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TWO CLASSES OF DARBOUX-LIKE, BAIRE ONE FUNCTIONS OF TWO VARIABLES

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Abstract. Among the many characterizations of the class of Baire one, Darboux real-valued functions of one real variable, the 1907 characterization of Young and the 1997 characterization of Agronsky, Ceder, and Pearson are particularly intriguing in that they yield interesting classes of functions when interpreted in the two-variable setting. We examine the relationship between these two subclasses of the real-valued Baire one defined on the unit square.

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1. Introduction: Motivation and notation

One of the most studied and interesting classes of real functions defined on $\mathbb{I} \equiv [0,1]$ is the family of Baire one, Darboux functions. Two reasons for this, of course, are that this class contains the collection of derivatives and that functions of this class share several of the properties of continuous functions. There are numerous characterizations of Baire one, Darboux functions and a list of many can be found in [2]. Most such characterizations suggest one or more possible subclasses of Baire one functions of two variables via interpreting the characterization in the two variable setting. Here we shall investigate the relationship between two of what we consider the more interesting such subclasses.

In 1907 Young [10] showed that a Baire one function $f\colon \mathbb{I} \to \mathbb{R}$ has the Darboux property if and only if f is *bilaterally approachable*, i.e., for each $x \in (0,1)$ there exist sequences $\{z_n\}$ and $\{y_n\}$ in $(0,1)$ such that $z_n \uparrow x$, $y_n \downarrow x$, and $\lim\limits_{n\to\infty} f(z_n) = \lim\limits_{n\to\infty} f(y_n) = f(x)$, with the one-sided versions applying at the endpoints 0 and 1. More recently, in 1997 Agronsky, Ceder, and Pearson [1] have characterized the class of Baire one, Darboux functions $f\colon \mathbb{I} \to \mathbb{R}$ as those that are *strongly polygonally approximable*. This means that there is a sequence of partitions $\{\mathcal{P}_n = \{0 = x_0 < x_1 < x_2 < \ldots < x_{m_n} = 1\}\}$ such that the sequence $\{L_n\}$ of continuous piecewise linear functions obtained by interpolating between $f(x_0), f(x_1), \ldots, f(x_{m_n})$ converges pointwise to f, and the *f-graph mesh* $\|\mathcal{P}_n\|_f \to 0$, where $\|\mathcal{P}_n\|_f \equiv \max\{\mathrm{dist}((x_i, f(x_i)), (x_{i-1}, f(x_{i-1}))\colon i = 1, 2, \ldots, m_n\}$. Both the Young and the Agronsky-Ceder-Pearson conditions yield very interesting families of functions in the two variable setting and these are the classes we investigate in this work.

One way to extend the Young condition to the unit square $\mathbb{I}^2 \equiv \mathbb{I} \times \mathbb{I}$ is as follows:

Definition 1.1. A function $f\colon \mathbb{I}^2 \to \mathbb{R}$ is *sectorially approachable* if

- for each $x \in \mathrm{int}(\mathbb{I}^2)$ and for each open sector S with vertex at x there is a sequence of points $\{z_n\}$ in S converging to x such that $\{f(z_n)\}$ converges to $f(x)$;
- for each $x \in \partial(\mathbb{I}^2)$ and for each open sector S with vertex at x for which $S \cap \mathbb{I}^2 \neq \emptyset$ there is a sequence of points $\{z_n\}$ in $S \cap \mathbb{I}^2$ converging to x such that $\{f(z_n)\}$ converges to $f(x)$.

We shall use SA to denote the class of Baire one, sectorially approachable real-valued functions on $\mathbb{I}^2$.

In [8] Malý shows that gradients of differentiable functions map closed convex sets with non-empty interiors to connected sets; as a consequence, partial derivatives of differentiable functions map closed convex sets with non-empty interiors to intervals. In [5] we prove the following theorem showing that for Baire one functions, SA is precisely the class that have this property.

Theorem 1.1. *A Baire one function $f\colon \mathbb{I}^2 \to \mathbb{R}$ is sectorially approachable if and only if it preserves the connectivity of every closed convex set having nonempty interior.*

This fact adds a great deal of credibility to SA's claim to be a "natural" generalization of Baire 1, Darboux for real valued functions defined in the plane.

