Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 60 (2010), No. 3, 589–596

Persistent URL: http://dml.cz/dmlcz/140591

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ON QUASINILPOTENT EQUIVALENCE OF FINITE RANK ELEMENTS IN BANACH ALGEBRAS

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(Received March 9, 2009)

Abstract. We characterize elements in a semisimple Banach algebra which are quasinilpotent equivalent to maximal finite rank elements.

Keywords: maximal finite rank elements, quasinilpotent equivalence

MSC 2010: 46H05, 46H10

1. Introduction

The notion of quasinilpotent equivalence for linear operators is due to Colojoară and Foiaş [3], [4]. This notion has been extended to general Banach algebras by Razpet in [12]. For all unexplained notation and terminology in this paper we refer the reader to [12].

Throughout this paper A is a complex Banach algebra with unit 1 and $\mathbb{C}$ are the complex numbers. The spectrum of $a \in A$ will be denoted by $\sigma(a, A)$ and the spectral radius of $a \in A$ by $r(a, A)$. Whenever there is no ambiguity we shall drop the A in σ and r. An element $a \in A$ is said to be *quasinilpotent* if $\sigma(a) = \{0\}$, equivalently $\lim\limits_n \|a^n\|^{1/n} = 0$. The set of these elements will be denoted by $\mathrm{QN}(A)$. An element $a \in A$ is called *Riesz* w.r.t. a closed ideal J in A if the coset $a + J$ is in $\mathrm{QN}(A/J)$.

For each $a, b \in A$ we introduce associate operators L_a, R_b and $C_{a,b}$ acting on A by the relations

$$L_a x = ax, \quad R_b x = xb \quad \text{and} \quad C_{a,b} x = (L_a - R_b)x$$

for all $x \in A$. It is easy to see that L_a, R_b and $C_{a,b}$ are bounded linear operators on A, i.e., $L_a, R_b, C_{a,b} \in \mathcal{L}(A)$.

2. Quasinilpotent equivalence

Let $a, b \in A$. Since the operators L_a and R_b commute,

$$C^n_{a,b}x = \sum_{k=0}^{n}(-1)^k \binom{n}{k} a^{n-k}xb^k \tag{2.1}$$

for all $x \in A$. We have

$$C^{n+1}_{a,b}x = a(C^n_{a,b}x) - (C^n_{a,b}x)b \tag{2.2}$$

and if also $c \in A$ one can prove

$$C^n_{a,b}(xy) = \sum_{k=0}^{n} \binom{n}{k} (C^{n-k}_{a,c}x)(C^k_{c,b}y) \tag{2.3}$$

for all $x, y \in A$, see [6] for a proof. Let

$$\varrho(a,b) = \limsup_n \|C^n_{a,b}1\|^{1/n}. \tag{2.4}$$

Note that in general the numbers $\varrho(a,b)$ and $\varrho(b,a)$ seem to be different. If, however, a and b commute then by (2.4) $\varrho(a,b) = \varrho(b,a) = r(a-b)$.

Define

$$d(a,b) = \max\{\varrho(a,b), \varrho(b,a)\}. \tag{2.5}$$

The identity in (2.3) is important because one needs it to prove that the function d is a semimetric on A. It is called the *spectral semidistance* from a to b. It is not a metric on A, see the remarks preceding Proposition 2.2 in [12]. In view of [12], elements $a, b \in A$ are called *quasinilpotent equivalent* if $d(a,b) = 0$.

As remarked above, the original idea of "quasinilpotent equivalence" goes back to Colojoară and Foiaş [3], [4]: Operators S and T on a Banach space X are *quasinilpotent equivalent* provided

$$d(S,T) = \max\{\varrho(S,T), \varrho(T,S)\} = 0, \tag{2.6}$$

where

$$\varrho(S,T) = \limsup_n \|(S-T)^{[n]}\|^{1/n} \tag{2.7}$$

with

$$(2.8) \qquad (S-T)^{[n]} = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} S^k T^{n-k}.$$

Note that (2.7) is not really a function of $S - T$. If S and T commute, this reduces to the condition that the difference $S - T$ is quasinilpotent. This applies to the left and right multiplications L_a and R_b on a Banach algebra $X = A$, whether or not the elements a and b commute. One can therefore define

$$(2.9) \qquad \varrho(a,b) = \limsup_n \|(L_a - R_b)^n(1)\|^{1/n}$$

reproducing (2.7) without the spurious dependence on $a - b$, and verifying that the definition of Razpet [12] is a valid generalization of the original operator condition.

