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ON THE INHOMOGENEOUS NONLINEAR SCHRÖDINGER EQUATION WITH HARMONIC POTENTIAL AND UNBOUNDED COEFFICIENT

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Abstract. By deriving a variant of interpolation inequality, we obtain a sharp criterion for global existence and blow-up of solutions to the inhomogeneous nonlinear Schrödinger equation with harmonic potential

$$\mathrm{i}\varphi_t = -\Delta\varphi + |x|^2\varphi - |x|^b|\varphi|^{p-2}\varphi.$$

We also prove the existence of unstable standing-wave solutions via blow-up under certain conditions on the unbounded inhomogeneity and the power of nonlinearity, as well as the frequency of the wave.

Keywords: interpolation inequality, inhomogeneous nonlinear Schrödinger equation, harmonic potential, blow-up, global existence, standing waves, strong instability

MSC 2010: 35J20, 35Q55

1. Introduction

Let $\varphi = \varphi(x,t)\colon \mathbb{R}^N \times \mathbb{R}_+ \to \mathbb{C}$ be a complex-valued function. The nonlinear Schrödinger equation with the harmonic potential and inhomogeneous nonlinearity (INLS-equation henceforth)

$$\mathrm{i}\varphi_t = -\Delta\varphi + |x|^2\varphi - K(x)|\varphi|^{p-2}\varphi, \quad x \in \mathbb{R}^N,\ t > 0, \tag{1.1}$$

arises in various physical contexts from the description of nonlinear waves such as propagation of a laser beam and plasma waves. For example, when $K(x) = 1$, the

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INLS-equation (1.1) is a model describing the Bose-Einstein condensate with attractive inter-particle interactions under magnetic trap [1], [27], [28]. Without the harmonic potential, the INLS-equation describes the beam propagation in an inhomogeneous medium, where φ is the electric field in laser optics and K is proportional to the electric density [14], [18].

From physical point of view, a basic question to the INLS-equation (1.1) is: when can the condensate be unstable to collapse (blow-up) or exist for all time (global existence)?

Another important issue that is often considered is whether or not global existence obtained for arbitrary classes of initial data is the stability of the standing-wave solutions $\mathrm{e}^{\mathrm{i}\omega t}\varphi(x)$ of (1.1). The localized solutions φ (ground-state solutions) are known in many circumstances to play a distinguished role in the long-time evolution of the initial disturbance. Therefore, the orbital and asymptotic stability of these special solutions has been a central theme of development for more than three decades (cf. [2], [4], [7], [8], [17], [24], [30], etc). Often, when nonlinear wave equations have solutions that lose regularity in finite time, the translation to singularity formation is associated with a standing wave going unstable.

The Cauchy problem and the issue of stability of standing waves of the INLS-equation have been studied extensively. For example, when $K(x) = 1$, Fukuizumi [11], Rose and Weinstein [22] and Zhang [32] obtained some results on the stability and instability of standing waves as well as global existence of (1.1) for various initial profiles. It is observed that there is a basic estimate used in their papers, i.e. the Gagliardo-Nirenberg inequality (i.e. Lemma 2.1 in the case of $\gamma = 0$ and $\beta = 0$). On the other hand, for the INLS-equation (K is not a constant) without harmonic potential,

$$\mathrm{i}\varphi_t + \Delta\varphi + K(x)|\varphi|^{p-2}\varphi = 0, \quad x \in \mathbb{R}^N, \tag{1.2}$$

Merle [20] proved the existence and nonexistence of blow-up solutions of Eq. (1.2) in the case of critical power $p = 2 + 4/N$ and $k_1 \leqslant K(x) \leqslant k_2$ with k_1 and k_2 being positive constants. Recently, Fibich, Liu, and Wang [10], [19] proved the stability and instability of standing waves of Eq. (1.2) under the assumptions on $p \geqslant 2 + 4/N$, $K(\varepsilon|x|)$ with ε small, and $K \in C^4(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Since $K(x)$ is bounded on $\mathbb{R}^N$ in their papers, the Gagliardo-Nirenberg inequality still plays a key role. However, when $K(x)$ is unbounded on $\mathbb{R}^N$, for example, $K(x) = |x|^b$, it seems that the standard Gagliardo-Nirenberg inequality can *not* be used any more. Let us note that Fukuizumi and Ohta [13] obtained the instability of standing waves for Eq. (1.2) when the inhomogeneity K of nonlinearity behaves like $|x|^{-b}$ at infinity with $0 < b < 2$. In [13], Hardy-Sobolev inequality plays an important role. For

$b > 0$, neither the standard Gagliardo-Nirenberg inequality nor the Hardy-Sobolev inequality can be used and the issue of whether or not particular initial data generate a blow-up solution of Eq. (1.1) is more subtle. In [8], Chen and Guo established a variant of interpolation inequality for $N \geqslant 2$, $b \geqslant 0$ and $2 + 2b/(N-1) < p < 2(N+b)/(N-1)$ and used it to study Eq. (1.2). Obviously, the nonlinear growth of p in [8] is not optimal and the results in [8] do not cover a large class of nonlinear Schrödinger equation.

In the present paper, we will derive an optimal version of Gagliardo-Nirenberg-type inequality (see Theorem 2.3) and use it to study the INLS-equation (1.1) with $K(x) = |x|^b$, $b > 0$. More precisely, we propose and analyze here an inhomogeneous nonlinear Schrödinger equation with harmonic potential of the form

$$\mathrm{i}\varphi_t = -\Delta\varphi + |x|^2\varphi - |x|^b|\varphi|^{p-2}\varphi, \tag{1.3}$$

where $b > 0$ (since the case of $b = 0$ has been studied extensively), $x \in \mathbb{R}^N$, $N \geqslant 2$. It is our purpose here to find the conditions on the initial data for global existence and blow-up of solutions to Eq. (1.3). Another goal of the paper is to show that those standing-wave solutions $\mathrm{e}^{\mathrm{i}\omega t}\varphi(x)$ are indeed unstable via blow-up when considered as solutions of the INLS-equation (1.3). In other words, there are perturbations arbitrary close to φ which, when posed as initial data for (1.3), lead to solutions blow-up in finite time.

To establish the results in view, we use the method of cross-invariant sets by Berestycki-Cazenave [3] and Shatah-Strauss [24], which was developed by many authors (see, e.g. [19, 32]). A crucial ingredient in the proof of blow-up of solutions and instability of standing waves by blow-up is a variant of the inequality of the Gagliardo-Nirenberg-type interpolation (Theorem 2.3).

The plan of the paper is as follows. In Section 2, we derive a variant interpolation inequality, which is found to play an essential role in the whole paper. In Section 3, the Cauchy problem for Eq. (1.3) is studied. In Section 4, the result on existence of the standing-wave solutions $\mathrm{e}^{\mathrm{i}\omega t}\varphi(x)$ of Eq. (1.3) is presented, where φ is the ground state solution of the related elliptic problem. Section 5 is devoted to proving blow-up of solutions in finite time and global existence of solutions to the initial-value problem for (1.3). Finally, in Section 6, the instability of the standing wave $\mathrm{e}^{\mathrm{i}\omega t}\varphi(x)$ for (1.3) by blow-up is obtained for some suitable frequency $\omega > 0$, the inhomogeneity associated with b and the power of nonlinearity p.

Notation. As above and henceforth, we denote the norm of the space $L^s(\mathbb{R}^n)$ by $|\cdot|_{L^s}$, $1 \leqslant s \leqslant \infty$ and denote the integral $\int_{\mathbb{R}^N} \mathrm{d}x$ simply by $\int$ unless stated otherwise. We also denote various positive constants by C or C_j and $p_c = 2(N+2+b)/N$. The function space in which we work is the Sobolev space

$$H^1_r(\mathbb{R}^N) = \{u \in H^1(\mathbb{R}^N),\ u(x) = u(|x|)\}$$

with the standard norm $\|u\|_1$ in $H^1(\mathbb{R}^N)$. We regard $H^1_r(\mathbb{R}^N)$ as a real Hilbert space with the inner product $(u, v)_{H^1_r} = \Re \int (\nabla u \nabla \overline{v} + u\overline{v})$.