Motivated by Agronsky, Ceder, and Pearson [1], we say that a function $f\colon \mathbb{I} \to \mathbb{R}$ is *polygonally approximable* if there is a sequence of partitions $\{\mathcal{P}_n = \{0 = x_0 < x_1 < x_2 < \ldots < x_{m_n} = 1\}\}$ such that the sequence $\{L_n\}$ of continuous piecewise linear functions obtained by interpolating between $f(x_0), f(x_1), \ldots, f(x_{m_n})$ converges pointwise to f. Among the several natural ways to extend this notion to the two-variable setting perhaps the most straightforward is that set out in [4]. Given a function $f\colon \mathbb{I}^2 \to \mathbb{R}$, we say that a continuous, piecewise linear function $L\colon \mathbb{I}^2 \to \mathbb{R}$ is *f-based* if all the vertices of L lie on the graph of the f. Specifically, an f-based, continuous, piecewise linear function L can be determined by triangulations of $\mathbb{I}^2 \equiv \mathbb{I} \times \mathbb{I}$, i.e., triangular partitions of $\mathbb{I}^2$ as follows: If one of the partitioning triangles has vertices z_1, z_2, z_3, we denote that triangle by $T(z_1, z_2, z_3)$ and require that $L(z_i) = f(z_i)$, $i = 1, 2, 3$, and that L be linear on $T(z_1, z_2, z_3)$. We say that the given triangulation determines or *supports* the continuous piecewise linear f-based L. The function $f\colon \mathbb{I}^2 \to \mathbb{R}$ is *polygonally approximable* if there is a sequence of triangulations $\{\mathcal{T}_n\}$ of $\mathbb{I}^2$ such that the corresponding sequence $\{L_n\}$ of f-based continuous piecewise linear functions supported by $\{\mathcal{T}_n\}$ converges pointwise to f. Whereas Agronsky, Ceder, and Pearson [1] showed that $f\colon \mathbb{I} \to \mathbb{R}$ is Baire one if and only if f is polygonally approximable, we showed in [4] that $f\colon \mathbb{I}^2 \to \mathbb{R}$ is Baire one if and only if f is polygonally approximable.

Extending the notion of strongly polygonally approximable to the two variable setting is more delicate than extending the notion of polygonally approximable. Certainly, one natural way to proceed is to add the graph mesh condition to the notion of polygonally approximable functions. A difficulty arises because the boundary of $\mathbb{I}^2$ is much richer than the boundary of $\mathbb{I}$. In the one variable case the boundary of the domain is a two point set and the graph-mesh condition plays no additional role at those points. In the two dimensional setting, requiring a partition of $\mathbb{I}^2$ entails a one dimensional partition of the boundary of $\mathbb{I}^2$. As a consequence, any corresponding graph-mesh condition would require that the function f restricted to each edge of $\mathbb{I}^2$ be strongly polygonally approximable as a function of one variable and thus, according to [1], be Baire one, Darboux as a function of one variable on that edge. We consider this condition too restrictive; in particular, partial derivatives of differentiable functions need not have this property (see, e.g., [8]). Thus, we prefer a less constraining version for the two variable case, but, of course, want one that will still be equivalent to the Agronsky-Ceder-Pearson condition when interpreted in the one variable setting. We'll define and investigate both notions below, beginning by revisiting the one dimensional setting.

For a function $f\colon \mathbb{I} \to \mathbb{R}$ we define its *symmetric periodic extension* $F\colon \mathbb{R} \to \mathbb{R}$ by first defining $F\colon [0, 2] \to \mathbb{R}$ via

$$F(x) = \begin{cases} f(x) & \text{if } x \in \mathbb{I}, \\ f(2-x) & \text{if } x \in [1, 2], \end{cases}$$

and then extending F to all of $\mathbb{R}$ periodically with period 2.

