f^\#(x)|\big)\,\mathrm{d}x \leqslant \int_\Omega \Phi\big(|\nabla f(x)|\big)\,\mathrm{d}x \leqslant 1.$$

Hence we can suppose without loss of generality that $f(x)=g(|x|)$, g is non-increasing, classically differentiable almost everywhere and, moreover, $\Omega=B(0,R)$. Since $f\in WL_0^\Phi(\Omega)$ we have $g(R)=0$.

Thanks to (4) and (38) we can find $\varepsilon_1>0$ and $\eta>0$ small enough such that

$$K > (1+\eta)D\frac{1}{\omega_{n-1}}. \tag{47}$$

Put $\mathrm{d}\mu(y)=\omega_{n-1}y^{n-1}\,\mathrm{d}y$. Given $t\in(0,R)$ set

$$A=\{y\in(t,R)\colon\ |g'(y)|>G\}$$

(recall that the constant G comes from (28)). From (28) we obtain

$$\int_A \Phi_1(|g'(y)|)\omega_{n-1}y^{n-1}\,\mathrm{d}y \leqslant \frac{\omega_{n-1}}{n}\int_A \Phi(|g'(y)|)y^{n-1}\,\mathrm{d}y$$
$$\leqslant \frac{\omega_{n-1}}{n}\int_0^R \Phi(|g'(y)|)y^{n-1}\,\mathrm{d}y = \frac{1}{n}\int_{B(0,R)} \Phi(|\nabla f(x)|)\,\mathrm{d}x \leqslant \frac{1}{n} = \Phi_1(1).$$

Thus $\|g'(y)\|_{L^{\Phi_1}(A,\,\mathrm{d}\mu)} \leqslant 1$.

Hence (7) and Lemma 3.2 give for $0 < t < t_0$ that

$$\begin{aligned}g(t) &\leqslant \int_t^R |g'(y)|\,\mathrm{d}y = \int_{y\in(t,R)\setminus A} |g'(y)|\,\mathrm{d}y + \int_A |g'(y)| \frac{1}{\omega_{n-1}y^{n-1}}\,\mathrm{d}\mu(y)\\ &\leqslant GR + \frac{1}{\omega_{n-1}} \|g'(y)\|_{L^{\Phi_1}(A,\,\mathrm{d}\mu)} \Big\| \frac{1}{y^{n-1}} \Big\|_{L^{\Psi}((t,R),\,\mathrm{d}\mu)} \leqslant GR + \frac{D}{\omega_{n-1}} \log_{[k]}^{1/\gamma}\Big(\frac{1}{t}\Big).\end{aligned}$$

Thus there is t_1, $0 < t_1 < t_0 < 1$, such that for $0 < t < t_1$ we have

$$g(t) \leqslant (1+\eta)\frac{D}{\omega_{n-1}} \log_{[k]}^{1/\gamma}\Big(\frac{1}{t}\Big).$$

Since g is non-increasing and (47) implies $(1+\eta)D/\omega_{n-1}K < 1$ we have

$$\begin{aligned}\int_{B(0,R)} \exp_{[k]}\Big(\Big(\frac{|f(x)|}{K}\Big)^{\gamma}\Big)\,\mathrm{d}x &= \omega_{n-1}\int_0^R \exp_{[k]}\Big(\Big(\frac{g(y)}{K}\Big)^{\gamma}\Big)\, y^{n-1}\,\mathrm{d}y\\ &\leqslant C\int_{t_1}^R \exp_{[k]}\Big(\Big(\frac{g(t_1)}{K}\Big)^{\gamma}\Big)\, y^{n-1}\,\mathrm{d}y\\ &\qquad + C\int_0^{t_1} \exp_{[k]}\Big(\Big(\frac{(1+\eta)D}{\omega_{n-1}K}\Big)^{\gamma} \log_{[k]}\Big(\frac{1}{y}\Big)\Big) y^{n-1}\,\mathrm{d}y\\ &\leqslant C\int_{t_1}^R y^{n-1}\,\mathrm{d}y + C\int_0^{t_1} y^{-1}y^{n-1}\,\mathrm{d}y = C.\end{aligned}$$

□

4. On sharp embedding into multiple exponential spaces

4.1. Counterexample.

We first prove that for any function $\Phi\colon \mathbb{R}^+ \to \mathbb{R}^+$ which satisfies assumptions (48) the statement of Theorem 1.1 is not valid in the borderline case $K = B^{-(n-1)/n}\omega_{n-1}^{-1/n}$.

Theorem 4.1. *Let $k \in \mathbb{N}$, $k \geqslant 2$, $\alpha < n-1$ and $a < \min\{1, B\}$. Suppose Φ is a Young function and there are $L_1 > 0$ and $L_2 > \exp_{[k]}(1)$ such that Φ satisfies*

$$\Phi(t) \leqslant \begin{cases} L_1 t^n & \text{for } t \in [0, L_2],\\ t^n\Big(\prod_{j=1}^{k-1} \log_{[j]}^{n-1}(t)\Big) \log_{[k]}^{\alpha}(t)(1-\log_{[k]}^{-a}(t)) & \text{for } t \in [L_2, \infty).\end{cases} \tag{48}$$

Let

$$K = B^{-(n-1)/n}\omega_{n-1}^{-1/n}.$$

Then for every $m \in \mathbb{N}$ there is $f \in WL_0^{\Phi}(B(0,R))$ such that $\int_{B(0,R)} \Phi(|\nabla f|)\,\mathrm{d}x \leqslant 1$ but

$$\int_{B(0,R)} \exp_{[k]}\Big(\Big(\frac{|f(x)|}{K}\Big)^{\gamma}\Big)\,\mathrm{d}x > m.$$

P r o o f. Fix $T > \exp_{[k]}(1)$. For $s > T$ we define $f_s(x) = g_s(|x|)$ where

$$g_s(y) = \begin{cases} \big(-\frac{2}{R}y+2\big)K\log_{[k]}^B(T+2)s^{1/\gamma-B}\big(1+\frac{\log(s)}{s}\big)^{1/\gamma} & \text{for } y \in [\frac{R}{2}, R], \\ K\log_{[k]}^B\big(T+\frac{R}{y}\big)s^{1/\gamma-B}\big(1+\frac{\log(s)}{s}\big)^{1/\gamma} & \text{for } y \in [R\exp_{[k]}^{-1/n}(s), \frac{R}{2}], \\ K\log_{[k]}^B\big(T+\exp_{[k]}^{1/n}(s)\big)s^{1/\gamma-B}\big(1+\frac{\log(s)}{s}\big)^{1/\gamma} & \text{for } y \in [0, R\exp_{[k]}^{-1/n}(s)]. \end{cases}$$