2. An interpolation inequality

In this section, we will derive a variant of the well known Gagliardo- Nirenberg inequality. This inequality will play an essential role in the study of the inhomogeneous nonlinear Schrödinger equation (1.3). First, we need the following two lemmas.

Lemma 2.1 ([5]) [Gagliardo-Nirenberg inequality]. *Let $2 < q < q^*$, where $q^* = 2N/(N-2)$ when $N \geqslant 3$ and $q^* = +\infty$ when $N = 2$. Then there is a positive constant C such that for any $u \in H^1(\mathbb{R}^N)$,*

$$\int |u|^q \leqslant C\Big(\int |\nabla u|^2\Big)^{N(q-2)/4}\Big(\int |u|^2\Big)^{(2q-N(q-2))/4}. \tag{2.1}$$

Lemma 2.2 [Strauss' inequality]. *Let $N \geqslant 2$. For any $u \in H^1_r$, there is a constant $C_N > 0$ such that*

$$|x|^{\frac{N-1}{2}}|u(x)| \leqslant C_N\Big(\int |u|^2\Big)^{\frac14}\Big(\int |\nabla u|^2\Big)^{\frac14} \quad \textit{for a.e. } x \in \mathbb{R}^N. \tag{2.2}$$

Proof. See [26] or [31, Page 76, Lemma 4.5]. □

Theorem 2.3. *Assume $N \geqslant 2$, $b \geqslant 0$ and $2 + 2b/(N-1) < p < \tilde{p}$, where*

$$\tilde{p} = \begin{cases} \dfrac{2N}{N-2} + \dfrac{2b}{N-1}, & \text{if } N \geqslant 3,\\ +\infty, & \text{if } N = 2.\end{cases}$$

Then there is a constant $C > 0$ depending only on N, p and b such that for any $u \in H^1_r$,

$$\int |x|^b|u|^p \leqslant C\Big(\int |\nabla u|^2\Big)^{\frac{Np-2N-2b}{4}}\Big(\int |u|^2\Big)^{\frac{2N+2b+2p-Np}{4}}. \tag{2.3}$$

Proof. First, we use Lemma 2.2 to get

$$\int |x|^b|u|^p = \int (|x|^{\frac{N-1}{2}}|u(x)|)^{\frac{2b}{N-1}}|u|^{p-\frac{2b}{N-1}} \leqslant C_N\Big(\int |u|^2\Big)^{\frac{b}{2(N-1)}}\Big(\int |\nabla u|^2\Big)^{\frac{b}{2(N-1)}}\int |u|^{p-\frac{2b}{N-1}}. \tag{2.4}$$

Next, since $2+2b/(N-1) < p < \tilde{p}$, we have that $2 < p-2b/(N-1) < q^*$. It is deduced from Lemma 2.1 that

$$\int |u|^{p-\frac{2b}{N-1}} \leqslant C\Big(\int |\nabla u|^2\Big)^{\frac{N(p-\frac{2b}{N-1}-2)}{4}}\Big(\int |u|^2\Big)^{\frac{2(p-\frac{2b}{N-1})-N(p-\frac{2b}{N-1}-2)}{4}}. \tag{2.5}$$

Since

$$\frac{b}{2(N-1)}+\frac{N(p-2b/(N-1)-2)}{4}=\frac{N(p-2)-2b}{4}$$

and

$$\frac{b}{2(N-1)}+\frac{2(p-2b/(N-1))-N(p-2b/(N-1)-2)}{4}=\frac{2p-(N(p-2)-2b)}{4},$$

we obtain from (2.4) and (2.5) that

$$\int |x|^b|u|^p \leqslant C\Big(\int |\nabla u|^2\Big)^{\frac{N(p-2)-2b}{4}}\Big(\int |u|^2\Big)^{\frac{2p-(N(p-2)-2b)}{4}}.$$

The proof is complete. □

Remark. From the proof of Theorem 2.3 one can see that we require $u \in H_r^1$. It is still a question whether or not Theorem 2.3 holds for $u \in H^1(\mathbb{R}^N)$ in the case of $b > 0$.

3. Cauchy problem

In this section we use Theorem 2.3 to establish the existence of local and global solutions of (1.3). To this end, we introduce a subspace of H_r^1. Let $\omega > 0$ and $H_\omega = \{u \in H_r^1;\ \int |x|^2|u|^2 < +\infty\}$ with the norm $\|u\|_{H_\omega}^2 = \int(|\nabla u|^2+|x|^2|u|^2+\omega|u|^2)$. Clearly, H_ω is a Hilbert space whose inner product is defined by $\langle u, v\rangle = \Re\int(\nabla u\nabla\overline{v}+|x|^2u\overline{v}+\omega u\overline{v})$, where $\overline{v}$ is the complex conjugate of v and $\Re$ means taking real part.

Proposition 3.1. *Let $N \geqslant 2$, $b \geqslant 0$ and $2 + 2b/(N-1) < p < \tilde{p}$, where $\tilde{p}$ is defined in Theorem 2.3. For any $\varphi_0 \in H_\omega$ there is $T > 0$ and a unique solution φ of (1.3) with $\varphi \in C([0,T), H_\omega)$ and $\varphi(0) = \varphi_0$. Moreover, we have the conserved particle number*

$$\int |\varphi|^2 \equiv \int |\varphi_0|^2 \tag{3.1}$$

and the conserved energy

$$E(\varphi) = \frac{1}{2}\int (|\nabla\varphi|^2 + |x|^2|\varphi|^2) - \frac{1}{p}\int |x|^b|\varphi|^p \tag{3.2}$$

for all $t \in [0,T)$, where either $T = +\infty$ or $T < +\infty$ and $\lim\limits_{t\to T^-} \|\varphi\|_{H_\omega} = +\infty$.

P r o o f. The proof follows bythe standard method (see e.g. [6], [15], [21] and the references therein) with help of Theorem 2.3. □

Next, we have the following virial identity which originated from Glassey [16]

Proposition 3.2. *Let $N \geqslant 2$, $b \geqslant 0$ and $2 + 2b/(N-1) < p < \tilde{p}$, where $\tilde{p}$ is defined in Theorem 2.3. Let $\varphi(t) \in C^1([0,T(\varphi_0)), H_\omega)$ be a solution of Eq. (1.1) with initial condition $\varphi(0) = \varphi_0(x) \in H_\omega$. Denote*

$$h(t) = \|x\varphi(t)\|_2^2. \tag{3.3}$$

Then for $0 \leqslant t < T(\varphi_0)$ one has

$$h''(t) = 8\int \Big(|\nabla\varphi(t)|^2 - |x|^2|\varphi(t)|^2 - \frac{Np-2N-2b}{2p}|x|^b|\varphi(t)|^p\Big). \tag{3.4}$$

P r o o f. We only prove Eq. (3.4) formally. Since φ satisfies Eq. (1.3), we have

$$\varphi_t = \mathrm{i}(\Delta\varphi - |x|^2\varphi + |x|^b|\varphi|^{p-2}\varphi).$$

Therefore

$$h'(t) = 2\mathrm{Re}\int |x|^2\bar{\varphi}\varphi_t = 4\mathrm{Im}\int \bar{\varphi}x\nabla\varphi$$

and

$$\begin{aligned} h''(t) &= 4\mathrm{Im}\int (\bar{\varphi}_t x\nabla\varphi + \bar{\varphi}x\nabla\varphi_t) \\ &= 4\mathrm{Im}\int \bar{\varphi}_t x\nabla\varphi - 4\mathrm{Im}\int \varphi_t(N\bar{\varphi} + x\nabla\bar{\varphi}) \\ &= -4\mathrm{Im}\int \varphi_t(N\bar{\varphi} + 2x\nabla\bar{\varphi}) \\ &= -4\mathrm{Re}\int (N\bar{\varphi} + 2x\nabla\bar{\varphi})(\Delta\varphi - |x|^2\varphi + |x|^b|\varphi|^{p-2}\varphi). \end{aligned}$$

