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ON THE MULTIPLICITY OF LAPLACIAN EIGENVALUES OF GRAPHS

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Abstract. In this paper we investigate the effect on the multiplicity of Laplacian eigenvalues of two disjoint connected graphs when adding an edge between them. As an application of the result, the multiplicity of 1 as a Laplacian eigenvalue of trees is also considered.

Keywords: Laplacian eigenvalue, multiplicity, tree, characteristic polynomial

MSC 2010: 05C50

1. Introduction

Let $G = (V, E)$ be a graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set E. Let $A(G)$ and $D(G) = \operatorname{diag}(d(v_1), d(v_2), \ldots, d(v_n))$ be the adjacency matrix and the diagonal matrix of vertex degrees of G, respectively, where $d(v_i)$ is the degree of vertex $v_i \in V(G)$. The matrix $L(G) = D(G) - A(G)$ is called the *Laplacian matrix* of the graph G because it is a discrete analogue of the Laplace differential operator (see [6]). It is well known that $L(G)$ is positive semidefinite, symmetric and singular. Moreover, since G is connected, $L(G)$ is irreducible. Denote its eigenvalues by

$$\mu_1(G) \geqslant \mu_2(G) \geqslant \ldots \geqslant \mu_n(G) = 0,$$

which are always enumerated in non-increasing order and repeated according to their multiplicity. We shall use the notation $\mu_k(G)$ to denote the k-th *Laplacian eigenvalue* of the graph G.

A vertex in a graph is called a quasipendant vertex if it is adjacent to a pendant vertex. Denote by $p(G)$ and $q(G)$ the number of pendant vertices and quasipendant

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vertices of G, respectively. We use $m_G(\lambda)$ to denote the multiplicity of λ as a Laplacian eigenvalue of G.

In [2], *Faria* proved that for any graph G,

$$m_G(1) \geqslant p(G) - q(G).$$

In [3], Grone, Merris, and Sunder proved that for a tree T with n vertices, if $\lambda > 1$ is an integer Laplacian eigenvalue of T, then $\lambda | n$ (i.e., λ divides n) and $m_T(\lambda) = 1$. In that paper, it is also pointed out that "there is an abundance of examples that leads the authors to believe there can be no simple graph theoretic interpretation for $m_T(1)$".

Recently, Shao, Guo and Shan [7] investigated the effect on the multiplicity of Laplacian eigenvalues of connected graphs when adding edges. In the present paper, we first consider the effect on the multiplicity of Laplacian eigenvalues of two disjoint connected graphs when adding an edge between them. Then we characterize all trees with the property $n - 6 \leqslant m_T(1) \leqslant n$.

2. Lemmas and results

Let G be a graph and let $G' = G + e$ be the graph obtained from G by inserting a new edge e into G. We have the following

Lemma 2.1 [3]. *The Laplacian eigenvalues of* G *and* G' *interlace, that is,*

$$\mu_1(G') \geqslant \mu_1(G) \geqslant \mu_2(G') \geqslant \mu_2(G) \geqslant \ldots \geqslant \mu_n(G') = \mu_n(G) = 0.$$

The following inequalities are known as *Cauchy's* inequalities and the whole theorem is also known as the interlacing theorem [1].

Lemma 2.2. *Let* A *be a Hermitian matrix with eigenvalues* $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n$ *and let* B *be a principal submatrix of order* m; *let* B *have eigenvalues* $\mu_1 \geqslant \mu_2 \geqslant \ldots \geqslant \mu_m$. *Then the inequalities* $\lambda_{n-m+i} \leqslant \mu_i \leqslant \lambda_i$ $(i = 1, 2, \ldots, m)$ *hold.*

Lemma 2.3 [4]. *For any tree* T *with* n *vertices, we have* $\mu(T) \leqslant n$, *with equality if and only if* $T \cong K_{1,n-1}$, *a star on* n *vertices.*

If $v \in G$, let $L_v(G)$ be the principal submatrix of $L(G)$ formed by deleting the row and column corresponding to vertex v.