As $|(\log(T+R/y))'| = (T+R/y)^{-1}R/y^2 = (R+Ty)^{-1}R/y$, on $\big[R\exp_{[k]}^{-1/n}(s), \frac{1}{2}R\big]$ we have

$$(49)\ |g_s'(y)| = KB\log_{[k]}^{B-1}\Big(T+\frac{R}{y}\Big)\Big(\prod_{j=1}^{k-1}\log_{[j]}^{-1}\Big(T+\frac{R}{y}\Big)\Big)\frac{1}{R+Ty}\frac{R}{y}s^{1/\gamma-B}\Big(1+\frac{\log(s)}{s}\Big)^{1/\gamma}.$$

Using (9) from Lemma 2.2 we obtain $s_1 > T$ such that for $s > s_1$ we have

$$\left(\frac{\log_{[k]}\big(\exp_{[k]}^{1/n}(s)\big)}{s}\right)^{B\gamma}\big(s+\log(s)\big) \geqslant \Big(1-\frac{C}{s}\Big)^{B\gamma}\big(s+\log(s)\big) \geqslant s+\frac{1}{2}\log(s)$$

and easy computation gives

$$\begin{aligned} &\int_{B(0,R)} \exp_{[k]}\Big(\Big(\frac{|f_s(x)|}{K}\Big)^{\gamma}\Big) \\ &\qquad \geqslant \int_{B\left(0,R\exp_{[k]}^{-1/n}(s)\right)} \exp_{[k]}\Big(\Big(\frac{|f_s(x)|}{K}\Big)^{\gamma}\Big)\,\mathrm{d}x \\ &\qquad \geqslant C\exp_{[k]}^{-1}(s)\exp_{[k]}\left(\left(\frac{\log_{[k]}\big(\exp_{[k]}^{1/n}(s)\big)}{s}\right)^{B\gamma}\big(s+\log(s)\big)\right) \\ &\qquad \geqslant C\exp_{[k]}^{-1}(s)\exp_{[k]}\Big(s+\frac{1}{2}\log(s)\Big) \underset{s\to\infty}{\longrightarrow} \infty. \end{aligned}$$

It remains to prove that $\int_{B(0,R)} \Phi(|\nabla f_s|) \leqslant 1$ for s large enough.

Set $M = M(s) = R/s^{\log(s)}$. Plainly there is $s_2 > s_1$ such that for $s > s_2$ we have

$$R\exp_{[k]}^{-1/n}(s) < M < \frac{R}{T}$$

and therefore

$$\int_0^1 \Phi(|g_s'(y)|)y^{n-1}\,\mathrm{d}y = \int_{R\exp_{[k]}^{-1/n}(s)}^{M} + \int_M^{\frac{R}{T}} + \int_{\frac{R}{T}}^{R} = I_1 + I_2 + I_3. \tag{50}$$

Obviously $1/\gamma - B < 0$ and $|g_s'(y)| \leqslant Cs^{1/\gamma - B}$ for $y \in (\frac{1}{2}R, R)$. Further, (49) implies $|g_s'(y)| < Cs^{1/\gamma - B}$ for $y \in (R/T, R/2)$. Hence there is $s_3 > s_2$ such that for every $s > s_3$ and $y \in (R/T, R)$ we have $|g_s'(y)| \leqslant Cs^{1/\gamma - B} < L_2$. Therefore it follows from (48) that

$$I_3 \leqslant C\int_{\frac{R}{T}}^{R} |g_s'(y)|^n y^{n-1}\,\mathrm{d}y \leqslant Cs^{(1/\gamma - B)n} \leqslant Cs^{-B}. \tag{51}$$

Using (49) for $y \in (M, R/T)$ we obtain

$$\begin{aligned} |\log(|g_s'(y)|)|^n &\leqslant C\Big(\log^n(s) + \log^n\Big(\frac{R}{y}\Big)\Big) \\ &\leqslant C\Big(\log^n(s) + \log^n\Big(\frac{R}{M}\Big)\Big) \leqslant C\log^{2n}(s). \end{aligned} \tag{52}$$

Further, for $y \in (M, R/T)$ we have $R/y \leqslant T + R/y \leqslant 2R/y$ and thus the following inequalities are satisfied:

$$\begin{aligned} \log_{[j]}^{-n}\Big(T + \frac{R}{y}\Big) &\leqslant C\log_{[j]}^{-n}\Big(\frac{R}{y}\Big) \qquad \text{for } j \in \{1, \dots, k-1\}, \\ \log_{[k]}^{B-1}\Big(T + \frac{R}{y}\Big) &\leqslant C\log_{[k]}^{B-1}\Big(\frac{R}{y}\Big). \end{aligned} \tag{53}$$

From (48) we obtain $\Phi(t) \leqslant Ct^n\big(1 + |\log(t)|^n\big)$ on $[0, \infty)$, hence (49), (52) and (53) imply

$$\begin{aligned} I_2 &\leqslant C\int_M^{\frac{R}{T}} |g_s'(y)|^n\big(1 + |\log(|g_s'(y)|)|^n\big)y^{n-1}\,\mathrm{d}y \\ &\leqslant Cs^{(1/\gamma - B)n}\int_M^{\frac{R}{T}} \log_{[k]}^{(B-1)n}\Big(\frac{R}{y}\Big)\Big(\prod_{j=1}^{k-1}\log_{[j]}^{-n}\Big(\frac{R}{y}\Big)\Big)\Big(1 + \log^{2n}(s)\Big)\frac{\mathrm{d}y}{y} \\ &\leqslant Cs^{(1/\gamma - B)n}\log^{2n}(s)\int_M^{\frac{R}{T}} \log_{[k]}^{(B-1)n}\Big(\frac{R}{y}\Big)\Big(\prod_{j=1}^{k-1}\log_{[j]}^{-1}\Big(\frac{R}{y}\Big)\Big)\frac{\mathrm{d}y}{y}. \end{aligned}$$

Now, if $(B-1)n \neq -1$, then

$$\begin{aligned} \int_M^{\frac{R}{T}} \log_{[k]}^{(B-1)n}\Big(\frac{R}{y}\Big)\Big(\prod_{j=1}^{k-1}\log_{[j]}^{-1}\Big(\frac{R}{y}\Big)\Big)\frac{\mathrm{d}y}{y} &= \frac{1}{(B-1)n+1}\Big[-\log_{[k]}^{(B-1)n+1}\Big(\frac{R}{y}\Big)\Big]_M^{\frac{R}{T}} \\ &\leqslant C + C\log_{[k]}^{(B-1)n+1}\Big(\frac{R}{M}\Big) \end{aligned}$$

and if $(B-1)n=-1$, then

$$\int_M^{\frac{R}{T}} \log_{[k]}^{(B-1)n}\Big(\frac{R}{y}\Big)\Big(\prod_{j=1}^{k-1}\log_{[j]}^{-1}\Big(\frac{R}{y}\Big)\Big)\frac{\mathrm{d}y}{y} = \Big[-\log_{[k+1]}\Big(\frac{R}{y}\Big)\Big]_M^{\frac{R}{T}} \leqslant C + C\log_{[k+1]}\Big(\frac{R}{M}\Big).$$