Note that if two elements in a Banach algebra differ by a commuting quasinilpotent element, then they are quasinilpotent equivalent. The converse, however, fails: Let X be a Banach space and $Y = X \oplus X$. Define operators T and S on Y as follows: $T(x_1, x_2) = (0, -x_1)$ and $S(x_1, x_2) = (x_2, 0)$ for all $(x_1, x_2) \in Y$. Notice that $T^2 = S^2 = 0$ and so $(S-T)^{[n]} = (T-S)^{[n]} = 0$ whenever $n \geqslant 3$. By (2.7) and (2.6) S and T are quasinilpotent equivalent. But $S - T$ is not quasinilpotent because $(S-T)^2 = I$.

Let $a, b \in A$ and suppose there is $\lambda_0 \in \mathbb{C}$ such that $\sigma(a) = \sigma(b) = \{\lambda_0\}$. Then a and b are quasinilpotent equivalent: By (2.4), $\varrho(a,b) \leqslant r(C_{a,b})$ and since L_a and R_b commute, $\sigma(C_{a,b}) \subset \sigma(a) - \sigma(b)$. If we combine these two facts, $\varrho(a,b) = 0$. It follows likewise that $\varrho(b,a) = 0$. In this regard also see [12, Corollary 2.3]. In particular, it follows from these remarks that quasinilpotent elements in A are quasinilpotent equivalent.

In [6] it is proved that if two elements a and b in A are quasinilpotent equivalent then $\sigma(a) = \sigma(b)$. Since we are going to use this fact repeatedly, we provide a Banach algebra proof of this fact using the notion of the joint spectrum of two commuting elements, see [10, Definition 2.14 and Theorem 2.20].

Theorem 2.1. *Let A be a Banach algebra and $a, b \in A$. If $d(a,b) = 0$ then $\sigma(a) = \sigma(b)$.*

Proof. If $a, b \in A$ commute and $\lambda \in \sigma(a)$ then there is $\mu \in \sigma(b)$ with $(\lambda, \mu) \in \sigma(a,b)$; now $\lambda - \mu \in \sigma(a - b) = \{0\}$ giving $\lambda = \mu \in \sigma(b)$. Similarly if $\mu \in \sigma(b)$ then $\mu = \lambda \in \sigma(a)$. In general if $a, b \in A$ then L_a and R_b commute in $\mathcal{L}(A)$, and hence $\sigma(a) = \sigma(L_a) = \sigma(R_b) = \sigma(b)$. □

As we remarked above the converse of Theorem 2.1 holds when $\sigma(a) = \sigma(b) = \{\lambda\}$ is a singleton. It fails in general, however: just let $a \neq b$ be distinct nontrivial idempotents. Then by (2.4) $\varrho(a, b) = \varrho(b, a) = 1$.

The previous observation implies that if $a, b \in A$ are both idempotents and $\varrho(a, b) = 0$ then $a = b$. This is a seemingly stronger statement than [12, Corollary 3.1]. Also, our proof only uses (2.4) while Corollary 3.1 in [12] relies on [12, Theorem 3.1].

Another consequence of Theorem 2.1 in quotient algebras is

Corollary 2.2. *Let A be a Banach algebra and I a closed ideal in A. If $a \in A$ is Riesz relative to I then $b \in A$ is Riesz relative to I if and only if $d(a + I, b + I) = 0$.*

Let A be a Banach algebra and I a closed ideal in A. Suppose $a, b \in A$. In view of $\|a + I\| \leqslant \|a\|$ it follows from the definition of the spectral semidistance that $d(a + I, b + I) \leqslant d(a, b)$. One can show by an example that this inequality may be strict. Recall that T is a Riesz operator on a Banach space X if $T + \mathcal{K}(X) \in \mathrm{QN}(\mathcal{L}(X)/\mathcal{K}(X))$ where $\mathcal{K}(X)$ is the closed ideal of compact operators on X. These remarks imply

Corollary 2.3. *Let S and T be operators on a Banach space X. If T is a Riesz operator and $d(T, S) = 0$ then S is a Riesz operator.*

3. Finite rank elements

In this section we will require that A is a *semiprime* Banach algebra, i.e., $xAx = \{0\}$ implies that $x = 0$ holds for all $x \in A$. It can be shown that all semisimple Banach algebras are semiprime. Following Puhl [11] we call an element $0 \neq a \in A$ *rank one* if $aAa \subset \mathbb{C}a$. Denote the set of these elements by $\mathcal{F}_1$. By [11, Lemma 2.7] we have $\mathcal{F}_1 A$, $A\mathcal{F}_1 \subset \mathcal{F}_1$. An idempotent belonging to $\mathcal{F}_1$ is called a *minimal idempotent*. Let $\mathcal{F}$ denote the set of all $u \in A$ of the form $u = \sum_{i=1}^{n} u_i$ with $u_i \in \mathcal{F}_1$. We will call $\mathcal{F}$ the set of *finite rank* elements of A. $\mathcal{F}$ is a twosided ideal in A and it coincides with the socle of A, i.e., $\mathrm{Soc}(A) = \mathcal{F}$.