Recall that $\operatorname{div}(|x|^b x) = (N+b)|x|^b$ and by direct computations, one has

$$\begin{aligned}
\operatorname{Re}\int (N\bar\varphi + 2x\nabla\bar\varphi)\Delta\varphi &= -2\int |\nabla\varphi|^2;\\
\operatorname{Re}\int (N\bar\varphi + 2x\nabla\bar\varphi)|x|^2\varphi &= -2\int |x|^2|\varphi|^2;\\
\operatorname{Re}\int (N\bar\varphi + 2x\nabla\bar\varphi)|x|^b|\varphi|^{p-2}\varphi &= N\int |x|^b|\varphi|^p + \operatorname{Re}\int 2x|x|^b|\varphi|^{p-2}\varphi\nabla\bar\varphi\\
&= N\int |x|^b|\varphi|^p + \frac{2}{p}\int x|x|^b\nabla(|\varphi|^p)\\
&= N\int |x|^b|\varphi|^p - \frac{2}{p}\int |\varphi|^p(N|x|^b + b|x|^b)\\
&= \frac{N(p-2)-2b}{p}\int |x|^b|\varphi|^p.
\end{aligned}$$

Therefore

$$h''(t) = 8\Bigl(\int (|\nabla\varphi|^2 - |x|^2|\varphi|^2) - \frac{N(p-2)-2b}{2p}\int |x|^b|\varphi|^p\Bigr).$$

Proposition 3.3. *Let $N \geqslant 2$, $b \geqslant 0$, $2 + 2b/(N-1) < p < \tilde p$ and $\varphi_0 \in H_\omega$. If $2 + 2b/(N-1) < p < p_c$, then the existence time T obtained in Proposition* 3.1 *must be infinite; if $p = p_c$, then the existence time T is infinite for $\|\varphi_0\|_{L^2}$ sufficiently small; if $p_c < p < \tilde p$, then the existence time T is infinite for $\|\varphi_0\|_{H_\omega}$ sufficiently small.*

Proof. From Proposition 3.1 we know that for $\varphi_0 \in H_\omega$,

$$\int (|\nabla\varphi|^2 + |x|^2|\varphi|^2) = 2E(\varphi_0) + \frac{2}{p}\int |x|^b|\varphi|^p.$$

It then follows from Theorem 2.3 that

$$\int (|\nabla\varphi|^2 + |x|^2|\varphi|^2) \leqslant 2E(\varphi_0) + C\Bigl(\int |\nabla\varphi|^2\Bigr)^{\frac{Np-2N-2b}{4}}\Bigl(\int |\varphi|^2\Bigr)^{\frac{2N+2b+2p-Np}{4}}. \tag{3.5}$$

If $2 + 2b/(N-1) < p < p_c$, then an application of the Young inequality yields

$$\int (|\nabla\varphi|^2 + |x|^2|\varphi|^2) \leqslant 2E(\varphi_0) + \delta\int |\nabla\varphi|^2 + C_\delta\Bigl(\int |\varphi|^2\Bigr)^{\frac{2N+2b+2p-Np}{2N+2b+4-Np}}. \tag{3.6}$$

Choosing $\delta < 1$, we conclude from (3.1) and (3.6) that $\int (|\nabla\varphi|^2 + |x|^2|\varphi|^2)$ is controlled from above by a positive constant, which implies that $T = +\infty$. If $N > 1 + \frac{1}{2}b$ and $p = p_c$, we deduce from (3.5) that

$$\int (|\nabla\varphi|^2 + |x|^2|\varphi|^2) \leqslant 2E(\varphi_0) + C\int |\nabla\varphi|^2\Bigl(\int |\varphi_0|^2\Bigr)^{\frac{2+b}{N}}.$$

If $\|\varphi_0\|_{L^2}$ is sufficiently small, we obtain again that the existence time $T = +\infty$. Finally, if $p_c < p < \tilde{p}$, then again using (3.5), one gets that the existence time T is infinite for $\|\varphi_0\|_{H_\omega}$ sufficiently small. $\square$

Remark. When $p = p_c$, the expression "sufficiently small" appearing in Proposition 3.3 is vague. We will give a qualitative description on how a small φ_0 ensures the global existence of a solution to (1.3) below.

4. Ground-state solutions

The focus of this section is the standing-wave solutions to Eq. (1.3) of the form $\varphi(x,t) = \mathrm{e}^{\mathrm{i}\omega t}\varphi(x)$ ($\omega > 0$) with $\varphi \in H_\omega$, a real-valued function.

Substituting $\mathrm{e}^{\mathrm{i}\omega t}\varphi(x)$ into Eq. (1.3), we have the elliptic equation

$$-\Delta\varphi + |x|^2\varphi + \omega\varphi - |x|^b|\varphi|^{p-2}\varphi = 0, \quad \varphi \in H_w. \tag{4.1}$$

Define a functional J on H_ω by

$$J(u) = \frac{1}{2}\int\Big(|\nabla u|^2 + |x|^2|u|^2 + \omega|u|^2\Big) - \frac{1}{p}\int |x|^b|u|^p. \tag{4.2}$$

By Theorem 2.3, the functional J is a well-defined on H_ω when $b \geqslant 0, \omega > 0$, and $2+2b/(N-1) < p < \tilde{p}$ with $N \geqslant 2$. It is clear that there is one-to-one correspondence between the weak solution of (4.1) and the critical point of (4.2). Define another functional I in H_ω by

$$I(u) = \int(|\nabla u|^2 + |x|^2|u|^2 + \omega|u|^2 - |x|^b|u|^p). \tag{4.3}$$

Define the set $\mathcal{N} = \{u \in H_\omega\,;\ I(u) = 0, u \neq 0\}$ and

$$d_{\mathcal{N}} = \inf_{u\in\mathcal{N}} J(u). \tag{4.4}$$

Definition 4.1. We say that $\varphi \in H_\omega$ is a ground-state solution of Eq. (4.1) if $\varphi \neq 0$, $J'(\varphi) = 0$, $J(\varphi) = d_{\mathcal{N}}$ and $J(\varphi) \leqslant J(\psi)$ for any $\psi \in \{\psi \in H_\omega\,;\ J'(\psi) = 0\}$.

Theorem 4.2. *Assume $\omega > 0$, $N \geqslant 2$ and $b \geqslant 0$. If $2 + 2b/(N-1) < p < \tilde{p}$, then $d_{\mathcal{N}} > 0$ and $d_{\mathcal{N}}$ is achieved at a ground-state solution φ of Eq.* (4.1).

Remark. When $N \geqslant 3$, $\omega > 0$, $b \geqslant 0$ and $2 + 2b/(N-1) < p < 2N/(N-2) + 2b/(N-1)$, one can use Rother's inequality [23] to prove Theorem 4.2 directly following an argument of Sintzoff et al [25]. However, it seems that the method by Sintzoff et al [25] cannot be applied to the case of $N = 2$ since Rother's inequality holds only for $N \geqslant 3$.

Proof of Theorem 4.2. Due to the above remark, it suffices to prove this theorem in the case of $N = 2$. On the other hand, when $b = 0$, the existence of the ground state solution of Eq. (4.1) has been studied extensively, cf. [11]. So we only consider the case $b > 0$, $N = 2$ and $2 + 2b < p < +\infty$. The proof is divided into five steps.

Step 1. We prove that $d_{\mathcal{N}} > 0$. In fact, for any $u \in \mathcal{N}$ we have from Theorem 2.3 that

$$\|u\|_{H_\omega}^2 = \int |x|^b |u|^p \leqslant C\Big(\int |\nabla u|^2\Big)^{\frac{p-2-b}{2}} \Big(\int |u|^2\Big)^{\frac{2+b}{2}} \leqslant C_1 \|u\|_{H_\omega}^p.$$

So $\|u\|_{H_\omega} \geqslant C_2 > 0$ and $J(u) = (\frac{1}{2} - 1/p)\|u\|_{H_\omega}^2 \geqslant C_3 > 0$. It then turns out that $d_{\mathcal{N}} > 0$.