Let B be a real square matrix. We denote by $\Phi(B) = \Phi(B; x) = \det(xI - B)$ the characteristic polynomial of B. In particular, if $B = L(G)$, we write $\Phi(G)$ instead of $\Phi(L(G))$ and call $\Phi(G)$ the Laplacian characteristic polynomial of G. For the Laplacian characteristic polynomial of a graph with a cut edge we have the following

Lemma 2.4 [5]. *Let G_1 and G_2 be two vertex-disjoint graphs and let $G = G_1u : vG_2$ be the graph obtained by joining the vertex u of the graph G_1 to the vertex v of the graph G_2 by an edge. Then*

$$\Phi(G) = \Phi(G_1)\Phi(G_2) - \Phi(G_1)\Phi(L_v(G_2)) - \Phi(G_2)\Phi(L_u(G_1)).$$

Let $T_n^*(s,t)$ $(s \geqslant t)$ be a tree on n vertices obtained from a star $K_{1,s}$ by joining t pendant vertices of $K_{1,s}$ to t new isolated vertices by t edges. It is easy to see that $n = s + t + 1$.

Corollary 2.1. *For $s \geqslant t \geqslant 1$ and $n \neq 3$ we have*

$$m_{T_n^*(s,t)}(1) = \begin{cases} s - t - 1, & s \geqslant t + 1; \\ 0, & s = t. \end{cases}$$

P r o o f. From Lemma 2.4, we have

$$\begin{aligned} \Phi(T_n^*(s,t)) = x(x-1)^{s-t-1}&(x^2 - 3x + 1)^{t-1} \\ &\times [x^3 - (s+4)x^2 + (3s+4)x - n]. \end{aligned}$$

Let $f_1(x) = x^3 - 3x + 1$, $f_2(x) = x^3 - (s+4)x^2 + (3s+4)x - n$. It is easy to see that $f_1(1) = -1 \neq 0$, $f_2(1) = s - t$. Thus, if $s \geqslant t + 1$, then $m_{T^*(s,t)}(1) = s - t - 1$; if $s = t$, then

$$\begin{aligned} \Phi(T_n^*(s,t)) &= x(x-1)^{-1}(x^2 - 3x + 1)^{t-1} \\ &\quad \times [x^3 - (s+4)x^2 + (3s+4)x - 2s - 1] \\ &= x(x^2 - 3x + 1)^{t-1}[x^2 - (s+3)x + 2s + 1] \\ &\triangleq x(x^2 - 3x + 1)^{t-1} f_3(x). \end{aligned}$$

Since $n \neq 3$, we have $s \neq 1$. Then $f_3(1) = s - 1 \neq 0$. So, $m_{T_n^*(s,t)}(1) = 0$ for $s = t$ and $n \neq 3$. The proof is complete. $\square$

Let $G_1u : v$ be the graph obtained by joining the vertex u of G_1 to a new isolated vertex v, and let $G_1u : vw$ be the graph obtained from G_1 and a new path $P_2 : vw$ by joining the vertex u to the vertex v. We have

Theorem 2.1. *Let G_1 and G_2 be two disjoint graphs and $G = G_1u : vG_2$ the graph obtained from G_1 and G_2 by joining an (arbitrary) vertex u of G_1 to an (arbitrary) vertex v of G_2. Then we have:*

(1) *If $m_{G_2}(\lambda) = m_{L_v(G_2)}(\lambda) + 1$, then*

$$m_G(\lambda) = m_{G_1}(\lambda) + m_{G_2}(\lambda) - 1.$$

(2) *If $m_{G_2}(\lambda) = m_{L_v(G_2)}(\lambda) - 1$, then*

$$m_G(\lambda) = m_{L_v(G_1u:v)}(\lambda) + m_{G_2}(\lambda).$$

and

$$m_G(\lambda) = m_{G_1u:vw}(\lambda) + m_{G_2}(\lambda)$$

(3) *If $m_{G_1}(\lambda) = m_{L_u(G_1)}(\lambda)$ and $m_{G_2}(\lambda) = m_{L_v(G_2)}(\lambda)$, then*

$$m_G(\lambda) \geqslant m_{G_1}(\lambda) + m_{G_2}(\lambda).$$

Proof. We first prove that (1) is true. From Lemma 2.1 we have

$$m_G(\lambda) \geqslant m_{G-uv}(\lambda) - 1 = m_{G_1}(\lambda) + m_{G_2}(\lambda) - 1. \tag{2.1}$$