By a partition $\mathcal{P}$ of $\mathbb{R}$ we mean a countable set $\{\ldots x_{-m} < x_{-m+1} < \ldots < x_{-1} < x_0 < x_1 < \ldots < x_m < x_{m+1}\}$ where $-\infty = \lim\limits_{m\to\infty} x_{-m}$ and $\lim\limits_{m\to\infty} x_m = \infty$. We could then say that a function $f\colon \mathbb{I} \to \mathbb{R}$ is strongly polygonally approximable if there exists a sequence of partitions $\{\mathcal{P}_n = \{x_i\colon i \in \mathbb{Z}\}\}$ of $\mathbb{R}$ such that the sequence $\{L_n\}$ of continuous piecewise linear functions obtained by interpolating between $f(x_i)$, $f(x_{i+1})$, $i \in \mathbb{Z}$ converges pointwise to f, and the *f-graph mesh* $\|\mathcal{P}_n\|_f \to 0$, where $\|\mathcal{P}_n\|_f \equiv \sup\{\mathrm{dist}((x_i, f(x_i)), (x_{i-1}, f(x_{i-1}))\colon\ i = 1, 2, \ldots, m_n\}$. It is an easy matter to see that this is equivalent to the definition given above. As an aside let us also notice that an equivalent, if cumbersome, way to say that a function $f\colon\ \mathbb{I} \to \mathbb{R}$ is bilaterally approachable is to say that its symmetric periodic extension has the property that for each $x \in \mathbb{R}$ there exist sequences $\{z_n\}$ and $\{y_n\}$ such that $z_n \uparrow x$, $y_n \downarrow x$, and $\lim\limits_{n\to\infty} f(z_n) = \lim\limits_{n\to\infty} f(y_n) = f(x)$.

For a function $f\colon\ \mathbb{I}^2 \to \mathbb{R}$ we define its *symmetric periodic extension* $F\colon\ \mathbb{R}^2 \to \mathbb{R}$ similarly, as follows: First define $F\colon\ [0,2] \times [0,1]$ via

$$F(x,y) = \begin{cases} f(x,y) & \text{if } x \in \mathbb{I}, \\ f(2-x,y) & \text{if } x \in [1,2]. \end{cases}$$

Then extend F periodically in the x variable with period 2 to all of $\mathbb{R} \times [0,1]$. Next, define $F\colon\ \mathbb{R} \times [0,2]$ via

$$F(x,y) = \begin{cases} F(x,y) & \text{if } y \in \mathbb{I}, \\ F(x,2-y) & \text{if } y \in [1,2]. \end{cases}$$

Finally, extend F periodically in the y variable with period 2 to all of $\mathbb{R} \times \mathbb{R}$.

Note that if a Baire one $f\colon\ \mathbb{I}^2 \to \mathbb{R}$, then $f \in \mathrm{SA}$ according to Definition 1.1 iff the symmetric periodic extension of f is sectorially approachable at every point. In the sequel, we use the notation $f \in \mathrm{SA}$ in this sense.

Definition 1.2. Let $f\colon\ \mathbb{I}^2 \to \mathbb{R}$ and let $F\colon\ \mathbb{R}^2 \to \mathbb{R}$ be its symmetric periodic extension. We say that f is *strongly polygonally approximable* if there exists a sequence $\{\mathcal{T}_n\}$ of triangulations of $\mathbb{R}^2$ such that the sequence $\{L_n\}$ of F-based continuous piecewise linear functions supported by $\mathcal{T}_n$ converges pointwise to F on $\mathbb{R}^2$ and the *F-graph mesh* of $\mathcal{T}_n$, $\|\mathcal{T}_n\|_F$ tends to zero, where $\|\mathcal{T}_n\|_F$ is the supremum of the lengths of the edges of the triangles $L_n(T)$ for $T \in \mathcal{T}_n$. We shall use SPA to denote the class of strongly polygonally approximable functions on $\mathbb{I}^2$.

Definition 1.3. Let $f\colon\ \mathbb{I}^2 \to \mathbb{R}$. We say that f is *strongly polygonally approximable in the restricted sense* if there exists a sequence $\{\mathcal{T}_n\}$ of triangulations of $\mathbb{I}^2$ such that the sequence $\{L_n\}$ of f-based continuous piecewise linear functions supported by $\mathcal{T}_n$ converges pointwise to f on $\mathbb{I}^2$ and the *f-graph mesh* of $\mathcal{T}_n$, $\|\mathcal{T}_n\|_f$ tends to zero. We shall use SPA^* to denote the class of strongly polygonally approximable in the restricted sense functions on $\mathbb{I}^2$.

Clearly, $\mathrm{SPA}^* \subseteq \mathrm{SPA}$. Also, it is easy (e.g., see [6]) to construct functions in $\mathrm{SPA} \setminus \mathrm{SPA}^*$. In [6] we showed that SPA functions need not preserve the connectivity of closed convex sets having nonempty interiors and, thus, they need not be SA functions. However, in that same paper we showed that SPA functions do preserve the connectivity of open connected sets. Furthermore, SPA functions are quite nice in that their graphs can be approximated (e.g., graphically represented on a computer screen) with interpolating triangular patches of arbitrarily small diameter.