For another approach to rank one and finite rank elements see [1], [8]. However, if A is a semisimple Banach algebra then the notion of rank one and finite rank elements in the sense of Puhl [11] coincides with the notion of rank one and finite rank elements in the sense of Aupetit/Mouton [1], see [8, Theorem 4] and [1, Theorem 2.12].

Let A be a semiprime Banach algebra and $a, b \in A$. Suppose $a, b \in \mathcal{F}_1$ and $d(a, b) = 0$. If $a \in \mathrm{QN}(A)$ then by Theorem 2.1 $\sigma(a) = \sigma(b) = \{0\}$. In view of [11], Section 2 and Lemma 2.8] $a^2 = b^2 = 0$. If we suppose $a, b \in \mathcal{F}$, $d(a,b) = 0$ and $a \in \mathrm{QN}(A)$ then again by Theorem 2.1 $\sigma(a) = \sigma(b) = \{0\}$. In view of [9, Lemma 3.10] there is a natural number m such that $a^m = b^m = 0$.

Theorem 3.1. *Let A be a semiprime Banach algebra and suppose both $a, b \in A$ are rank one. If $d(a,b) = 0$ and a is not quasinilpotent then $a = b$.*

P r o o f. If a is not quasinilpotent then by [11, Lemma 2.8] and Theorem 2.1, $\sigma(a) = \{0, \lambda\} = \sigma(b)$. But then $\lambda^{-1}a$ and $\lambda^{-1}b$ are minimal idempotents. In view of [12, Corollary 2.1] they are quasinilpotent equivalent. By [12, Corollary 3.1], $\lambda^{-1}a = \lambda^{-1}b$ and so $a = b$. □

Let A be a semisimple Banach algebra and $a \in A$. Following Aupetit and Mouton [1] we define the *rank* of a by

$$\operatorname{rank}(a) = \sup_{x \in A} \#(\sigma(xa) \setminus \{0\}) \tag{3.1}$$

where # denotes the number of elements in a set. An element $a \in A$ is said to be of *maximal finite rank* if

$$\operatorname{rank}(a) = \#(\sigma(a) \setminus \{0\}). \tag{3.2}$$

Theorem 3.2. *Let A be a semisimple Banach algebra with $a, b \in A$. If both a and b are of maximal finite rank and $d(a,b) = 0$ then $a = b$.*

P r o o f. Since a is of maximal finite rank we can by [1, Theorem 2.8] assume $a = \lambda_1 p_1 + \ldots + \lambda_n p_n$ with $\lambda_1, \ldots, \lambda_n$ the nonzero distinct spectral values of a and $p_1, \ldots, p_n$ orthogonal minimal idempotents. Likewise we can assume that $b = \mu_1 e_1 + \ldots + \mu_m e_m$ with $\mu_1, \ldots, \mu_m$ the nonzero distinct spectral values of b and $e_1, \ldots, e_m$ orthogonal minimal idempotents. Since $d(a,b) = 0$, by Theorem 2.1 we can suppose $\sigma(a) \setminus \{0\} = \{\lambda_1, \ldots, \lambda_n\} = \sigma(b) \setminus \{0\}$. Note that $ap_i = \lambda_i p_i$ and $be_i = \lambda_i e_i$ $(i = 1, \ldots, n)$. If $f(z) = z(z - \lambda_1) \ldots (z - \lambda_n)$, then f is an entire function with simple zeros such that $f(a) = f(b) = 0$. In view of [12, Theorem 3.1], $a = b$. □

In the above proof note that if A is semisimple and infinite dimensional and if $a \in A$ is of maximal finite rank then $0 \in \sigma(a)$. However, if A is finite dimensional then it is possible to give examples of maximal finite rank elements which are invertible.

In our next result we are going to characterize elements b in a semisimple Banach algebra which are quasinilpotent equivalent to maximal finite rank elements.

Theorem 3.3. *Let A be an infinite dimensional semisimple Banach algebra with $a \in A$ a nonzero maximal finite rank element and $b \in A$. Then $d(a,b) = 0$ if and only if $b - a$ is quasinilpotent and commutes with a.*