Step 2. Let $\{u_n\} \subset \mathcal{N}$ be a minimizing sequence of $d_{\mathcal{N}}$. We obtain from Ekeland's variational principle [31, Page 39, Theorem 2.4] that there is $\{\varphi_n\} \subset \mathcal{N}$ such that:

$$J(\varphi_n) \leqslant d_{\mathcal{N}} + \frac{1}{n^2}, \quad \|J'(\varphi_n)\|_{(H_\omega)^*} < \frac{1}{n}, \quad \|\varphi_n - u_n\|_{H_\omega} < \frac{1}{n}.$$

Therefore

$$J(\varphi_n) \to d_{\mathcal{N}}, \quad J'(\varphi_n) \to 0, \qquad n \to +\infty. \tag{4.5}$$

From (4.5) we know that for n large

$$d_{\mathcal{N}} + o(1) = \Big(\frac{1}{2} - \frac{1}{p}\Big)\|\varphi_n\|_{H_\omega}^2, \tag{4.6}$$

which implies that $\{\varphi_n\}$ is bounded in H_ω. Going if necessary to a subsequence, still denoted by $\{\varphi_n\}$, we may assume that $\varphi_n \rightharpoonup \varphi$ weakly in H_ω and $\varphi_n \to \varphi$ a.e. in $\mathbb{R}^N$. Hence for any $\psi \in H_\omega$ we have

$$\int (\nabla\varphi_n \nabla\psi + |x|^2 \varphi_n \psi + \omega\varphi_n\psi) \to \int (\nabla\varphi\nabla\psi + |x|^2\varphi\psi + \omega\varphi\psi).$$

Moreover, from $|x|^b|\varphi_n|^{p-2}\varphi_n \to |x|^b|\varphi|^{p-2}\varphi$ a.e. in $\mathbb{R}^N$ and the Lebesgue dominated convergence theorem we get that

$$\int |x|^b|\varphi_n|^{p-2}\varphi_n\psi \to \int |x|^b|\varphi|^{p-2}\varphi\psi.$$

Consequently,

$$\langle J'(\varphi_n), \psi\rangle \to \int (\nabla\varphi\nabla\psi + |x|^2\varphi\psi + \omega\varphi\psi - |x|^b|\varphi|^{p-2}\varphi\psi),$$

which implies that $J'(\varphi) = 0$.

Step 3. We prove that $\varphi \neq 0$, arguing by a contradiction. Assume that $\varphi = 0$, i.e., $\varphi_n \rightharpoonup 0$ weakly in H_ω. We claim that $\int |x|^b|\varphi_n|^p \to 0$ $(n \to +\infty)$. Indeed, for any $\varepsilon > 0$, we write

$$\int |x|^b|\varphi_n|^p = \Big(\int_{|x|\leqslant\varepsilon} + \int_{\varepsilon\leqslant|x|\leqslant 1/\varepsilon} + \int_{|x|\geqslant 1/\varepsilon}\Big)|x|^b|\varphi_n|^p.$$

We first consider the term $\int_{|x|\leqslant\varepsilon}|x|^b|\varphi_n|^p$. Since $b > 0$, we obtain from the standard Gagliardo-Nirenberg inequality that

$$\begin{aligned}\int_{|x|\leqslant\varepsilon}|x|^b|\varphi_n|^p &\leqslant \varepsilon^b\int_{|x|\leqslant\varepsilon}|\varphi_n|^p\\ &\leqslant C\varepsilon^b\Big(\int|\nabla\varphi_n|^2\Big)^{\frac{p-2}{2}}\Big(\int|\varphi_n|^2\Big)\\ &\leqslant C_1\varepsilon^b,\end{aligned}\tag{4.7}$$

where use has been made of the fact that $\{\varphi_n\}$ is bounded in H_ω. It follows that $\int_{|x|\leqslant\varepsilon}|x|^b|\varphi_n|^p$ tends to 0 uniformly in n as $\varepsilon \to 0$. On the other hand, as $\{\varphi_n\}$ is bounded in H_ω and $N = 2$, we deduce from Strauss' inequality that

$$\begin{aligned}\int_{|x|\geqslant 1/\varepsilon}|x|^b|\varphi_n|^p &= \int_{|x|\geqslant 1/\varepsilon}|x|^{b-2-\frac{p-2}{2}}|x|^{\frac{p-2}{2}}|\varphi_n|^{p-2}|x|^2|\varphi_n|^2\\ &\leqslant C\Big(\int|\nabla\varphi_n|^2\Big)^{\frac{p-2}{4}}\Big(\int|\varphi_n|^2\Big)^{\frac{p-2}{4}}\int_{|x|\geqslant 1/\varepsilon}|x|^{b-2-\frac{p-2}{2}}|x|^2|\varphi_n|^2\\ &\leqslant C\Big(\frac{1}{\varepsilon}\Big)^{b-2-\frac{p-2}{2}}.\end{aligned}\tag{4.8}$$

It follows from $b - 2 - \frac{1}{2}(p-2) < 0$ and $\{\varphi_n\}$ being bounded in H_ω that the above integral tends to 0 uniformly in n as $\varepsilon \to 0$.

For the term $\int_{\varepsilon\leqslant|x|\leqslant 1/\varepsilon}|x|^b|\varphi_n|^p$, because $\{\varphi_n\}$ is bounded in H_ω, we have first from the Rellich compact embedding theorem that

$$\int_{\varepsilon\leqslant|x|\leqslant\frac{1}{\varepsilon}}|\varphi_n|^2\to 0\quad\text{as } n\to+\infty.$$

Using Strauss' inequality we get

$$\int_{\varepsilon\leqslant|x|\leqslant 1/\varepsilon}|x|^b|\varphi_n|^p\leqslant C|\varphi_n|_\infty^{p-2}\int_{\varepsilon\leqslant|x|\leqslant 1/\varepsilon}|\varphi_n|^2\to 0\quad\text{as } n\to+\infty. \tag{4.9}$$

Since ε is arbitrary, we obtain from (4.7), (4.8) and (4.9) that $\int|x|^b|\varphi_n|^p\to 0$ as $n\to+\infty$. However, Step 1 and Step 2 imply that there is a positive constant C_0 such that $\int|x|^b|\varphi_n|^p=\|\varphi_n\|_{H_\omega}^2\geqslant C_0$. This is a contradiction. Hence, we conclude that $\varphi\neq 0$.

Step 4. The value $d_{\mathcal{N}}$ is achieved at φ. Indeed, from Step 2 and Step 3, we have that $\varphi\in\mathcal{N}$. Now for n large,

$$\begin{aligned}d_{\mathcal{N}}+o(1)=J(\varphi_n)&=\frac{1}{2}\|\varphi_n-\varphi\|_{H_\omega}^2+\frac{1}{2}\|\varphi\|_{H_\omega}-\frac{1}{p}\int|x|^b|\varphi|^p+o(1)\\&\geqslant J(\varphi)\geqslant d_{\mathcal{N}}.\end{aligned}$$

It follows that $J(\varphi)=d_{\mathcal{N}}$.

Step 5. We prove that φ is a ground state solution of Eq. (4.1). Indeed, from the previous arguments we know that $\varphi\neq 0$, $J'(\varphi)=0$ and $J(\varphi)=d_{\mathcal{N}}$. Now for any nonzero ψ satisfying $J'(\psi)=0$ we have that $I(\psi)=\langle J'(\psi),\psi\rangle=0$. So $\psi\in\mathcal{N}$. The definition of $d_{\mathcal{N}}$ implies that $J(\varphi)\leqslant J(\psi)$. This completes the proof of the theorem. $\square$

5. Invariant sets and applications

Having established local well-posedness for the initial-value problem under study and obtained the existence of ground-state solutions for Eq. (4.1), attention is paid to whether the locally defined solution can be extended to the entire time interval.

To this end, we construct several cross-invariant sets under the flow generated by Eq. (1.3). Using these cross-invariant sets enables us to establish a criterion for global existence and blow-up of solutions to Eq. (1.3).

From now on, we always assume that $N\geqslant 2$ and $\omega>0$.

Define a functional on H_ω by

$$Q(u) = \int \Big(|\nabla u|^2 - |x|^2|u|^2 - \frac{Np - 2N - 2b}{2p} |x|^b |u|^p \Big).$$

Let

$$d_{\mathcal{M}} = \inf\{J(u);\ u \in \mathcal{M}\},$$

where

$$\mathcal{M} = \{u \in H_\omega;\ I(u) < 0,\ Q(u) = 0\}.$$

To construct the cross-invariant sets, we need a series of lemmas.