In this context, the current paper is related to the so-called "Woodcutters Problem" which provided some of the motivation for the current study. Cavaretta, Dahmen, and Micchelli begin their comprehensive study of algorithmic methods for viewing curves and surfaces, [3], as follows:

> Subdivision methods on computer graphics constitute a large class of recursive schemes for computing curves and surfaces... The algorithms begin with some initial set of discrete data, called the control points [here called *vertices*], which one can visualize as the vertices of a given polyhedral surface. A particular algorithm is determined by a few simple linear rules which, used repeatedly, successively generate 'denser' data sets from the initial set of control points. If the rules are well chosen, these ever more dense data sets will approach some continuous curve or surface in the limit. At any particular stage, the limiting surface can be approximated by a polyhedral surface which interpolates the newly generated data.

Subsequently they comment that

> Despite the simplicity of the algorithms themselves, the analysis of the limiting curve or surface associated with any given algorithm seems to be formidable.

Also in [9], Micchelli and Prautzsch comment

> However, it is often not clear what kind of curves and surfaces are produced by these procedures.

In this paper we shed some light on the answers to these later questions. First, it is clear that unless a given algorithm guarantees uniform convergence, there are always discontinuous limiting curves or surfaces associated with that algorithm. Deeper insight is entailed by the proof of Theorem 4.1 showing that every sectorially approachable function is strongly polygonally approximable. The methods we develop in this paper and employ in the proof of Theorem 4.1 are highly algorithmic, but our proof also hinges on a theorem of Kuratowski concerning exhibiting a given Baire Class 1 function as the uniform limit of Baire Class 1 functions having discrete range. This sequence of functions having discrete range can be obtained algorithmically in some cases (e.g. if the original function is continuous), but in others it is not so clear. In any case, the proof of Theorem 4.1 and its satellite lemmas provide insight into the nature of the Woodcutters Problem. See [3] and [9] for details. Our goal for the present work is not to explore this question, but to establish that SA $\subseteq$ SPA.

Before beginning our work we need yet a bit more notation:

If $g\colon \mathbb{R}^2 \to \mathbb{R}$ and $T = T(v_0, v_1, v_2)$ is a triangle, we define the *vertex-oscillation of g on T* as

$$\mathrm{v} - \mathrm{osc}(g, T) = \max\{|g(v_i) - g(v_j)|\colon\ i, j = 0, 1, 2\}.$$

For each natural number m we let $\mathcal{P}_m$ denote the regular partition of $\mathbb{R}^2$ into the squares $R^m_{i,j} = [(i-1)/2^m, i/2^m] \times [(j-1)/2^m, j/2^m]$; $i, j \in \mathbb{Z}$. From this we obtain *the basic triangulation $\mathcal{B}_m$ of* $\mathbb{R}^2$ by dividing each $R^m_{i,j}$ into two right triangles, by inserting the diagonal joining the upper left vertex to the lower right. Each such triangle is called a *basic triangle* of $\mathcal{B}_m$. If $T(v_0, v_1, v_2)$ is a basic triangle, we denote its edges as $E_0 = [v_0, v_1]$, $E_1 = [v_1, v_2]$, and $E_2 = [v_2, v_0]$. Given a finite collection $\mathcal{K} = \{K_1, K_2, \ldots K_n\}$ of disjoint compact sets, we shall say that the basic triangulation $\mathcal{B}_m$ of $\mathbb{R}^2$ *separates* $\mathcal{K}$ if whenever T is a basic triangle in $\mathcal{B}_m$ with $T \cap K_i \neq \emptyset$, and S is a basic triangle in $\mathcal{B}_m$ for which $T \cap S \neq \emptyset$, then $S \cap K_l = \emptyset$ for all $l \neq i$. Clearly, given $\mathcal{K}$, it is always possible to find an m large enough so that $\mathcal{B}_m$ separates $\mathcal{K}$.

Next, when we say that one triangulation $\mathcal{T}'$ of $\mathbb{R}^2$ is a *refinement* of another triangulation $\mathcal{T}$ of $\mathbb{R}^2$ we mean, as is customary with partitions, that each triangle in $\mathcal{T}$ has been triangulated by triangles in $\mathcal{T}'$ with the interior of each triangle of T' lying in the interior of exactly one triangle in $\mathcal{T}$. Less restrictively, when we say that one triangulation $\mathcal{T}'$ of $\mathbb{R}^2$ is an *enhancement* of another triangulation $\mathcal{T}$ of $\mathbb{R}^2$ we mean that all vertices of triangles in $\mathcal{T}$ occur as vertices of triangles in $\mathcal{T}'$ with the interior of each triangle of T' lying in the union of at most two triangles in $\mathcal{T}$.