Thus in both cases for $s>s_3$ we obtain

$$I_2 \leqslant Cs^{(1/\gamma-B)n}\log^{2n}(s)\Big(C + C\log_{[k]}^{|(B-1)n+1|+1}\Big(\frac{R}{M}\Big)\Big) \leqslant Cs^{-B}\log^{2n}(s)(1+\log_{[k]}^{|(B-1)n+1|+1}(s^{\log(s)})) \leqslant Cs^{-B}\log^{2n+1}(s). \tag{54}$$

As $M \ll s^{1/\gamma-B} \ll 1$, by (49) there is $s_4>s_3$ such that for all $s>s_4$ and for $y\in[R\exp_{[k]}^{-1/n}(s), M]$ we have

$$|g_s'(y)|>E.$$

Hence we obtain from (48)

$$I_1 \leqslant \int_{R\exp_{[k]}^{-1/n}(s)}^{M} |g_s'(y)|^n\Big(\prod_{j=1}^{k-1}\log_{[j]}^{n-1}(|g_s'(y)|)\Big) \times (1-\log_{[k]}^{-a}(|g_s'(y)|))\log_{[k]}^{\alpha}(|g_s'(y)|)y^{n-1}\,\mathrm{d}y. \tag{55}$$

Further, for $y\in[R\exp_{[k]}^{-1/n}(s), M]$ we trivially have

$$\frac{1}{R+Ty}\frac{R}{y} \leqslant \frac{1}{y}, \tag{56}$$

$$\log_{[j]}^{-n}\Big(T+\frac{R}{y}\Big) \leqslant \log_{[j]}^{-n}\Big(\frac{R}{y}\Big) \qquad \text{for } j\in\{1,\dots,k-1\}. \tag{57}$$

Since $1/\gamma-B<0$, by (49) we can find $s_5>s_4$ such that for all $s>s_5$ we have

$$y\in[R\exp_{[k]}^{-1/n}(s), M] \Rightarrow \frac{R}{y}\log^{-2}\Big(\frac{R}{y}\Big)s^{1/\gamma-B} < |g_s'(y)| < \frac{R}{y} \tag{58}$$

and thus for $y\in[R\exp_{[k]}^{-1/n}(s), M]$ we obtain

$$\log_{[j]}^{n-1}(|g_s'(y)|) \leqslant \log_{[j]}^{n-1}\Big(\frac{R}{y}\Big) \qquad \text{for } j\in\{2,\dots,k-1\}. \tag{59}$$

Further, let us show that we can find $s_6 > s_5$ such that for all $s > s_6$ we have

$$y \in [R \exp_{[k]}^{-1/n}(s), M] \Rightarrow \log_{[k]}^{(B-1)n}\Big(T + \frac{R}{y}\Big) \leqslant \Big(1 + \frac{1}{4s^a}\Big) \log_{[k]}^{(B-1)n}\Big(\frac{R}{y}\Big). \tag{60}$$

If $(B-1)n < 0$, then (60) is obviously satisfied. Otherwise we use the following estimates that are satisfied for s large enough:

$$\log\Big(T + \frac{R}{y}\Big) \leqslant \log\Big(\frac{R}{y}\Big) + \log\Big(1 + \frac{TM}{R}\Big) \leqslant \log\Big(\frac{R}{y}\Big) + \frac{1}{s} \leqslant \log\Big(\frac{R}{y}\Big)\Big(1 + \frac{1}{s}\Big),$$

$$\log_{[2]}\Big(T + \frac{R}{y}\Big) \leqslant \log\Big(\log\Big(\frac{R}{y}\Big)\Big(1 + \frac{1}{s}\Big)\Big) \leqslant \log_{[2]}\Big(\frac{R}{y}\Big) + \frac{1}{s} \leqslant \log_{[2]}\Big(\frac{R}{y}\Big)\Big(1 + \frac{1}{s}\Big)$$

and induction yields

$$\log_{[k]}\Big(T + \frac{R}{y}\Big) \leqslant \log_{[k]}\Big(\frac{R}{y}\Big)\Big(1 + \frac{1}{s}\Big). \tag{61}$$

Therefore (60) follows from $a < 1$ and (61).

Finally, we need to prove that there is $s_7 > s_6$ such that for every $s > s_7$ we have

$$\begin{aligned} &y \in [R \exp_{[k]}^{-1/n}(s), M] \\ &\quad \Rightarrow \Big(1 - \log_{[k]}^{-a}(|g_s'(y)|)\Big) \log_{[k]}^{\alpha}(|g_s'(y)|) \leqslant \Big(1 - \frac{1}{2s^a}\Big) \log_{[k]}^{\alpha}\Big(\frac{R}{y}\Big). \end{aligned} \tag{62}$$

If $\alpha \geqslant 0$, then the estimate follows from (58) and

$$|g_s'(y)| \leqslant \lim_{y \to (R \exp_{[k]}^{-1/n}(s))_+} |g_s'(y)| \leqslant \exp_{[k]}(s). \tag{63}$$

Otherwise set $M_1 = R/\exp(s)$ and $\delta = \frac{1}{2}(1-a)$. On $[M_1, M]$ we have by (58)

$$1 - \log_{[k]}^{-a}(|g_s'(y)|) \leqslant 1 - \log_{[k]}^{-a}\Big(\frac{R}{M_1}\Big) = 1 - \log_{[k-1]}^{-a}(s) \leqslant 1 - \frac{1}{\log^a(s)}. \tag{64}$$

Further, for s large enough and $y \in [M_1, M]$ we obtain using $\log(R/y) \geqslant \log^2(s)$ and (58)

$$\log(|g_s'(y)|) \geqslant \log\Big(\frac{R}{y}\Big) - 2\log_{[2]}\Big(\frac{R}{y}\Big) + \Big(\frac{1}{\gamma} - B\Big)\log(s) \geqslant \log\Big(\frac{R}{y}\Big)\Big(1 - \frac{1}{\log^{1-\delta}(s)}\Big),$$

$$\begin{aligned} \log_{[2]}(|g_s'(y)|) &\geqslant \log\Big(\log\Big(\frac{R}{y}\Big)\Big(1 - \frac{1}{\log^{1-\delta}(s)}\Big)\Big) \geqslant \log_{[2]}\Big(\frac{R}{y}\Big) - \frac{2}{\log^{1-\delta}(s)} \\ &\geqslant \log_{[2]}\Big(\frac{R}{y}\Big)\Big(1 - \frac{1}{\log^{1-\delta}(s)}\Big). \end{aligned}$$

Induction implies

$$\log_{[k]}(|g_s'(y)|) \geqslant \log_{[k]}\Big(\frac{R}{y}\Big)\Big(1 - \frac{1}{\log^{1-\delta}(s)}\Big). \tag{65}$$

Since $1-\delta > a$, (62) is proved for $y \in [M_1, M]$ and $\alpha < 0$ by (64) and (65).