From Lemma 2.4 we have

$$\begin{aligned} \Phi(G) &= \Phi(G_2)(\Phi(G_1) - \Phi(L_u(G_1))) - \Phi(G_1)\Phi(L_v(G_2)) \\ &= \Phi(G_2)\Phi(L_v(G_1u : v)) - \Phi(G_1)\Phi(L_v(G_2)). \end{aligned} \tag{2.2}$$

From Lemma 2.2 we have

$$\begin{aligned} m_G(\lambda) \leqslant m_{L_v(G)}(\lambda) + 1 &= m_{L_v(G_1u:v)}(\lambda) + m_{L_v(G_2)}(\lambda) + 1 \\ &= m_{L_v(G_1u:v)}(\lambda) + m_{G_2}(\lambda). \end{aligned} \tag{2.3}$$

From Eqs. (2.2) and (2.3) we have

$$m_G(\lambda) \leqslant m_{G_1}(\lambda) + m_{L_v(G_2)}(\lambda) = m_{G_1}(\lambda) + m_{G_2}(\lambda) - 1. \tag{2.4}$$

Eqs. (2.1) and (2.4) complete the proof of (1).

Next, we prove that (2) holds. From Lemma 2.2 we have

$$\begin{aligned} m_G(\lambda) \geqslant m_{L_v(G)}(\lambda) - 1 &= m_{L_v(G_1u:v)}(\lambda) + m_{L_v(G_2)}(\lambda) - 1 \\ &= m_{L_v(G_1u:v)}(\lambda) + m_{G_2}(\lambda). \end{aligned} \tag{2.5}$$

From Lemma 2.1 we have

$$m_G(\lambda) \leqslant m_{G-uv}(\lambda) + 1 = m_{G_1}(\lambda) + m_{G_2}(\lambda) + 1 = m_{G_1}(\lambda) + m_{L_v(G_2)}(\lambda). \tag{2.6}$$

From Eqs. (2.2) and (2.6), we have

$$m_{L_v(G_1u:v)}(\lambda) + m_{G_2}(\lambda) \geqslant m_G(\lambda). \tag{2.7}$$

Thus, from Eqs. (2.5) and (2.7) we have

$$m_G(\lambda) = m_{L_v(G_1u:v)}(\lambda) + m_{G_2}(\lambda).$$

In particular, taking $G_2 = vw$ we have $m_{G_1u:vw}(\lambda) = m_{L_v(G_1u:v)}(\lambda)$. The proof of (2) is complete.

From Lemma 2.4 it is easy to see that (3) holds. □

From Theorem 2.1 we obtain the following known result.

Corollary 2.2 [3]. *Let G_1 be a graph on $n \geqslant 1$ vertices, and let G be a graph obtained from G_1 and $K_{1,s}$ by joining an (arbitrary) vertex of G_1 to a vertex v of $K_{1,s}$. Then we have $m_G(s+1) = m_{G_1}(s+1)$.*

Proof. Without loss of generality, we distinguish the following two cases:

Case 1. v is the center of $K_{1,s}$. Since

$$\Phi(K_{1,s}) = x(x-s-1)(x-1)^{s-1} \tag{2.8}$$

and

$$\Phi(L_v(K_{1,s})) = (x-1)^s. \tag{2.9}$$

Eqs. (2.8) and (2.9) yield

$$m_{K_{1,s}}(s+1) = m_{L_v(K_{1,s})}(s+1) + 1.$$

From (1) of Theorem 2.1 we have $m_G(s+1) = m_{G_1}(s+1)$.

Case 2. v is a pendant vertex of $K_{1,s}$. It is easy to see that

$$\Phi(L_v(K_{1,s})) = (x-1)^{s-2}(x^2-(s+1)x+1). \tag{2.10}$$

From Eqs. (2.8) and (2.10) we have

$$m_{K_{1,s}}(s+1) = m_{L_v(K_{1,s})}(s+1) + 1.$$

From (1) of Theorem 2.1 we have $m_G(s+1) = m_{G_1}(s+1)$. □

Furthermore, we have the following

Corollary 2.3. *Let G_1 be a graph on $n \geqslant 1$ vertices, and let G be the graph obtained from G_1 and $K_{1,s}$ $(s \geqslant 2)$ by joining a vertex u of G_1 to a vertex v of $K_{1,s}$. Then we have:*

(1) *If v is a pendant vertex of $K_{1,s}$, then*

$$m_G(1) = m_{G_1}(1) + s - 2;$$

(2) *if v is the center of $K_{1,s}$, then*

$$m_G(1) = m_{G_1 u:vw}(1) + s - 1.$$

Proof. If v is a pendant vertex of $K_{1,s}$, then from Eqs. (2.8) and (2.10) we have

$$m_{K_{1,s}}(1) = m_{L_v(K_{1,s})}(1) + 1 = s - 1.$$

From (1) of Theorem 2.1 we have

$$m_G(1) = m_{G_1}(1) + m_{K_{1,s}}(1) - 1 = m_{G_1}(1) + s - 2.$$

Hence (1) follows.