Lemma 5.1. *If $d_{\mathcal{N}}$ is achieved for φ, then $Q(\varphi) = 0$.*

P r o o f. Let $\varphi^\eta(x) = \eta^{\frac{N}{2}} \varphi(\eta x)$. Since $\varphi \in H_\omega$ is a ground-state solution of Eq. (4.1), we have $\partial_\eta \varphi^\eta \in H_\omega$ and $\partial_\eta J(\varphi^\eta)\big|_{\eta=1} = \big\langle J'(\varphi), \partial_\eta \varphi^\eta\big|_{\eta=1}\big\rangle = 0$. Note that

$$J(\varphi^\eta) = \int \Big(\frac{\eta^2}{2} |\nabla \varphi|^2 + \frac{\eta^{-2}}{2} |x|^2 |\varphi|^2 + \frac{\omega}{2} |\varphi|^2 - \frac{1}{p} \eta^{\frac{Np-2b-2N}{2}} |x|^b |\varphi|^p \Big).$$

We obtain $Q(\varphi) = \partial_\eta J(\varphi^\eta)\big|_{\eta=1} = 0$. $\square$

Lemma 5.2. *Assume $N \geqslant 2$ and $\omega > 0$. If $b \geqslant 0$, then $\mathcal{M}$ is not empty provided one of the following assumptions holds.*

(A1) *$N > 1 + \frac{1}{2}b$ and $p_c \leqslant p < \tilde{p}$;*

(A2) *$N \leqslant 1 + \frac{1}{2}b$ and $2 + 2b/(N-1) < p < \tilde{p}$.*

Remark. Note that when $N \leqslant 1 + \frac{1}{2}b$, then $2 + 2b/(N-1) \geqslant p_c$ holds. So $p > 2 + 2b/(N-1)$ implies that $p > p_c$. Hence we only need to prove Lemma 5.2 under assumption (A1).

P r o o f of Lemma 5.2. Assume (A1). Let φ be a minimizer of $d_{\mathcal{N}}$. It follows from Lemma 5.1 that $Q(\varphi) = 0$, which implies that

$$\int |\nabla \varphi|^2 - \frac{Np - 2N - 2b}{2p} \int |x|^b |\varphi|^p = \int |x|^2 |\varphi|^2 > 0.$$

Choosing ξ_0 such that

$$1 < \xi_0 < \Big(\int |\nabla \varphi|^2 \Big/ \frac{Np - 2N - 2b}{2p} \int |x|^b |\varphi|^p \Big)^{\frac{1}{p-2}}$$

and denoting $v = \xi_0 \varphi$, we get from $I(\varphi) = 0$ and $Q(\varphi) = 0$ that

$$I(v) < 0 \quad \text{and} \quad Q(v) < 0.$$

Let $v_\mu(x) = \mu^{(2+b)/(p-2)} v(\mu x)$. We then have

$$I(v_\mu) = \mu^{\frac{2(N+b)-(N-2)p}{p-2}} \int \Big(|\nabla v|^2 - |x|^b |v|^p \Big) + \omega \mu^{\frac{2(2+b)-N(p-2)}{p-2}} \int |v|^2 + \mu^{\frac{8+2N+2b-(N+2)p}{p-2}} \int |x|^2 |v|^2$$

and

$$Q(v_\mu) = \mu^{\frac{2(N+b)-(N-2)p}{p-2}} \int \Big(|\nabla v|^2 - \frac{N(p-2)-2b}{2p} |x|^b |v|^p \Big) - \mu^{\frac{8+2N+2b-(N+2)p}{p-2}} \int |x|^2 |v|^2.$$

Since $p \geqslant p_c$, we infer that $Q(v_\mu) \to Q(v) < 0 \quad \text{as} \quad \mu \to 1$. On the other hand, from the choice of $v = \xi_0 \varphi$ we know that

$$\int \Big(|\nabla v|^2 - \frac{Np-2N-2b}{2p} |x|^b |v|^p \Big) > 0.$$

In view of $2(N+b)-(N-2)p > 0$ and $8+2N+2b-(N+2)p < 0$, it is easy to see that

$$Q(v_\mu) \to +\infty \quad \text{as} \quad \mu \to +\infty.$$

It then turns out that there is $\mu_* > 1$ such that $Q(v_{\mu_*}) = 0$.

Now we turn to $I(v_{\mu_*})$. Since $\mu_* > 1$ and $2(2+b)-N(p-2) < 2(N+b)-(N-2)p$ and $8+2N+2b-(N+2)p < 2(N+b)-(N-2)p$, it implies that

$$I(v_{\mu_*}) < \mu_*^{\frac{2(N+b)-(N-2)p}{p-2}} I(v) < 0.$$

Hence $v_{\mu_*} \in \mathcal{M}$. The proof of Lemma 5.2 is complete. $\square$

Lemma 5.3. *Under the same assumptions as in Lemma* 5.2, $d_{\mathcal{M}} > 0$.

Proof. From the remark to Lemma 5.2, we only need to prove this lemma under assumption (A1). To this end, we first treat the case $p_c < p < \tilde{p}$. For any $u \in \mathcal{M}$, we have from $I(u) < 0$ and Theorem 2.3 that

$$\|u\|_{H_\omega}^2 < \int |x|^b |u|^p \leqslant C \Big(\int |\nabla u|^2 \Big)^{\frac{Np-2N-2b}{4}} \Big(\int |u|^2 \Big)^{\frac{2N+2b+2p-Np}{4}} \leqslant C \|u\|_{H_\omega}^p.$$

Since $p > 2$, it follows that $\|u\|_{H_\omega} \geqslant C_0 > 0$. In view of $Q(u) = 0$ and $p > p_c$, we deduce that

$$J(u) \geqslant \Big(\frac{1}{2} - \frac{2}{Np - 2N - 2b}\Big)\|u\|_{H_\omega}^2 \geqslant C > 0.$$

Thus we have that $d_{\mathcal{M}} > 0$ for $p_c < p < \tilde{p}$.

Next, we consider the case $p = p_c$. Arguing by a contradiction, if $d_{\mathcal{M}} = 0$ then there is a sequence $\{u_n\} \subset \mathcal{M}$ such that

$$Q(u_n) = 0, \qquad I(u_n) < 0 \quad \text{for all } n$$

and $J(u_n) \to 0$ as $n \to \infty$. Since $p = p_c$ and

$$J(u_n) = \int (|x|^2|u_n|^2 + \frac{\omega}{2}|u_n|^2),$$

it follows from $J(u_n) \to 0$ that $\int |x|^2|u_n|^2 \to 0$ and $\int |u_n|^2 \to 0$. Again using Theorem 2.3 yields

$$\int |x|^b|u_n|^p \leqslant C_0 \int |\nabla u_n|^2 \Big(\int |u_n|^2\Big)^{\frac{p-2}{2}}$$

for all $n \geqslant 1$, where C_0 is independent of u. On the other hand, in view of $I(u_n) < 0$ and Theorem 2.3 and by the fact that $\int |x|^2|u_n|^2 \to 0$ and $\int |u_n|^2 \to 0$ we get that for C_0 as above and a large n

$$\int |x|^b|u_n|^p > \|u_n\|_{H_\omega}^2 > C_0 \int |\nabla u_n|^2 \Big(\int |u_n|^2\Big)^{\frac{p-2}{2}}.$$

This is a contradiction. The proof of Lemma 5.3 is complete. □

5.1. Cross-invariant sets.

A set S is said to be invariant under the flow generated by Eq. (1.3) if $\varphi(t) \in S$ for $t \in [0, T)$ as long as $\varphi_0 \in S$. Denote $\min\{d_{\mathcal{N}}, d_{\mathcal{M}}\} := d > 0$. Let

$$\begin{aligned}
\mathcal{K}_- &= \{u \in H_\omega;\ J(u) < d,\ I(u) < 0,\ Q(u) < 0\},\\
\mathcal{K}_+ &= \{u \in H_\omega;\ J(u) < d,\ I(u) < 0,\ Q(u) > 0\},\\
\mathcal{S}_- &= \{u \in H_\omega;\ J(u) < d,\ I(u) < 0\},
\end{aligned}$$

and

$$\mathcal{S}_+ = \{u \in H_\omega;\ J(u) < d,\ I(u) > 0\}.$$

Proposition 5.4. *Assume $N \geqslant 2$, $b \geqslant 0$ and $\omega > 0$. If assumption* (A1) *or* (A2) *is satisfied, then $\mathcal{K}_{\mp}$ as well as $\mathcal{S}_{\mp}$ are invariant sets under the flow generated by Eq.* (1.3). *We refer to $\mathcal{K}_{\mp}$ and $\mathcal{S}_{\mp}$ as the cross-invariant sets under the flow of* (1.3).