If $x \in \mathbb{R}^2$ and θ is an angle, we let $r(x, \theta)$ denote the ray emanating from x in the direction θ and $l(x, \theta)$ denote the line through x with direction θ. If $x_1 \neq x_2$ are both in $\mathbb{R}^2$, we denote the ray emanating from x_1 and passing through x_2 by $r(x_1, x_2)$, the line segment between x_1 and x_2 by $[x_1, x_2]$, and the line containing both the points by $l(x_1, x_2)$. If $\delta > 0$ and $x \in \mathbb{R}^2$, we denote the open ball of radius δ about x by $B_\delta(x)$. Finally, if $S \subseteq \mathbb{R}^2$, we use $\mathrm{hull}(S)$ to denote the closed convex hull of S.

2. A preliminary triangulation lemma

Throughout this section we shall assume that $f\colon \mathbb{I}^2 \to \mathbb{R}$ is sectorially approachable, that $F\colon \mathbb{R}^2 \to \mathbb{R}$ is its symmetric periodic extension, that both $\varepsilon > 0$ and a constant real number k are fixed, that C is a closed set, and that $C \subseteq W_\varepsilon$, where for each $\eta > 0$ we let $W_\eta = \{x \in \mathbb{R}^2\colon\ |F(x) - k| < \eta\}$.

The goal of this section is to establish the following:

Lemma 2.1. *For each m, there is a triangulation $\mathcal{S}_m$ of $\mathbb{R}^2$ which is an enhancement of $\mathcal{B}_m$ such that*

1. *Every triangle $T \in \mathcal{S}_m$ hits at most two basic triangles of $\mathcal{T}_m$ at points other than vertices of those basic triangles.*
2. *Every triangle $T \in \mathcal{S}_m$ is one of the following three types.*
   (a) *All three vertices of T are in $W_{2\varepsilon}$.*
   (b) *Two vertices of T are in $W_{2\varepsilon}$ and the third vertex is a vertex of a basic triangle.*
   (c) *One vertex of T is in $W_{2\varepsilon}$ and the other two vertices are vertices of a basic triangle.*

2.1. Proof of Lemma 2.1 for the special case where C intersects no edge of any basic triangle

Fix an m. Here we shall be considering the special case where C intersects no edge of any basic triangle in $\mathcal{B}_m$. In this situation, our triangulation of $\mathbb{R}^2$ will result from a triangulation of each basic triangle. Fix a $T \in \mathcal{B}_m$. If $C \cap T = \emptyset$, T itself is one of our triangulating triangles.

Next suppose that $\emptyset \neq C \cap T \subset \operatorname{int}(T)$. Let $K = \operatorname{hull}(C \cap T)$. For ease of exposition, let us assume that E_0 is horizontal. Denote $\theta_i = \min(\{\arg(x - v_i)\colon x \in K\})$ for $i = 0, 1, 2$, where $-\pi \leqslant \arg(z) < \pi$. For each i, $r(v_i, \theta_i) \cap K$ is a (possibly degenerate) line segment which we denote as $[p_i, q_i]$ where $d(v_i, p_i) \leqslant d(v_i, q_i)$. Note that since C is closed, p_i and q_i must be in C. Let $\theta_i^* = \arg(q_{i-1} - v_i)$ (when $i = 0$, we set $\theta_0^* = \arg(q_2 - v_0)$. Then for every $\theta \in [\theta_i, \theta_i^*]$, $r(v_i, \theta) \cap K$ is a line segment $[\alpha(\theta), \underline{(}\theta)]$ where $d(\alpha(\theta), v_i) \leqslant d(\underline{(}\theta), v_i)$. As K is convex, the function $\alpha\colon [\theta_i, \theta_i^*] \to \partial K$ is a continuous parameterization of a portion of the boundary of K.

Our initial goal is to show that for each $\theta \in [\theta_0, \theta_0^*]$, there are two points $a(\theta)$, $b(\theta)$ in $W_{2\varepsilon}$ such that both $\arg(a(\theta) - v_0) < \theta < \arg(b(\theta) - v_0)$ and $T(v_0, a(\theta), b(\theta)) \cap K = \emptyset$.