If v is the center of $K_{1,s}$, then from Eqs. (2.8) and (2.9), we have

$$m_{K_{1,s}}(1) = m_{L_v(K_{1,s})}(1) - 1 = s - 1.$$

From (2) of Theorem 2.1 we have

$$m_G(1) = m_{G_1 u:vw}(1) + s - 1.$$

Hence (2) follows. □

From (1) of Corollary 2.3 we immediately have the following known result.

Corollary 2.4 [3]**.** *Let G be a graph obtained from G_1 and a new path P_3 by joining a vertex of G_1 to a pendant vertex of P_3. Then we have $m_G(1) = m_{G_1}(1)$.*

From Corollary 2.1 and (2) of Theorem 2.1 we have

Corollary 2.5. *Let u be a vertex of the graph G_1 and let G be a graph obtained from G_1 and $T_n^*(s,t)$ by joining the vertex u to the vertex v (with degree s) of $T_n^*(s,t)$. If $s-t \geqslant 1$, then we have*

$$m_G(1) = m_{G_1 u:vw}(1) + s - t - 1.$$

Next we investigate the multiplicity of 1 as a Laplacian eigenvalue of trees. We first introduce the following definition.

Let $[0,n] = \{0,1,\ldots,n\}$. A subset N of $[0,n]$ is said to be (Laplacian) 1-realizable for trees with n vertices provided that for any $k \in N$, there exists at least one tree T on n vertices such that $m_T(1) = k$.

Theorem 2.2. *For any tree T on $n \geqslant 4$ vertices we have*

$$m_T(1) \neq n,\ m_T(1) \neq n-1 \ \text{ and } \ m_T(1) \neq n-3.$$

P r o o f. Since $\mu_n(T) = 0$, $m_T(1) \neq n$ is obvious. If $m_T(1) = n-1$, from the well known fact $\sum\limits_{i=1}^{n-1} \mu_i(T) = 2(n-1)$ we have $n=1$, a contradiction to $n \geqslant 4$. If $m_T(1) = n-3$, then $T \neq K_{1,n-1}$ (since $M_{K_{1,n-1}}(1) = n-2$). So T contains P_4 as a subgraph. From Lemma 2.1 we have $\mu_{n-1}(T) \leqslant \mu_3(P_4) \approx 0.586 < 1$. From $\sum\limits_{i=1}^{n-1} \mu_i(T) = 2(n-1)$ we have

$$\mu_{n-1}(T) + \mu_1(T) = n+1.$$

Thus, we have $\mu_1(T) > n$, a contradiction. □

Theorem 2.3. *The set $N = \{0,1,2,\ldots,n-4,n-2\}$ is 1-realizable for trees with $n \geqslant 4$ vertices.*

P r o o f. In order to obtain the result, we only need to prove that for any $k \in N$ there exists a tree T on n vertices such that $m_T(1) = k$. We distinguish the following four cases:

Case 1. If $k = n-2$, take $T = K_{1,n-1}$. The result follows from $m_{K_{1,n-1}}(1) = n-2$.

Case 2. If $k = 0$, take $T = T^*(s,t)$ $(0 \leqslant s-t \leqslant 1,\ s+t+1 = n)$. The result follows from Corollary 2.1.

Case 3. $k = n-6$ $(n \geqslant 7)$. Let T' be the tree obtained from $K_{1,n-5}$ and the path P_4 by joining the center of $K_{1,n-5}$ to a nonpendant vertex of P_4. From Corollaries 2.1 and 2.5 we have $m_{T'}(1) = n-6$.