Proof. Similarly to the proof of Lemma 5.2, it suffices to prove this proposition under assumption (A1). We only prove that $\mathcal{K}_-$ is an invariant set since the proof of the other cases is similar. Let $\varphi(t)$ be the solution of (1.3) with initial data $\varphi_0 \in \mathcal{K}_-$. In view of the conserved identities $\int |\varphi|^2 \equiv \int |\varphi_0|$ and

$$E(\varphi) = \int \Big(\frac{1}{2}(|\nabla\varphi|^2 + |x|^2|\varphi|^2) - \frac{1}{p}|x|^b|\varphi|^p\Big) \equiv E(\varphi_0),$$

we get immediately that $J(\varphi) \equiv J(\varphi_0)$. Thus $J(\varphi) < d$.

We now claim that $I(\varphi(t)) < 0$ for $t \in [0, T)$. If this were not true, then by the continuity there would be a $t_0 \in (0, T)$ such that $I(\varphi(t_0)) = 0$. Then, since $\varphi(t_0) \neq 0$, we know that $J(\varphi(t_0)) \geqslant d_{\mathcal{N}} \geqslant d$. This contradicts the inequality $J(\varphi(t)) < d$ for all $t \in (0, T)$. Therefore $I(\varphi(t)) < 0$ for $t \in [0, T)$.

Finally, we show $Q(\varphi(t)) < 0$ for $t \in [0, T)$. If this were not true, then by the continuity there would be a $t_1 \in (0, T)$ such that $Q(\varphi(t_1)) = 0$. Since $I(\varphi(t_1)) < 0$, we have that $\varphi(t_1) \in \mathcal{M}$. This implies that $J(\varphi(t_1)) \geqslant d_{\mathcal{M}} \geqslant d$, which contradicts the inequality $J(\varphi(t)) < d$ $\forall t \in [0, T)$. Consequently, $Q(\varphi(t)) < 0$ $\forall t \in [0, T)$. $\square$

5.2. Sharp global existence.

Theorem 5.5. *Assume that $N \geqslant 2$, $b \geqslant 0$, $\omega > 0$, and $2 + 2b/(N-1) < p < \tilde{p}$. If $\varphi_0 \in \mathcal{K}_+ \cup \mathcal{S}_+$, then the solution $\varphi(t)$ of Eq.* (1.3) *with initial data φ_0 exists globally in time.*

Proof. In view of Proposition 3.3, we only need to prove this theorem in the case that $b \geqslant 0$ and assumption (A1) or (A2) holds. Moreover, from the remark after Lemma 5.2, it suffices to prove this theorem under assumption (A1). Due to the presence of inhomogeneous nonlinearity, we need a quite different scaling argument. First, let $\varphi_0 \in \mathcal{K}_+$. Proposition 5.4 implies that the solution $\varphi(t)$ of (1.3) belongs to $\mathcal{K}_+$ for any $t \in [0, T)$. Now for any fixed $t \in [0, T)$ we have $J(\varphi) < d$ and $Q(\varphi) > 0$. It follows from the expressions of J and Q that

$$\int \Big(\frac{Np-2N-2b-4}{2(Np-2N-2b)}|\nabla\varphi|^2 + \frac{Np-2N-2b+4}{2(Np-2N-2b)}|x|^2|\varphi|^2 + \frac{\omega}{2}|\varphi|^2\Big) < d. \tag{5.1}$$

Now we treat the case $p = p_c$. In this case, relation (5.1) implies

$$\int \Big(|x|^2|\varphi|^2 + \frac{\omega}{2}|\varphi|^2\Big) < d. \tag{5.2}$$

Put $\varphi_\mu = \mu^{(N+b)/p}\varphi(\mu x)$. Then using $p = p_c$ we get

$$(5.3)\quad Q(\varphi_\mu) = \mu^{\frac{4+2b}{N+2+b}}\int|\nabla\varphi|^2 - \mu^{-2-\frac{2N}{N+2+b}}\int|x|^2|\varphi|^2 - \int\frac{Np-2N-2b}{2p}|x|^b|\varphi|^p.$$

It follows from $Q(\varphi) > 0$ and $Q(\varphi_\mu) < 0$ for μ small enough that there exists $0 < \mu_1 < 1$ such that $Q(\varphi_{\mu_1}) = 0$. Again using the expressions for J and Q, we have

$$(5.4)\quad J(\varphi_{\mu_1}) = \int(|x|^2|\varphi_{\mu_1}|^2 + \frac{\omega}{2}|\varphi_{\mu_1}|^2)\,\mathrm{d}x = \mu_1^{-2-\frac{2N}{N+2+b}}\int|x|^2|\varphi|^2 + \mu_1^{\frac{-2N}{N+2+b}}\int\frac{\omega}{2}|\varphi|^2.$$

Since $0 < \mu_1 < 1$ and $-2 - 2N/(N+2+b) < -2N/(N+2+b)$, it follows from (5.2) that

$$(5.5)\quad J(\varphi_{\mu_1}) < \mu_1^{-2-\frac{2N}{N+2+b}}d.$$

Next, we turn to $I(\varphi_{\mu_1})$, which has two possibilities. In the case $I(\varphi_{\mu_1}) < 0$, the equality $Q(\varphi_{\mu_1}) = 0$ and Lemma 5.3 imply that

$$(5.6)\quad J(\varphi_{\mu_1}) \geqslant d_{\mathcal{M}} \geqslant d > J(\varphi).$$

It follows that $J(\varphi) - J(\varphi_{\mu_1}) < 0$, that is

$$(5.7)\quad \left(1-\mu_1^{\frac{4+2b}{N+2+b}}\right)\int\frac{1}{2}|\nabla\varphi|^2 + \left(1-\mu_1^{-2-\frac{2N}{N+2+b}}\right)\int\frac{1}{2}|x|^2|\varphi|^2 + \left(1-\mu_1^{\frac{-2N}{N+2+b}}\right)\int\frac{\omega}{2}|\varphi|^2 < 0.$$

This implies that $\int|\nabla\varphi|^2 < C\left(\int|\varphi|^2 + \int|x|^2|\varphi|^2\right)$. Combining this estimate with (5.2), we obtain

$$(5.8)\quad \int|\nabla\varphi|^2 < C.$$

For the other case of $I(\varphi_{\mu_1}) \geqslant 0$, we have from (5.5) that

$$(5.9)\quad J(\varphi_{\mu_1}) - \frac{1}{p}I(\varphi_{\mu_1}) < \mu_1^{-2-\frac{2N}{N+2+b}}d.$$

It follows that

$$(5.10)\quad \mu_1^{\frac{4+2b}{N+2+b}}\int|\nabla\varphi|^2 + \mu_1^{-2-\frac{2N}{N+2+b}}\int|x|^2|\varphi|^2 + \mu_1^{\frac{-2N}{N+2+b}}\int\omega|\varphi|^2 < \frac{2p}{p-2}\,\mu_1^{-2-\frac{2N}{N+2+b}}d.$$

Thus

$$(5.11)\quad \int|\nabla\varphi|^2 < C.$$

It follows from (5.8) and (5.11) that in the case $p = p_c$, $\int |\nabla\varphi|^2$ is bounded for all $t \in [0, T)$. Therefore, it follows from (5.2) and Proposition 3.1 that $\varphi(t)$ exists globally in time.

For the case $p_c < p < \tilde{p}$, we also get from (5.1) that

$$\int (|\nabla\varphi|^2 + |x|^2|\varphi|^2) < C. \tag{5.12}$$

Proposition 3.1 again implies that $\varphi(t)$ exists globally in time.

Up to now, we have proved that for $\varphi_0 \in \mathcal{K}_+$, Theorem 5.5 holds. The proof of $\varphi_0 \in \mathcal{S}_+$ is similar but quite simpler. We omit the details. The proof of Theorem 5.5 is complete. □

Now we are in a position to give a qualitative answer to how a small φ_0 can ensure the existence of global solutions (1.3) in the case of $p = 2 + (4 + 2b)/N$.