The proof on $[R\exp_{[k]}^{-1/n}(s), M_1]$ is similar. We have for s large enough

$$\log(|g_s'(y)|) \geqslant \log\Big(\frac{R}{y}\Big)\bigg(1 - \frac{2\log_{[2]}(R/y)}{\log(R/y)} + \Big(\frac{1}{\gamma} - B\Big)\frac{\log(s)}{\log(R/y)}\bigg) \geqslant \log\Big(\frac{R}{y}\Big)\Big(1 - \frac{1}{s^{1-\delta}}\Big)$$

and the induction implies

$$\log_{[k]}(|g_s'(y)|) \geqslant \log_{[k]}\Big(\frac{R}{y}\Big)\Big(1 - \frac{1}{s^{1-\delta}}\Big). \tag{66}$$

Hence (62) is proved for $y \in [R\exp_{[k]}^{-1/n}(s), M_1]$ and $\alpha < 0$ by (63), (66) and $1-\delta > a$.

Therefore we go on estimating I_1 and from (49), (55), (56), (57), (59), (60) and (62) we obtain

$$\begin{aligned} I_1 &\leqslant K^n B^n s^{(1/\gamma - B)n}\Big(1 + \frac{\log(s)}{s}\Big)^{n/\gamma}\Big(1 - \frac{1}{2s^a}\Big)\Big(1 + \frac{1}{4s^a}\Big) \\ &\qquad \times \int_{R\exp_{[k]}^{-1/n}(s)}^{M} \log_{[k]}^{(B-1)n+\alpha}\Big(\frac{R}{y}\Big)\bigg(\prod_{j=1}^{k-1}\log_{[j]}^{-1}\Big(\frac{R}{y}\Big)\bigg)\frac{\mathrm{d}y}{y} \\ &\leqslant \Big(1 - \frac{1}{4s^a}\Big)K^n B^n s^{(1/\gamma - B)n}\Big(1 + \frac{\log(s)}{s}\Big)^{n/\gamma} \\ &\qquad \times \frac{1}{(B-1)n+\alpha+1}\Big[-\log_{[k]}^{(B-1)n+\alpha+1}\Big(\frac{R}{y}\Big)\Big]_{R\exp_{[k]}^{-1/n}(s)}^{M}. \end{aligned}$$

From (2) we can see that $(B-1)n+\alpha+1 = B = -(1/\gamma - B)n$. Further, since $s_7 > s_1 > T$ we have $\log_{[k]}(1/M) > 0$ for all $s > s_7$. Therefore

$$\Big[-\log_{[k]}^{(B-1)n+\alpha+1}\Big(\frac{R}{y}\Big)\Big]_{R\exp_{[k]}^{-1/n}(s)}^{M} \leqslant \log_{[k]}^{(B-1)n+\alpha+1}\Big(\exp_{[k]}^{1/n}(s)\Big) \leqslant s^B.$$

Hence as $a < 1$ there is $s_8 > s_7$ such that for all $s > s_8$ we have

$$\begin{aligned} I_1 &\leqslant \Big(1 - \frac{1}{4s^a}\Big)K^n B^{n-1}\Big(1 + \frac{\log(s)}{s}\Big)^{n/\gamma} \\ &\leqslant \frac{1}{\omega_{n-1}}\Big(1 - \frac{1}{4s^a}\Big)\Big(1 + \frac{\log(s)}{s}\Big)^{n/\gamma} \leqslant \frac{1}{\omega_{n-1}}\Big(1 - \frac{1}{8s^a}\Big). \end{aligned} \tag{67}$$

Using (51), (54), (67) and $a < B$ for s large enough conclude that

$$\int_{B(0,1)} \Phi(|\nabla f_s(x)|)\,\mathrm{d}x = \omega_{n-1}\int_0^R \Phi(|g_s'(y)|)y^{n-1}\,\mathrm{d}y = \omega_{n-1}(I_1 + I_2 + I_3)$$
$$\leqslant \Big(1 - \frac{1}{8s^a}\Big) + Cs^{-B}(1 + \log^{2n+1}(s)) \leqslant 1.$$

□

4.2. Sharp embedding.

From the previous section we know that if we want to have the statement of Theorem 1.1 for the borderline case (5) then we need to require something more from Φ than (48).

Theorem 4.2. *Let $\alpha < n-1$. Suppose that Φ is a Young function and there are $A_1 > \exp_{[k]}(1)$ and $a \in (0, \min\{1, 1/\gamma\})$ such that Φ satisfies*

$$\Phi(t) \geqslant t^n \Big(\prod_{j=1}^{k-1} \log_{[j]}^{n-1}(t)\Big) \log_{[k]}^{\alpha}(t)(1 + \log_{[k]}^{-a}(t)) \qquad \text{for } t > A_1. \tag{68}$$

Let

$$K = B^{-(n-1)/n}\omega_{n-1}^{-1/n}.$$

Suppose that $f \in WL_0^{\Phi}(\Omega)$ and $\int_\Omega \Phi(|\nabla f|)\,\mathrm{d}x \leqslant 1$. Then

$$\int_\Omega \exp_{[k]}\Big(\Big(\frac{|f(x)|}{K}\Big)^{\gamma}\Big)\,\mathrm{d}x < d \tag{69}$$

where d depends on n, k, α, $\mathcal{L}_n(\Omega)$ and Φ only.