Case 4. $1 \leqslant k \leqslant n-4$ and $k \neq n-6$. Let T'' be the tree obtained from $K_{1,k+2}$ and $T^*_{n-k-3}(s,t)$ ($0 \leqslant s-t \leqslant 1$, $s+t+1 = n-k-3$) by joining a pendant vertex of $K_{1,k+2}$ and a vertex of $T^*_{n-k-3}(s,t)$. Since $k \neq n-6$, we have $n-k-3 \neq 3$. From Corollary 2.1 and (1) of Corollary 2.3 we have

$$m_{T''}(1) = m_{T^*_{n-k-3}(s,t)}(1) + k = k.$$

□

Let $T_3(s,t)$ be the tree on n vertices with diameter 3 obtained from $K_{1,s}$ and $K_{1,t}$ by joining the center of $K_{1,s}$ to the center of $K_{1,t}$ (see Fig. 1). It is easy to see that $n = s+t+2$.

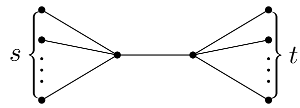

Fig. 1. $T_3(s,t)$

Let $T_4(s,r,t)$ be the tree on n vertices with diameter 4 obtained from P_5 : $v_1v_2v_3v_4v_5$, a path on 5 vertices, by attaching $s-1$, r, $t-1$ ($s,t \geqslant 1$, $r \geqslant 0$) pendant edges at vertices v_2, v_3, v_4 of P_5, respectively (see Fig. 2). It is easy to see that $n = s+t+r+3$.

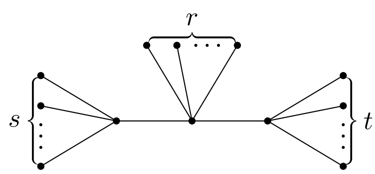

Fig. 2. $T_4(s,r,t)$

Let $T_5(s,t)$ be the tree on n vertices with diameter 5 obtained from $T_4(s,0,t)$ by subdividing one of the nonpendant edges of $T_4(s,0,t)$ (see Fig. 3). It is easy to see that $n = s+t+4$.

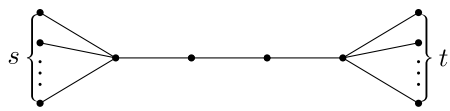

Fig. 3. $T_5(s,t)$

Let $d(T)$ be the diameter of T. In the following we characterize all trees satisfying $n-6 \leqslant m_T(1) \leqslant n-4$ and $m_T(1) = n-2$.

Theorem 2.4. *Let T be a tree on $n \geqslant 6$ vertices. Then we have*
(1) $m_T(1) = n-2$ *if and only if* $T \cong K_{1,n-1}$;
(2) $m_T(1) = n-4$ *if and only if* $T \cong T_3(s,t)$ $(s,t \geqslant 1; s+t+2=n)$;
(3) $m_T(1) = n-5$ *if and only if* $T \cong T_4(s,0,t)$ $(s,t \geqslant 1, s+t+3=n)$ *or* $T \cong T_5(s,t)$ $(s,t \geqslant 1; s+t+4=n)$;
(4) $m_T(1) = n-6$ *if and only if* $T \cong T_4(s,r,t)$ $(r \neq 0; s,t \geqslant 1; s+t+r+3=n)$.

Proof. We first prove that (1) holds. From $\sum_{i=1}^{n-1} \mu_i(T) = 2(n-1)$ we have $\mu_1(T) = n$. By Lemma 2.3 (1) holds.

Secondly, we prove that (2) holds. By virtue of Lemma 2.3 we can assume that $T \neq K_{1,n-1}$, that is $d(T) \geqslant 3$. If $d(T) \geqslant 4$, then T contains P_5 as a subgraph. It is easy to calculate that $\operatorname{spec}(P_5) = \{3.6180, 2.6180, 1.382, 0.382, 0\}$. From Lemma 2.1 we have $m_T(1,n] \geqslant 3$ and $m_T[0,1) \geqslant 2$. Thus,

$$m_T(1) = n - m_T(1,n] - m_T[0,1) \leqslant n-5.$$

So, if $m_T(1) = n-4$, then $d(T) = 3$. Then there exist two integers $s \geqslant 1$ and $t \geqslant 1$ such that $T \cong T_3(s,t)$. If $T \cong T_3(s,t)$, then Corollary 2.1 and (2) of Corollary 2.3 imply

$$m_{T_3(s,t)}(1) = s+t-2 = n-4.$$

The proof of (2) is complete.