Corollary 5.6. *Let $\varphi_0 \in H_\omega$, $\int(|\nabla\varphi_0|^2 + |x|^2|\varphi_0|^2 + \omega|\varphi_0|^2) < 2d$, $b \geqslant 0$, $N > 1 + \frac{1}{2}b$ and $p = p_c$. Then the solution $\varphi(t)$ of* (1.3) *with initial data φ_0 exists globally in time.*

Proof. Since $\int(|\nabla\varphi_0|^2 + |x|^2|\varphi_0|^2 + \omega|\varphi_0|^2) < 2d$, we know that $J(\varphi_0) < d$. Moreover, we claim that $I(\varphi_0) > 0$. Otherwise, there is a $\lambda \in (0, 1]$ such that $I(\lambda\varphi_0) = 0$. Thus $J(\lambda\varphi_0) \geqslant d$. On the other hand,

$$\int \lambda^2(|\nabla\varphi_0|^2 + |x|^2|\varphi_0|^2 + \omega|\varphi_0|^2) < 2\lambda^2 d \leqslant 2d.$$

It follows that $J(\lambda\varphi_0) < d$. This is a contradiction. Therefore we have $\varphi_0 \in \mathcal{S}_+$. The corollary follows from Theorem 5.5. □

5.3. Blow-up solutions.

In this subsection we will use the cross-invariant sets constructed in Section 5.1 to obtain a blow-up result for Eq. (1.3) with suitable initial data φ_0. The idea has been previously introduced in [3], [24], [19], [32], but we need a rather different scaling argument due to the unbounded inhomogeneity of nonlinearity.

Theorem 5.7. *Assume $\omega > 0$, $b \geqslant 0$, $N \geqslant 2$ and let assumption* (A1) *or* (A2) *hold. If $\varphi_0 \in \mathcal{K}_-$, then the solution $\varphi(t)$ of Eq.* (1.3) *with initial data φ_0 blows up in finite time, i.e. there is a $T > 0$ such that*

$$\lim_{t \to T^-} |\nabla\varphi(t)|_2 = +\infty. \tag{5.13}$$

P r o o f. As we mentioned before, it suffices to prove this theorem only under assumption (A1). By the previous argument, we always have $\varphi(t) \in \mathcal{K}_-$ as long as $\varphi_0 \in \mathcal{K}_-$. For any fixed $t \in [0,T)$, denote $u_\lambda(x) = \lambda^{(N+b)/p}\varphi(\lambda x, t)$ and $u(\cdot) = \varphi(\cdot, t)$. A direct calculation shows that

$$I(u_\lambda) = \int (\lambda^{\frac{2N+2b+2p-Np}{p}}|\nabla u|^2 + \lambda^{\frac{2N+2b-pN-2p}{p}}|x|^2|u|^2 + \omega\lambda^{\frac{2N+2b-Np}{p}}|u|^2 - |x|^b|u|^p)\,\mathrm{d}x \tag{5.14}$$

and

$$Q(u_\lambda) = \int \Big(\lambda^{\frac{2N+2b+2p-Np}{p}}|\nabla u|^2 - \lambda^{\frac{2N+2b-pN-2p}{p}}|x|^2|u|^2 - \frac{Np-2N-2b}{2p}|x|^b|u|^p\Big)\,\mathrm{d}x. \tag{5.15}$$

Since $2N+2b+2p-Np > 0$ and $2N+2b-pN-2p < 0$, we have that $Q(u_\lambda) \to +\infty$ as $\lambda \to +\infty$. On the other hand, $Q(u_\lambda) \to Q(u) < 0$ as $\lambda \to 1$. It then follows from the fact that $Q(u_\lambda)$ is continuous for $\lambda > 1$ that there is $\lambda^* > 1$ such that $Q(u_{\lambda^*}) = 0$ and when $\lambda \in [1, \lambda^*)$, then $Q(u_\lambda) < 0$. For $\lambda \in [1, \lambda^*)$, since $I(u) < 0$, there are two possibilities for the sign of $I(u_\lambda)$:

(i) $I(u_\lambda) < 0$ for all $\lambda \in [1, \lambda^*]$, or

(ii) $\exists\ \lambda_1$ with $1 < \lambda_1 \leqslant \lambda^*$ such that $I(u_{\lambda_1}) = 0$.

In the case (i), we have that $u_{\lambda^*} \in \mathcal{M}$. We conclude that $J(u_{\lambda^*}) \geqslant d_{\mathcal{M}} \geqslant d$. Moreover, we have from $\lambda^* > 1$ and the assumption on p that

$$\begin{aligned} J(u) - J(u_{\lambda^*}) &= \int \frac{1}{2}\big(\big(1 - \lambda^{*\frac{2N+2b+2p-Np}{p}}\big)|\nabla u|^2 + \big(1 - \lambda^{*\frac{2N+2b-pN-2p}{p}}\big)|x|^2|u|^2 \\ &\quad + \omega\big(1 - \lambda^{*\frac{2N+2b-Np}{p}}\big)|u|^2\big) \\ &\geqslant \frac{1}{2}\int \big(\big(1 - \lambda^{*\frac{2N+2b+2p-Np}{p}}\big)|\nabla u|^2 - \big(1 - \lambda^{*\frac{2N+2b-pN-2p}{p}}\big)|x|^2|u|^2\big)\,\mathrm{d}x \\ &= \frac{1}{2}(Q(u) - Q(u_{\lambda^*})) \geqslant \frac{1}{2}Q(u). \end{aligned} \tag{5.16}$$

In the case (ii), we have $I(u_{\lambda_1}) = 0$ and $Q(u_{\lambda_1}) \leqslant 0$, i.e. $u_{\lambda_1} \in \mathcal{N}$. So $J(u_{\lambda_1}) \geqslant d_{\mathcal{N}} \geqslant d$ and a similar computation shows that

$$J(u) - J(u_{\lambda_1}) \geqslant \frac{1}{2}(Q(u) - Q(u_{\lambda_1})) \geqslant \frac{1}{2}Q(u). \tag{5.17}$$

In both cases, we have

$$Q(u) < 2(J(u) - d).$$

Notice that for $V(t) = \frac{1}{2}\int |x|^2|\varphi|^2\,\mathrm{d}x$ we have from Proposition 3.2

$$V''(t) = 4Q(\varphi).$$

Now using $J(\varphi) = J(\varphi_0)$ and $\varphi_0 \in \mathcal{K}_-$, we obtain that

$$V''(t) < 8(J(\varphi) - d) = 8(J(\varphi_0) - d) := \delta_0 < 0. \tag{5.18}$$

Thus, combining the above estimate of V with $V(0) = \frac{1}{2}\int |x|^2|\varphi_0|^2\,\mathrm{d}x$, we deduce that there exists a $T_0 > 0$ such that

$$\lim_{t\to T_0^-} V(t) = 0. \tag{5.19}$$

Observing that

$$\int |\varphi|^2 \leqslant C\Big(\int |x|^2|\varphi|^2\,\mathrm{d}x\Big)^{\frac{1}{2}}\Big(\int |\nabla\varphi|^2\,\mathrm{d}x\Big)^{\frac{1}{2}},$$

we obtain from the conservation of particle number, i.e., $\int |\varphi|^2 \equiv \int |\varphi_0|^2 > 0$, that there is $T > 0$ such that (5.13) holds. The proof of Theorem 5.7 is complete. □

6. Strong instability of standing-wave solutions

In the present section, attention is now turned to proving the strong instability of standing waves of Eq. (1.3) via blow-up. Recall from Section 4 that, for any $\omega > 0$, $\mathrm{e}^{\mathrm{i}\omega t}\varphi$ is a standing-wave solution of Eq. (1.3).