P r o o f. Suppose $a \in A$ is maximal finite rank and a and $b \in A$ are quasinilpotent equivalent. Since a is Riesz relative to $\overline{\operatorname{Soc} A}$, it follows from Corollary 2.2 that b is Riesz relative to $\overline{\operatorname{Soc} A}$. We are going to show that b has the desired decomposition. It follows from [1, Theorem 2.8] that $a = \lambda_1 p_1 + \ldots + \lambda_n p_n$ with $\lambda_1, \ldots \lambda_n$ the nonzero distinct spectral values of a and p_i $(i = 1, \ldots, n)$ the Riesz idempotents associated with a and λ_i. Then $a = \lambda_0 p_0 + \lambda_1 p_1 + \ldots + \lambda_n p_n$ where $\lambda_0 = 0$ and p_0 is the Riesz idempotent corresponding to a and λ_0. Since the Riesz idempotents p_i commute with a, are orthogonal and minimal, it follows that $ap_0 = 0$, $ap_i = \lambda_i p_i$ $(i = 1, \ldots, n)$. In view of Theorem 2.1, $\sigma(b) = \{\lambda_0, \lambda_1, \ldots, \lambda_n\}$. By [2, Proposition 7.9] there are nonzero orthogonal idempotents $f_0, f_1, \ldots, f_n$ such that $1 = f_0 + f_1 + \ldots + f_n$. Hence

$$b = bf_0 + bf_1 + \ldots + bf_n = bf_0 + f_1 b f_1 + \ldots + f_n b f_n$$

because the f_i commute with b. Since a and b are quasinilpotent equivalent the series

$$\sum_{r=0}^{\infty} (\lambda - \lambda_k)^{-r-1} C^r_{b,a} 1$$

converges for all $\lambda \neq \lambda_k$. Put

$$F(\lambda) = \sum_{k=0}^{n} \left(\sum_{r=0}^{\infty} (\lambda - \lambda_k)^{-r-1} C^r_{b,a} 1 \right) p_k.$$

In view of (2.2)

$$(\lambda - b) C^r_{b,a} 1 = \bigl(C^r_{b,a} 1\bigr)(\lambda - a) - C^{r+1}_{b,a} 1.$$

Hence for all $\lambda \notin \sigma(b)$

$$\begin{aligned}(\lambda - b)F(\lambda) &= \sum_{k=0}^{n} \sum_{r=0}^{\infty} (\lambda - \lambda_k)^{-r-1} \bigl(C^r_{b,a} 1 (\lambda - \lambda_k) + (a - \lambda_k) - C^{r+1}_{b,a} 1\bigr) p_k \\ &= \sum_{k=0}^{n} p_k = 1,\end{aligned}$$

and so $F(\lambda) = (\lambda - b)^{-1}$. For $k = 0, 1, \ldots, n$ let Γ_k be a small circle around λ_k which contains no other elements of the spectrum of b. Then in view of the definition of $F(\lambda)$

$$f_k = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_k} (\lambda - b)^{-1} \, \mathrm{d}\lambda = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_k} F(\lambda) \, \mathrm{d}\lambda = p_k.$$

Hence $b = bf_0 + p_1bp_1 + \ldots + p_nbp_n$. By the minimality of p_i, $p_ibp_i = \mu_ip_i$ for some $\mu_i \in \mathbb{C}$ $(i = 1, \ldots, n)$. But the spectral mapping theorem implies that $\lambda_i = \mu_i$ $(i = 1, \ldots, n)$. Consequently, $b = bf_0 + a$ where $bf_0 \in \mathrm{QN}(A)$. The fact that b commutes with q implies that b commutes with a.

Conversely, suppose $b = a+q$ for some quasinilpotent element $q \in A$ with $aq = qa$. By remarks preceding Theorem 2.1 or [12, Proposition 2.4], $d(a, b) = r(q) = 0$. □

If A is an infinite dimensional semisimple Banach algebra and $a \in A$ is of maximal finite rank then it can happen that $d(a, b) = 0$ for some $b \in A$ that does not belong to the socle of A: Let $q \in \mathrm{QN}(A)$ and suppose q is not nilpotent and $aq = qa$. If we put $b = a + q$ then by [12, Proposition 2.4], $d(a, b) = 0$. Since q is not nilpotent we have by [9, Lemma 3.10] that b does not belong to the socle of A.

Corollary 3.4. *Let A be a finite dimensional semisimple Banach algebra with $a \in A$ of maximal finite rank and $b \in A$. If $d(a, b) = 0$ then $b = a + q$ for some nilpotent $q \in A$ with $aq = qa$.*

P r o o f. If a is invertible then $0 \notin \sigma(a)$ and so by Theorem 3.3 $a = b$. If a is not invertible then $0 \in \sigma(a)$ and again it follows from Theorem 3.3 that $b = a + q$ for some quasinilpotent q in A. Since A is finite dimensional, $A = \mathrm{Soc}\, A$ [5, Theorem 11] and hence $b - a = q$ is a quasinilpotent element in $\mathrm{Soc}\, A$. But by [5, Corollary 9] q is algebraic in A. If we combine these two facts it follows that q is nilpotent, see Remark 2 in [7]. □

Acknowledgement. The author wishes to thank the referee for his helpful comments.

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