Suppose that the function $\Phi\colon \mathbb{R}^+ \to \mathbb{R}^+$ satisfies (68). In a standard way we can prove that there is a function $\Phi_1\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that

(70)
$$\Phi_1 \text{ is a Young function,}$$
$$\Phi_1' \text{ is continuous and increasing on } (0,\infty),$$
$$\Phi_1(t) = \frac{1}{n}t^n \text{ for } t \in [0,1],$$
$$\text{there is } A_2 > A_1 \text{ such that for every } t > A_2 \text{ we have}$$
$$\Phi_1(t) = \frac{1}{n}t^n\Big(\prod_{j=1}^{k-1}\log_{[j]}^{n-1}(t)\Big)\log_{[k]}^{\alpha}(t)(1 + \log_{[k]}^{-a}(t)) \leqslant \frac{1}{n}\Phi(t).$$

Denote by Ψ the Young function associated with the function Φ_1. Clearly $\Psi(t) = t^{n'}/n'$ for $t \in [0,1]$. Hence $\Phi_1(1) + \Psi(1) = 1$; therefore (Φ_1, Ψ) is a normalized complementary Young pair and we can use inequality (7).

Let us first estimate the growth of Ψ.

Lemma 4.3. *There is $E > 0$ such that for every $t \in \mathbb{R}$ we have*

$$\Psi(t) < \hat{\Psi}(t) := Et^{n/(n-1)}\big(1 + |\log t|^E\big). \tag{71}$$

Moreover, there are $A_3 > A_2$ and $b \in (a, \min\{1, 1/\gamma\})$ such that for every $t \in [A_3, \infty)$ we have

$$\Psi(t) \leqslant \tilde{\Psi}(t) := \frac{(n-1)^2}{n} t^{n/(n-1)} \left(\prod_{j=1}^{k-1} \log_{[j]}^{-1}(t)\right) \log_{[k]}^{-\alpha/(n-1)}(t)(1 - \log_{[k]}^{-b}(t)). \tag{72}$$

P r o o f. Let us choose $b \in (a, \min\{1, 1/\gamma\})$ and $b_1 \in (a, b)$. Thus $0 < a < b_1 < b < 1$. Put

$$\tilde{\Psi}_1(t) = \frac{(n-1)^2}{n} t^{n/(n-1)} \left(\prod_{j=1}^{k-1} \log_{[j]}^{-1}(t)\right) \log_{[k]}^{-\alpha/(n-1)}(t)(1 - \log_{[k]}^{-b_1}(t)).$$

Denote $\tilde{\psi}_1 = \tilde{\Psi}_1'$, $\varphi = \Phi_1'$ and $\psi = \varphi^{-1}$, hence $\Psi(t) = \int_0^t \psi$.

By (70) there is $B_1 > A_2$ such that for every $t > B_1$ we have

$$\begin{aligned} \varphi(t) &= t^{n-1} \left(\prod_{j=1}^{k-1} \log_{[j]}^{n-1}(t)\right) \log_{[k]}^{\alpha}(t) \Bigg[1 + \log_{[k]}^{-a}(t) \\ &\quad + \sum_{j=1}^{k-1} \frac{n-1}{n} \frac{1 + \log_{[k]}^{-a}(t)}{\prod_{i=1}^{j} \log_{[i]}(t)} + \frac{\alpha}{n} \frac{1 + \log_{[k]}^{-a}(t)}{\prod_{j=1}^{k} \log_{[j]}(t)} - \frac{a}{n} \frac{\log_{[k]}^{-a}(t)}{\prod_{j=1}^{k} \log_{[j]}(t)}\Bigg] \\ &\geqslant t^{n-1} \left(\prod_{j=1}^{k-1} \log_{[j]}^{n-1}(t)\right) \log_{[k]}^{\alpha}(t) \Big(1 + \frac{1}{2} \log_{[k]}^{-a}(t)\Big) = \tilde{\varphi}(t). \end{aligned} \tag{73}$$

Analogously there is $B_2 > B_1$ such that for every $t > B_2$

$$\begin{aligned} \tilde{\psi}_1(t) &= (n-1)t^{1/(n-1)} \left(\prod_{j=1}^{k-1} \log_{[j]}^{-1}(t)\right) \log_{[k]}^{-\alpha/(n-1)}(t) \Bigg[1 - \log_{[k]}^{-b_1}(t) \\ &\quad - \sum_{j=1}^{k-1} \frac{n-1}{n} \frac{1 - \log_{[k]}^{-b_1}(t)}{\prod_{i=1}^{j} \log_{[i]}(t)} - \frac{\alpha}{n} \frac{1 - \log_{[k]}^{-b_1}(t)}{\prod_{j=1}^{k} \log_{[j]}(t)} + \frac{b_1(n-1)}{n} \frac{\log_{[k]}^{-b_1}(t)}{\prod_{j=1}^{k} \log_{[j]}(t)}\Bigg] \\ &\geqslant (n-1)t^{1/(n-1)} \left(\prod_{j=1}^{k-1} \log_{[j]}^{-1}(t)\right) \log_{[k]}^{-\alpha/(n-1)}(t)(1 - 2\log_{[k]}^{-b_1}(t)) = \tilde{\psi}(t). \end{aligned} \tag{74}$$

Using (74) and $\log_{[2]}(t)/\log(t) \ll 1/\log_{[k]}(t)$ we find $B_3 > B_2$ such that for $t > B_3$ we have

$$\begin{aligned}
\log^{n-1}\big(\tilde{\psi}(t)\big) &\geqslant \log^{n-1}\Big(t^{1/(n-1)}\frac{1}{\log^2(t)}\Big) = \Big(\frac{1}{n-1}\log(t) - 2\log_{[2]}(t)\Big)^{n-1} \\
&\geqslant \frac{1}{(n-1)^{n-1}}\log^{n-1}(t)\,\Big(1 - 2\frac{(n-1)\log_{[2]}(t)}{\log(t)}\Big)^{n-1} \\
&\geqslant \frac{1}{(n-1)^{n-1}}\log^{n-1}(t)\,\Big(1 - \frac{C}{\log_{[k]}(t)}\Big)^{n-1}.
\end{aligned} \tag{75}$$

By Lemma 2.2 there is $B_4 > B_3$ such that for $t > B_4$ we obtain

$$\log_{[j]}^{n-1}\big(\tilde{\psi}(t)\big) \geqslant \log_{[j]}^{n-1}\big(t^{1/n}\big) \geqslant \log_{[j]}^{n-1}(t)\Big(1-\frac{C}{\log_{[k]}(t)}\Big) \quad \text{for } j \in \{2,\ldots,k-1\}, \tag{76}$$

analogously

$$\log_{[k]}^{\alpha}\big(\tilde{\psi}(t)\big) \geqslant \log_{[k]}^{\alpha}(t)\Big(1 - \frac{C}{\log_{[k]}(t)}\Big) \tag{77}$$

and

$$1 + \frac{1}{2}\log_{[k]}^{-a}\big(\tilde{\psi}(t)\big) \geqslant 1 + \frac{1}{4}\log_{[k]}^{-a}(t). \tag{78}$$