To this end, we first consider the case where $\theta \in (\theta_0, \theta_0^*)$. Here there are two possibilities: $\alpha(\theta) \in C \cap T$ and $\alpha(\theta) \notin C \cap T$.

Suppose that $\alpha(\theta) \in C \cap T$. Let $q_2' = l(p_0, \alpha(\theta)) \cap l(v_0, q_2)$ and $p_0' = l(v_0, p_0) \cap l(q_2, \alpha(\theta))$. Then, since K is convex, we have both $T(v_0, q_2', \alpha(\theta)) \cap K = \emptyset$ and $T(v_0, p_0', \alpha(\theta)) \cap K = \emptyset$. Since $F \in \mathrm{SA}$ there exists a $b(\theta) \in \operatorname{int}(T(v_0, q_2', \alpha(\theta)) \cap W_\varepsilon$ and there exists an $a(\theta) \in \operatorname{int}(T(v_0, p_0', \alpha(\theta)) \cap W_\varepsilon$. Furthermore, $T(v_0, a(\theta), b(\theta)) \cap K = \emptyset$.

Next, suppose that $\theta \in (\theta_0, \theta_0^*)$, but $\alpha(\theta) \notin C \cap T$. As $C \cap T$ is compact, there exist two angles $\psi_1 < \theta$ and $\psi_2 > \theta$ such that both $\alpha(\psi_1)$ and $\alpha(\psi_2)$ belong to $C \cap T$ and for every $\gamma \in (\psi_1, \psi_2)$ we have $\alpha(\gamma) \in [\alpha(\psi_1), \alpha(\psi_2)] \setminus (C \cap T)$. We let $q_2' = l(v_0, q_2) \cap l(p_0, \alpha(\psi_2))$ and $p_0' = l(v_0, p_0) \cap l(q_2, \alpha(\psi_1))$. Again, as K is convex, both $T(v_0, q_2', \alpha(\psi_2)) \cap K = \emptyset$ and $T(v_0, p_0', \alpha(\psi_1)) \cap K = \emptyset$. Since $F \in \mathrm{SA}$ there exists a point $a(\theta) \in \mathrm{int}(T(v_0, p_0', \alpha(\psi_1)) \cap W_{2\varepsilon}$ and there exists a $b(\theta) \in \mathrm{int}(T(v_0, q_2', \alpha(\psi_2)) \cap W_{2\varepsilon}$. Note that $T(v_0, a(\theta), b(\theta)) \cap K = \emptyset$.

Next, we consider the case where $\theta = \theta_0$. Let $v_0' = r(p_0, -\pi/2) \cap [v_0, v_1]$. Then $\mathrm{int}(T(v_0, v_0', p_0)) \cap K = \emptyset$ and as $F \in \mathrm{SA}$, there exists a point $a(\theta_0)$ in $\mathrm{int}(T(v_0, v_0', p_0)) \cap W_{2\varepsilon}$. Defining $b(\theta_0)$ takes a bit more effort. Let z denote the (necessarily unique) point of $K \cap T(v_0, q_2, p_0)$ closest to $a(\theta_0)$.

If $z = p_0$, then $B_\delta(a(\theta_0)) \cap K = \emptyset$ where $\delta = |p_0 - a(\theta_0)|$. As K is convex, this entails that $\mathrm{int}(T(v_0, p_0, p_0')) \cap K = \emptyset$ where $p_0' = r(p_0, \pi) \cap [v_0, v_1]$. Since $F \in \mathrm{SA}$, there exists a $b(\theta_0) \in \mathrm{int}(T(v_0, p_0, p_0')) \cap W_{2\varepsilon}$. If $z \neq p_0$, but $z \in C \cap T$, then $\mathrm{int}(T(v_0, z, a(\theta_0))) \cap K = \emptyset$. Again as $F \in \mathrm{SA}$, there exists a $b(\theta_0) \in \mathrm{int}(T(v_0, z, a(\theta_0))) \cap W_{2\varepsilon}$. Finally, suppose $z \neq p_0$ and $z \notin C \cap T$. Here, there exist $z_1, z_2 \in \partial(K \cap C \cap T)$ such that $z \in [z_1, z_2]$. We suppose $\arg(z_1 - v_0) < \arg(z_2 - v_0)$. Since $F \in \mathrm{SA}$ there is a $b(\theta_0) \in \mathrm{int}(T(v_0, z_2, z)) \cap W_{2\varepsilon}$. Now, $\mathrm{int}(T(v_0, z_1, z_2)) \cap K = \emptyset$ and $\mathrm{int}(T(v_0, a(\theta_0), z)) \cap K = \emptyset$, and since $\mathrm{int}(T(v_0, a(\theta_0), b(\theta_0)))$ is contained in the union of those two triangles, we have $T(v_0, a(\theta_0), b(\theta_0)) \cap K = \emptyset$, as well. This completes the case for $\theta = \theta_0$.