Definition 6.1. A standing-wave solution $\mathrm{e}^{\mathrm{i}\omega t}\varphi$ is called H_ω-strongly unstable with respect to (1.3) if for any $\delta > 0$ there is $\varphi_0 \in H_\omega$ satisfying

$$\|\varphi_0 - \varphi\|_{H_\omega} < \delta,$$

but the solution $\varphi(t)$ of Eq. (1.3) with initial data $\varphi(x, 0) = \varphi_0$ blows up at finite time, i.e., there is a $T > 0$ such that

$$\lim_{t\to T^-} |\nabla\varphi|_2 = +\infty. \tag{6.1}$$

Theorem 6.2. *Assume that $N \geqslant 2$ and $b \geqslant 0$. If assumption* (A1) *or* (A2) *holds and $\omega > 0$ is such that $d_{\mathcal{M}} \geqslant d_{\mathcal{N}}$, then for the minimizer $\varphi \in H_\omega$ of $d_{\mathcal{N}}$, the standing-wave solution $\varphi(t,x) = \mathrm{e}^{\mathrm{i}\omega t}\varphi(x)$ of* (1.3) *is H_ω-strongly unstable.*

Proof. Again, we only need to consider assumption (A1). Since φ is the minimizer of $d_{\mathcal{N}}$ and $I(\varphi) = 0$, Lemma 5.1 implies that

$$\int |\nabla\varphi|^2 - \int |x|^2|\varphi|^2 = \frac{Np - 2N - 2b}{2p} \int |x|^b|\varphi(x)|^p > 0 \tag{6.2}$$

and $J(\varphi) = d_{\mathcal{N}} = d$. By the assumption on p, we deduce from $I(\varphi) = 0$ that

$$J(\lambda\varphi) < J(\varphi) = d \qquad \text{for any} \quad \lambda > 1.$$

Moreover, using the fact that $I(\varphi) = 0$ and (6.2) holds, we deduce that for any $\lambda > 1$,

$$I(\lambda\varphi) = (\lambda^2 - \lambda^p) \int (|\nabla\varphi|^2 + |x|^2|\varphi|^2 + \omega|\varphi|^2) < 0$$

and

$$Q(\lambda\varphi) = (\lambda^2 - \lambda^p) \int (|\nabla\varphi|^2 - |x|^2|\varphi|^2)\,\mathrm{d}x < 0,$$

which implies that $\lambda\varphi \in \mathcal{K}_-$ for $\lambda > 1$. Now for any $\delta > 0$ we take $\varphi_0 = \lambda\varphi$ with $\lambda > 1$ such that

$$\|\varphi_0 - \varphi\|_{H_\omega} < \delta. \tag{6.3}$$

Since $\varphi \in H_\omega$ is a solution of Eq. (4.1) and has an exponential decay at infinity, we know from Theorem 5.7 that the solution $\varphi(t)$ of Eq. (1.3) with initial data $\varphi_0 = \lambda\varphi$ blows up in finite time. The proof of Theorem 6.2 is complete. $\square$

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References

[1] *G. Baym and C. J. Pethick*: Ground state properties of magnetically trapped Bose-condensed rubidium gas. Phys. Rev. Lett. *76* (1996), 6–9.
[2] *T. B. Benjamin*: The stability of solitary waves. Proc. Royal Soc. London, Ser. A. *328* (1972), 153–183.
[3] *H. Berestycki and T. Cazenave*: Instabilité des états stationnaires dans les équations de Schrödinger et de Klein-Gordon non linéarires. C. R. Acad. Sci. Paris I *293* (1981), 489–492.
[4] *J. L. Bona*: On the stability theory of solitary waves. Proc. Royal Soc. London, Ser. A. *344* (1975), 363–374.

[5] *L. Caffarelli, R. Kohn and L. Nirenberg*: First order interpolation inequalities with weights. Compositio Math. *53* (1984), 259–275.
[6] *T. Cazenave*: An Introduction to Nonlinear Schrödinger Equations. Textos de Metodos Matematicos, 22, Rio de Janeiro, 1989.
[7] *T. Cazenave and P. L. Lions*: Orbital satbility of standing waves for some nonlinear Schrödinger equations. Comm. Math. Phys. *85* (1982), 549–561.
[8] *J. Chen and B. Guo*: Strong instability of standing waves for a nonlocal Schrödinger equation. Phys. D *227* (2007), 142–148.
[9] *J. Chen and B. Guo*: Sharp global existence and blowing up results for inhomogeneous Schrödinger equations. Discrete Contin. Dynam. Systems *8* (2007), 357–367.
[10] *G. Fibich and X. P. Wang*: Stability of solitary waves for nonlinear Schrödinger equations with inhomogeneous nonlinearity. Phys. D. *175* (2003), 96–108.
[11] *R. Fukuizumi*: Stability and instability of standing waves for the nonlinear Schrödinger equation with harmonic potential. Discrete Contin. Dyn. Syst. *7* (2001), 525–544.
[12] *R. Fukuizumi and M. Ohta*: Stability of standing waves for nonlinear Schrödinger equations with potentials. Differential Integral Equations *16* (2003), 111–128.
[13] *R. Fukuizumi and M. Ohta*: Instability of standing waves for nonlinear Schrödinger equations with inhomogeneous nonlinearities. J. Math. Kyoto Univ. *45* (2005), 145–158.
[14] *T. S. Gill*: Optical guiding of laser beam in nonuniform plasma. Pramana Journal of Physics *55* (2000), 845–852.
[15] *J. Ginibre and G. Velo*: On the class of nonlinear Schrödinger equations I, II. J. Funct. Anal. *32* (1979), 1–32, 33–71.
[16] *R. T. Glassey*: On the blowing-up of solutions to the Cauchy problem for the nonlinear Schrödinger equation. J. Math. Phys. *18* (1977), 1794–1797.
[17] *M. Grillakis, J. Shatah and W. Strauss*: Stability theory of solitary waves in the presence of symmetry, I. J. Funct. Anal. *74* (1987), 160–197.
[18] *C. S. Liu and V. K. Tripathi*: Laser guiding in an axially nonuniform plasma channel. Phys. Plasmas *1* (1994), 3100–3103.
[19] *Y. Liu, X. P. Wang and K. Wang*: Instability of standing waves of the Schrödinger equations with inhomogeneous nonlinearity. Trans. Amer. Math. Soc. *358* (2006), 2105–2122.
[20] *F. Merle*: Nonexistence of minimal blow up solutions of equations $iu_t = -\Delta u - K(x)$ $|u|^{4/N}u$ in $\mathbb{R}^N$. Ann. Inst. H. Poincaré, Phys. Théor. *64* (1996), 33–85.
[21] *Yong-Geun, Oh*: Cauchy problem and Ehrenfest's law of nonlinear Schrödinger equation with potentials. J. Differential Equations *81* (1989), 255–274.
[22] *H. A. Rose and M. I. Weinstein*: On the bound states of the nonlinear Schrödinger equation with linear potential. Phys. D *30* (1988), 207–218.
[23] *W. Rother*: Some existence results for the equation $-\Delta u + K(x)u^p = 0$. Comm. Partial Differential Equations *15* (1990), 1461–1473.
[24] *J. Shatah and W. Strauss*: Instability of nonlinear bound states. Comm. Math. Phys. *100* (1985), 173–190.
[25] *P. Sintzoff and M. Willem*: A semilinear elliptic equation on $\mathbb{R}^N$ with unbounded coefficients. Variational and topological methods in the study of nonlinear phenomena *49* (Pisa 2000), 105–113 Birkhauser, Boston, 2002.
[26] *W. Strauss*: Existence of solitary waves in higher dimensions. Comm. Math. Phys. *55* (1977), 149–162.
[27] *T. Tsurumi and M. Waditi*: Collapses of wave functions in multidimensional nonlinear Schrödinger equations under harmonic potential. J. Phys. Soc. Japan *66* (1997), 3031–3034.
[28] *T. Tsurumi and M. Waditi*: Instability of the Bose-Einstein condensate under magnetic trap. J. Phys. Soc. Japan *66* (1997), 3035–3039.

[29] *Y. Wang*: Strong instability of standing waves for Hartree equation with harmonic potential. Phys. D *237* (2008), 998–1005.
[30] *M. I. Weinstein*: Nonlinear Schrödinger equations and sharp interpolation estimates. Comm. Math. Phys. *87* (1983), 567–576.
[31] *M. Willem*: Minimax Theorems. Progress in Nonlinear Differential Equations and Their Applications, 24, Birkhäuser, Boston, 1996.
[32] *J. Zhang*: Sharp threshold for global existence and blowup in nonlinear Schrödinger equation with harmonic potential. Comm. Partial Differential Equations *30* (2005), 1429–1443.

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