Hence using $0 < a < b_1 < 1$, (73), (74), (75), (76), (77) and (78) we can see that there is $B_5 > B_4$ such that for all $t > B_5$ we have

$$\begin{aligned}
&\tilde{\varphi}(\tilde{\psi}(t)) \\
&\quad\geqslant \tilde{\psi}^{n-1}(t)\Big(\prod_{j=1}^{k-1}\log_{[j]}^{n-1}(\tilde{\psi}(t))\Big)\log_{[k]}^{\alpha}(\tilde{\psi}(t))\Big(1 + \frac{1}{2}\log_{[k]}^{-a}(\tilde{\psi}(t))\Big) \\
&\quad\geqslant \Big[(n-1)t^{1/(n-1)}\Big(\prod_{j=1}^{k-1}\log_{[j]}^{-1}(t)\Big)\log_{[k]}^{-\alpha/(n-1)}(t)\Big(1 - 2\log_{[k]}^{-b_1}(t)\Big)\Big]^{n-1} \\
&\qquad\times\frac{\log^{n-1}(t)}{(n-1)^{n-1}}\Big(\prod_{j=2}^{k}\log_{[j]}^{n-1}(t)\Big)\log_{[k]}^{\alpha}(t)\Big(1 - \frac{C}{\log_{[k]}(t)}\Big)^{n+k-2}\Big(1 + \frac{1}{4}\log_{[k]}^{-a}(t)\Big) \\
&\quad\geqslant t\Big(1 - 2\log_{[k]}^{-b_1}(t)\Big)^{n-1}\Big(1 - \frac{C}{\log_{[k]}(t)}\Big)^{n+k-2}\Big(1 + \frac{1}{4}\log_{[k]}^{-a}(t)\Big) > t.
\end{aligned}$$

It follows that $\varphi(\tilde{\psi}_1(t)) > t$ for $t > B_5$ and thus

$$t > B_5 \quad\Rightarrow\quad \tilde{\psi}_1(t) > \varphi^{-1}(t) = \psi(t). \tag{79}$$

Hence for $t > B_5$ we have

$$\Psi(t) < \tilde{\Psi}_1(t) + C.$$

Together with $b_1 < b$ this implies that there is $A_3 > B_5$ such that for all $t > A_3$ we have

$$\Psi(t) < \tilde{\Psi}(t).$$

Since Ψ is increasing and $\Psi(t) = t^{n'}/n'$ for $t \in [0,1]$, (72) obviously implies (71). $\square$

In the proof of Theorem 4.2 we use the generalized Hölder inequality (7) and thus we need to estimate the term $\|1/y^{n-1}\|_{L^\Psi((t,R),\omega_{n-1}y^{n-1}\,\mathrm{d}y)}$.

Lemma 4.4. *There are $t_0 \in (0,1)$ and $c \in (b, \min\{1, 1/\gamma\})$ such that if $0 < t \leqslant t_0$ then*

$$\Big\|\frac{1}{y^{n-1}}\Big\|_{L^\Psi((t,R),\omega_{n-1}y^{n-1}\,\mathrm{d}y)} \leqslant D\log_{[k]}^{1/\gamma}\Big(\frac{1}{t}\Big)\Big(1 - \log_{[k]}^{-c}\Big(\frac{1}{t}\Big)\Big), \tag{80}$$

$$\text{where } D = \Big(\frac{\omega_{n-1}}{B}\Big)^{(n-1)/n}. \tag{81}$$

Proof. Let us fix $c \in (b, \min\{1, 1/\gamma\})$. We want to prove that for

$$\lambda = D\log_{[k]}^{1/\gamma}\Big(\frac{1}{t}\Big)\Big(1 - \log_{[k]}^{-c}\Big(\frac{1}{t}\Big)\Big)$$

we have

$$\int_t^R \Psi\Big(\frac{1}{\lambda y^{n-1}}\Big)\omega_{n-1}y^{n-1}\,\mathrm{d}y \leqslant \Psi(1) = \frac{n-1}{n}$$

for $t > 0$ small enough.

For $t \in \big(0, \exp_{[k]}^{-1}(1)\big)$ set $M = M(t) = \exp\big(-\log_{[k]}^{\min\{1,1/\gamma\}/(E+2)(n-1)}(1/t)\big)$. Clearly, we can find $t_1 \in \big(0, \exp_{[k]}^{-1}(1)\big)$ such that for $0 < t < t_1$ we have

$$t < M < R \qquad \text{and} \qquad \frac{1}{\lambda M^{n-1}} > A_3. \tag{82}$$

Hence Lemma 4.3 yields that

$$\begin{aligned} &\int_t^R \Psi\Big(\frac{1}{\lambda y^{n-1}}\Big)y^{n-1}\,\mathrm{d}y \\ &\qquad \leqslant \int_t^M \tilde{\Psi}\Big(\frac{1}{\lambda y^{n-1}}\Big)y^{n-1}\,\mathrm{d}y + \int_M^R \hat{\Psi}\Big(\frac{1}{\lambda y^{n-1}}\Big)y^{n-1}\,\mathrm{d}y = I_1 + I_2. \end{aligned} \tag{83}$$

By (71) we have

$$I_2 \leqslant E \int_M^R \frac{1}{\lambda^{n/(n-1)}} \Big(1 + \Big|\log\Big(\frac{1}{\lambda y^{n-1}}\Big)\Big|^E\Big) \frac{\mathrm{d}y}{y}$$
$$\leqslant \frac{C}{\lambda^{n/(n-1)}} \int_M^R \Big(1 + |\log(\lambda)|^E + |\log(y)|^E\Big) \frac{\mathrm{d}y}{y} = J_1 + J_2,$$

where (we observe that $(1 - \log_{[k]}^{-c}(1/t))^{-1} < C$ on $(0, t_1) \subset (0, \exp_{[k]}^{-1}(1))$)

$$J_1 = \frac{C}{\lambda^{n/(n-1)}} \int_M^R \Big(1 + |\log(\lambda)|^E\Big) \frac{\mathrm{d}y}{y}$$
$$\leqslant C \frac{1}{\log_{[k]}^{n/(n-1)\gamma}(\frac{1}{t})} \Big(1 + \log^E\Big(\log_{[k]}\Big(\frac{1}{t}\Big)\Big)\Big)\Big(1 + \log\Big(\frac{1}{M}\Big)\Big)$$

and

$$J_2 = \frac{C}{\lambda^{n/(n-1)}} \int_M^R |\log(y)|^E \frac{\mathrm{d}y}{y} \leqslant \frac{C}{\log_{[k]}^{n/((n-1)\gamma)}(\frac{1}{t})} \Big(1 + \log^{E+1}\Big(\frac{1}{M}\Big)\Big).$$