Finally, consider $\theta = \theta_0^*$. Since $F \in \mathrm{SA}$ there is a $b(\theta_0^*) \in \mathrm{int}(T(v_0, q_2, q_2'')) \cap W_{2\varepsilon}$, where $q_2'' = r(q_2, \pi) \cap E_2$. Likewise there is an $a(\theta_0^*) \in \mathrm{int}(T(v_0, v_0'', q_2)) \cap W_{2\varepsilon}$, where $v_0'' = r(q_2, -\pi/2) \cap E_0$. Since $\mathrm{int}(T(v_0, q_2, q_2'')) \cap K = \emptyset$ and $\mathrm{int}(T(v_0, v_0', q_2)) \cap K = \emptyset$, it follows that

$$\mathrm{int}(T(v_0, a(\theta_0^*), b(\theta_0^*))) \cap K = \emptyset.$$

Recapping, to this point we have shown that for each $\theta \in [\theta_0, \theta_0^*]$, there are two points $a(\theta)$, $b(\theta)$ in $W_{2\varepsilon}$ such that $\arg(a(\theta) - v_0) < \theta < \arg(b(\theta) - v_0)$ and $T(v_0, a(\theta), b(\theta)) \cap K = \emptyset$. For each $\theta \in [\theta_0, \theta_0^*]$, let $\psi_a(\theta) = \arg(a(\theta) - v_0)$ and $\psi_b(\theta) = \arg(b(\theta) - v_0)$. Then the collection $\{(\psi_a(\theta), \psi_b(\theta))\colon \theta_0 \leqslant \theta \leqslant \theta_0^*\}$ is an open cover for $[\theta_0, \theta_0^*]$ and as such contains a finite chain cover, say $\{(\psi_a(\theta^i), \psi_b(\theta^i))\colon i = 1, 2, \ldots, P_0\}$, where $\psi_a(\theta^{i+1}) < \psi_b(\theta^i) < \psi_a(\theta^{i+2})$ for $i = 1, 2, \ldots P_0 - 2$. Using this, it is a straightforward matter to see that there is a set of triangles $T_j(v_0, A_{0,j}, A_{0,j+1})$, $j = 1, 2, \ldots, M_0 \leqslant P_0$ such that $\arg(A_{0,1} - v_0) = \psi_a(\theta^1)$, $\arg(A_{M_0+1} - v_0) = \psi_b(\theta^{P_0})$, $T(v_0, A_{0,j}, A_{0,j+1}) \cap K = \emptyset$, for each $j = 1, 2, \ldots, M_0$, each $A_{0,j} \in \{a(\theta^i), b(\theta^i)\colon i = 1, 2, \ldots, P_0\}$, and the polygon, H_0, having vertices $p_0, A_{0,1}, A_{0,2}, \ldots, A_{0,M_0}, q_2$ is convex.

We proceed to vertex v_1 and form the analogous convex polygon H_1 having vertices $p_1, A_{1,1}, A_{1,2}, \ldots, A_{1,M_1}, q_0$; then on to v_2 and form the analogous convex polygon H_2 having vertices $p_2, A_{2,1}, A_{2,2}, \ldots, A_{2,M_2}, q_1$. We then start to assemble our collection of triangles which will triangulate T. We start by first triangulating H_0 by connecting q_2 to each other vertex of H_0. Likewise, we triangulate H_1 by connecting q_0 to all vertices of H_1, and triangulate H_2 by connecting q_1 to all vertices of H_2. The portion of K not yet covered by this collection of triangles is either empty or is a polygon, which we may readily cover with a triangulation, all of whose triangles lie in K, and whose vertices are in $C \cap T$. To this point, we have started our triangulation of T by assembling triangles which cover K. Note that every triangle selected so far has all three of its vertices in $W_{2\varepsilon}$.