Finally, we prove that (3) and (4) hold. If $d(T) \geqslant 6$, then T contains P_7 as a subgraph. By a simple calculation we have

$$\operatorname{spec}(P_7) = \{3.801, 3.246, 2.445, 1.555, 0.754, 0.1999, 0\}.$$

Then by reasoning similar as above we have $m_T(1) \leqslant n-7$. So, in the following, we can assume that $4 \leqslant d(T) \leqslant 5$. Without loss of generality, we distinguish the following two cases:

Case 1. $d(T) = 5$. Let $P_6: v_1v_2v_3v_4v_5v_6$ be a path on 6 vertices and let P_6' be the tree on 7 vertices obtained from P_6 by attaching a new pendant edge v_3v_3' at v_3. By a simple calculation we have

$$\operatorname{spec}(P_6') = \{4.334, 3.099, 2.274, 1.406, 0.623, 0.261, 0\}.$$

Then by reasoning similar as above we have $m_T(1) \leqslant n-7$ $(n \geqslant 7)$. Thus, if $m_T(1) = n-6, d(T) = 5$ or $m_T(1) = n-5, d(T) = 5$, then there exists a tree $T_5(s,t)$ $(s,t \geqslant 1; s+t+4=n)$ such that $T \cong T_5(s,t)$. From (1) of Corollary 2.3 we have $m_{T_5(s,t)}(1) = s-1+t = n-5$. Thus we have if $d(T) = 5$, then $m_T(1) \neq n-6$, and $m_T(1) = n-5$ if and only if $T \cong T_5(s,t)$ $(s,t \geqslant 1; s+t+4=n)$.

Case 2. $d(T) = 4$. Let $P_5 : v_1v_2v_3v_4v_5$ be a path on 5 vertices and let P_5' be a tree on 7 vertices obtained from P_5 by attaching a new path v_3uw with length 2 at vertex v_3, where u and w are new different vertices. By a simple calculation we have $\operatorname{spec}(P_5') = \{4.414, 2.618, 2.618, 1.586, 0.382, 0.382, 0\}$. Then by reasoning similar as above, if T contains P_5' as a subgraph, then $m_T(1) \leqslant n-7$. Thus, if $d(T) = 4$ and $m_T(1) = n-5$ or $d(T) = 4$ and $m_T(1) = n-6$, then there exists a tree $T_4(s,r,t)$ $(s,t \geqslant 1; s+t+r+3 = n)$ such that $T \cong T_4(s,r,t)$.

If $r = 0$, then from (1) of Corollary 2.3 we have

$$m_{T_4(s,0,t)}(1) = s+1-2+t-1 = n-5.$$

If $r \neq 0$, then from Corollary 2.1 and (2) of Corollary 2.3 we have

$$m_{T_4(s,r,t)}(1) = s-1+t-1+r-1 = n-6.$$

The proofs of (3) and (4) are complete. □

At the end of this paper, we propose the following problem:

Characterize the trees on n vertices with the property

$$m_T(1) = 0.$$

References

[1] *D. M. Cvetković, M. Doob and H. Sachs*: Spectra of Graphs-Theory and Application. Deutscher Verlag der Wissenschaften - Academic Press, Berlin-New York, 1980; second edition 1982; third edition, Johann Ambrosius Barth Verlag, Heidelberg-Leipzig, 1995.

[2] *I. Faria*: Permanental roots and the star degree of a graph. Linear Algebra Appl. *64* (1985), 255–265.

[3] *R. Grone, R. Merris and V. S. Sunder*: The Laplacian spectrum of a graph. SIAM J. Matrix Anal. Appl. *11* (1990), 218–238.

[4] *R. Grone and R. Merris*: The Laplacian spectrum of a graph II*. SIAM J. Discrete Math. *7* (1994), 221–229.

[5] *J. M. Guo*: On the second largest Laplacian eigenvalue of trees. Linear Alg. Appl. *404* (2005), 251–261.

[6] *B. Mohar*: The Laplacian spectrum of graphs. Graph Theory, Combinatorics, and Applications *2* (1991), 871–898.

[7] *J. Y. Shao, J. M. Guo and H. Y. Shan*: The ordering of trees and connected graphs by algebraic connectivity. Linear Alg. Appl. *428* (2008), 1421–1438.

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