Hence we obtain

$$I_2 \leqslant C \frac{1}{\log_{[k]}^{n/((n-1)\gamma)}(\frac{1}{t})} \Big(1 + \log^E\Big(\log_{[k]}\Big(\frac{1}{t}\Big)\Big)\Big)\Big(1 + \log\Big(\frac{1}{M}\Big) + \log^{E+1}\Big(\frac{1}{M}\Big)\Big).$$

Thus there is $t_2 \in (0, t_1)$ such that if $0 < t < t_2$ then

$$I_2 < C \frac{1}{\log_{[k]}^{n/((n-1)\gamma)}(\frac{1}{t})} \log^{E+2}\Big(\frac{1}{M}\Big) \leqslant C \frac{1}{\log_{[k]}^{n/((n-1)\gamma) - \min\{1, 1/\gamma\}/(n-1)}(\frac{1}{t})}. \tag{84}$$

Since $b \in (0, 1)$ and thus $\log^{1-b}(1/M) \gg \log(\lambda) > 1$ for small $t > 0$, we can choose $t_3 \in (0, t_2)$ such that if $0 < t < t_3$ and $y \in [t, M]$ then

$$\log^{-1}\Big(\frac{1}{\lambda y^{n-1}}\Big) = \log^{-1}\Big(\frac{1}{y^{n-1}}\Big)\Big(1 + \frac{\log(1/\lambda)}{\log(1/y^{n-1})}\Big)^{-1} \tag{85}$$
$$\leqslant \frac{1}{n-1} \log^{-1}\Big(\frac{1}{y}\Big)\Big(1 + \frac{C\log(\lambda)}{\log(\frac{1}{y})}\Big)$$
$$\leqslant \frac{1}{n-1} \log^{-1}\Big(\frac{1}{y}\Big)\Big(1 + \frac{C\log(\lambda)}{\log^{1-b}(\frac{1}{M})} \frac{1}{\log^b(\frac{1}{y})}\Big)$$
$$\leqslant \Big(1 + \frac{\frac{1}{4}}{\log_{[k]}^b(\frac{1}{y})}\Big) \frac{1}{n-1} \log^{-1}\Big(\frac{1}{y}\Big).$$

Further, estimate (9) from Lemma 2.2 gives $t_4 \in (0, t_3)$ such that if $0 < t < t_4$ and $y \in [t, M]$, then we have

$$\log_{[j]}^{-1}\Big(\frac{1}{\lambda y^{n-1}}\Big) \leqslant \Big(1 + \frac{C}{\log_{[k]}(\frac{1}{y})}\Big) \log_{[j]}^{-1}\Big(\frac{1}{y}\Big) \quad \text{for } j \in \{2, \dots, k-1\}, \tag{86}$$

$$\log_{[k]}^{-\alpha/(n-1)}\Big(\frac{1}{\lambda y^{n-1}}\Big) \leqslant \Big(1 + \frac{C}{\log_{[k]}(\frac{1}{y})}\Big) \log_{[k]}^{-\alpha/(n-1)}\Big(\frac{1}{y}\Big), \tag{87}$$

$$1 - \log_{[k]}^{-b}\Big(\frac{1}{\lambda y^{n-1}}\Big) \leqslant 1 - \frac{1}{2}\log_{[k]}^{-b}\Big(\frac{1}{y}\Big) \tag{88}$$

and

$$\Big(1 - \log_{[k]}^{-c}\Big(\frac{1}{t}\Big)\Big)^{-n/(n-1)} \leqslant 1 + C\log_{[k]}^{-c}\Big(\frac{1}{t}\Big). \tag{89}$$

Hence (72), (83), (85), (86), (87), (88) and (89) give that

$$\begin{aligned} I_1 &\leqslant \frac{(n-1)^2}{n}\int_t^M \Big(\frac{1}{\lambda y^{n-1}}\Big)^{n/(n-1)} \Big(\prod_{j=1}^{k-1}\log_{[j]}^{-1}\Big(\frac{1}{\lambda y^{n-1}}\Big)\Big) \log_{[k]}^{-\alpha/(n-1)}\Big(\frac{1}{\lambda y^{n-1}}\Big) \\ &\qquad \Big(1 - \log_{[k]}^{-b}\Big(\frac{1}{\lambda y^{n-1}}\Big)\Big) y^{n-1}\,\mathrm{d}y \\ &\leqslant \frac{\frac{(n-1)^2}{n}\Big(1 + C\log_{[k]}^{-c}(\frac{1}{t})\Big)}{D^{n/(n-1)}\log_{[k]}^{n/((n-1)\gamma)}(\frac{1}{t})}\,\frac{1}{n-1}\int_t^M \Big(\prod_{j=1}^{k-1}\log_{[j]}^{-1}\Big(\frac{1}{y}\Big)\Big)\log_{[k]}^{-\alpha/(n-1)}\Big(\frac{1}{y}\Big) \\ &\qquad \Big(1 + \frac{1}{4}\log_{[k]}^{-b}\Big(\frac{1}{y}\Big)\Big)\Big(1 + \frac{C}{\log_{[k]}(\frac{1}{y})}\Big)^{k-1}\Big(1 - \frac{1}{2}\log_{[k]}^{-b}\Big(\frac{1}{y}\Big)\Big)\frac{\mathrm{d}y}{y}. \end{aligned}$$

Further, as $b < c < 1$ there is $t_5 \in (0, t_4)$ such that for $0 < t < t_5$ we conclude

$$\begin{aligned} &\Big(1 + C\log_{[k]}^{-c}\Big(\frac{1}{t}\Big)\Big)\Big(1 + \frac{C}{\log_{[k]}(\frac{1}{y})}\Big)^{k-1}\Big(1 + \frac{1}{4}\log_{[k]}^{-b}\Big(\frac{1}{y}\Big)\Big)\Big(1 - \frac{1}{2}\log_{[k]}^{-b}\Big(\frac{1}{y}\Big)\Big) \\ &\qquad \leqslant \Big(1 + C\log_{[k]}^{-c}\Big(\frac{1}{t}\Big)\Big)\Big(1 - \frac{1}{8}\log_{[k]}^{-b}\Big(\frac{1}{y}\Big)\Big) \\ &\qquad \leqslant \Big(1 + C\log_{[k]}^{-c}\Big(\frac{1}{t}\Big)\Big)\Big(1 - \frac{1}{8}\log_{[k]}^{-b}\Big(\frac{1}{t}\Big)\Big) \leqslant 1 - \frac{1}{16}\log_{[k]}^{-b}\Big(\frac{1}{t}\Big). \end{aligned}$$

Therefore (81) and $-\alpha/(n-1) = B - 1 \neq -1$ imply

$$\begin{aligned} I_1 &\leqslant \frac{n-1}{n}\,\frac{B}{\omega_{n-1}}\,\frac{1 - \frac{1}{16}\log_{[k]}^{-b}(\frac{1}{t})}{\log_{[k]}^{n/((n-1)\gamma)}(\frac{1}{t})}\int_t^M \Big(\prod_{j=1}^{k-1}\log_{[j]}^{-1}\Big(\frac{1}{y}\Big)\Big)\log_{[k]}^{-\alpha/(n-1)}\Big(\frac{1}{y}\Big)\frac{\mathrm{d}y}{y} \\ &\leqslant \frac{n-1}{n}\,\frac{B}{\omega_{n-1}}\,\frac{1 - \frac{1}{16}\log_{[k]}^{-b}(\frac{1}{t})}{\log_{[k]}^{n/((n-1)\gamma)}(\frac{1}{t})}\Big[-\frac{\log_{[k]}^{1-\alpha/(n-1)}(\frac{1}{y})}{1 - \frac{\alpha}{n-1}}\Big]_t^M. \end{aligned}$$

Since $1-\alpha/(n-1)=B=n/(n-1)\gamma$ and $\log_{[k]}(1/M)>0$, we have

$$I_1 \leqslant \frac{n-1}{\omega_{n-1}n}\Big(1-\frac{1}{16}\log_{[k]}^{-b}\Big(\frac{1}{t}\Big)\Big). \tag{90}$$

From (83), (84), (90) and $0<b<\min\{1,1/\gamma\}\leqslant n/\gamma(n-1)-\min\{1,1/\gamma\}/(n-1)$ we obtain that there is $t_0\in(0,t_5)$ such that for $0<t<t_0$ we have

$$\begin{aligned}\int_t^R \Psi\Big(\frac{1}{\lambda y^{n-1}}\Big)y^{n-1}\,\mathrm{d}y &\leqslant I_1+I_2\\ &\leqslant C\frac{1}{\log_{[k]}^{n/((n-1)\gamma)-\min\{1,\gamma\}/(n-1)}(\frac{1}{t})}+\frac{n-1}{\omega_{n-1}n}\Big(1-\frac{1}{16}\log_{[k]}^{-b}\Big(\frac{1}{t}\Big)\Big)\\ &\leqslant \frac{n-1}{\omega_{n-1}n}=\frac{1}{\omega_{n-1}}\Psi(1).\end{aligned}$$

□

P r o o f of Theorem 4.2. As in the proof of Theorem 1.1 we can suppose without loss of generality that $f(x)=g(|x|)$, where g is nonincreasing, classically differentiable almost everywhere, $g(R)=0$ and $\Omega=B(0,R)$. Put $\mathrm{d}\mu(y)=\omega_{n-1}y^{n-1}\,\mathrm{d}y$. Given $t\in(0,R)$ set

$$A=\{y\in(t,R)\colon\ |g'(y)|>A_2\}$$

(recall that the constant A_2 comes from (70)). Analogously to the proof of Theorem 1.1 we obtain, thanks to (70), that $\|g'(y)\|_{L^{\Phi_1}(A,\,\mathrm{d}\mu)}\leqslant 1$.

Therefore (7) and Lemma 4.4 with constant (81) give for $0<t<t_0$ that

$$\begin{aligned}g(t)\leqslant\int_t^R|g'(y)|\,\mathrm{d}y&=\int_{y\in(t,R)\setminus A}|g'(y)|\,\mathrm{d}y+\int_A|g'(y)|\frac{1}{\omega_{n-1}y^{n-1}}\,\mathrm{d}\mu(y)\\ &\leqslant A_2R+\frac{1}{\omega_{n-1}}\|g'(y)\|_{L^{\Phi_1}(A,\,\mathrm{d}\mu)}\Big\|\frac{1}{y^{n-1}}\Big\|_{L^{\Psi}(A,\,\mathrm{d}\mu)}\\ &\leqslant A_2R+\frac{D}{\omega_{n-1}}\log_{[k]}^{1/\gamma}\Big(\frac{1}{t}\Big)\Big(1-\log_{[k]}^{-c}\Big(\frac{1}{t}\Big)\Big).\end{aligned}$$

Further, (5) and (81) imply $D/\omega_{n-1}=K$ and thus for every $t\in[0,t_0]$ we have

$$g(t)\leqslant A_2R+K\log_{[k]}^{1/\gamma}\Big(\frac{1}{t}\Big)-K\log_{[k]}^{1/\gamma-c}\Big(\frac{1}{t}\Big). \tag{91}$$

Since $1/\gamma-c>0$, from (91) we observe that there is $t_1\in(0,t_0)$ such that $g(t)\leqslant K\log_{[k]}^{1/\gamma}(1/t)$ on $(0,t_1)$. Conversely, for $t\in[t_1,R]$ we have from (91)

$$g(t)\leqslant g(t_1)\leqslant A_2R+K\log_{[k]}^{1/\gamma}\Big(\frac{1}{t_1}\Big)\Big(1-\log_{[k]}^{-c}\Big(\frac{1}{t_1}\Big)\Big)\leqslant C.$$

Thus we obtain

$$\begin{aligned}
&\int_{B(0,R)} \exp_{[k]}\Big(\Big(\frac{|f(x)|}{K}\Big)^{\gamma}\Big)\,\mathrm{d}x \\
&\qquad = \omega_{n-1}\int_0^R \exp_{[k]}\Big(\Big(\frac{g(y)}{K}\Big)^{\gamma}\Big) y^{n-1}\,\mathrm{d}y \\
&\qquad \leqslant \omega_{n-1}\int_0^{t_1} \exp_{[k]}\Big(\log_{[k]}\Big(\frac{1}{y}\Big)\Big) y^{n-1}\,\mathrm{d}y + \omega_{n-1}\int_{t_1}^R \exp_{[k]}(C) y^{n-1}\,\mathrm{d}y \\
&\qquad \leqslant C\int_0^{t_1} y^{n-2}\,\mathrm{d}y + C\int_{t_1}^R y^{n-1}\,\mathrm{d}y \leqslant C.
\end{aligned}$$

□

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Authors' addresses: Robert Černý, Charles University, Faculty of Mathematics and Physics, Department of Mathematical Analysis, Sokolovská 83, 186 75 Praha 8, Czech Republic, e-mail: `rcerny@karlin.mff.cuni.cz`; Silvie Mašková, Charles University, Faculty of Mathematics and Physics, Department of Condensed Matter Physics, Ke Karlovu 5, 121 16 Praha 2, Czech Republic, e-mail: `maskova@mag.mff.cuni.cz`.