We next supplement this collection of triangles as follows: We add all triangles of the form $T(v_0, A_{0,j}, A_{0,j+1})$, $j = 1, 2, \ldots M_0 - 1$. Recall that none of these triangles intersect K and each has two vertices in $W_{2\varepsilon}$ while v_0 is the third vertex. Next we add the two triangles $T(A_{0,1}, p_0, q_0)$ and $T(A_{0,1}, q_0, A_{1,M_1})$, noting that this is just one triangle if $p_0 = q_0$. These two triangles have all three vertices in $W_{2\varepsilon}$. Next we note which of $A_{0,1}$ and A_{1,M_1} is closer to E_0. If it is $A_{0,1}$, or if there is a tie, we add the triangles $T(v_1, A_{0,1}, A_{1,M_1})$ and $T(v_0, A_{0,1}, v_1)$ to our collection noting that neither intersects any previously selected triangle. However, if A_{1,M_1} is the closer we instead add the triangles $T(v_0, A_{0,1}, A_{1,M_1})$ and $T(v_0, A_{1,M_1}, v_1)$ to our collection, again noting that neither intersects any previously selected triangle. We then move to v_1 repeating this process, and finally to v_2. At that point we will have completed the triangulation of T. Note that each edge of T is an edge of a triangulating triangle. Thus, as we do this for each $T \in \mathcal{T}_m$, we arrive at a triangulation of $\mathbb{R}^2$ and it has the desired properties, thereby completing the proof of Lemma 2.1, but only in the special case where C touches no edge of triangles in $\mathcal{B}_m$.

To obtain the proof of Lemma 2.1 for the general situation, we introduce the notion of "bridge pairs."

2.2. Bridge pairs for edges of basic triangles

Again, let m be fixed, and let $T = T(v_0, v_1, v_2)$ be a basic triangle of $\mathcal{B}_m$. We find it convenient to let u_i denote the midpoint of the edge of T opposite v_i. Depending on how $K \equiv \text{hull}(C \cap T)$ intersects an edge of T we will associate two, one, or no pairs of "bridge points" with that edge. For ease of notation suppose $v_0 = (0, 0)$, $v_1 = (1, 0)$, and $v_2 = (1, 0)$ and consider how to associate bridge pairs with E_0. We proceed by (not necessarily mutually exclusive) cases.

- If $K \cap E_0 = \emptyset$ or $K \cap E_0 = E_0$, then we associate no bridge points with E_0.
- If $K \cap E_0 \neq \emptyset$ and $v_0 \notin K \cap E_0$, then we associate a pair of bridge points $w(T, E_0, v_0), w'(T, E_0, v_0)$ with E_0. This pair will have the following properties.
  - $w(T, E_0, v_0) \in \text{int}(T)$, and $w'(T, E_0, v_0) \in \text{int}(T')$, where $T' \in \mathcal{T}_m$ and shares the edge E_0 with T. More specifically, we require both that $w(T, E_0, v_0) \in \text{int}(T(v_0, v_1, u_1))$ and $w'(T, E_0, v_0) \in \text{int}(T(v_0, v_1, u_1'))$, where u_1' is the midpoint of the edge of T' opposite v_1.
  - $|F(w(T, E_0, v_0)) - k| < 2\varepsilon$ and $|F(w'(T, E_0, v_0)) - k| < 2\varepsilon$.

– $T(v_0, w(T, E_0, v_0), w'(T, E_0, v_0)) \cap (K \cup K') = \emptyset$, where $K' = \mathrm{hull}(C \cap T')$. Let $p_0 = (c, 0)$ be the closest point to v_0 of $K \cap E_0$. We consider various possibilities based on whether the convex sets K and K' approach p_0 tangentially or non-tangentially:

– As a first case, suppose that neither approach is tangential. Hence, there exist points $z = (x, y) \in \mathrm{int}(T(v_0, v_1, u_1))$ and $z' = (x, -y) \in \mathrm{int}(T(v_0, v_1, u_1'))$, where $0 < x < c$, $y > 0$, $\|z\| < c$, and both $T(v_0, p_0, z) \cap K$ and $T(v_0, p_0, z') \cap K'$ equal $\{p_0\}$. Since $F \in \mathrm{SA}$ there are points $w'(T, E_0, v_0) \in \mathrm{int}(T(v_0, p_0, z'))$ and $w(T, E_0, v_0) \in \mathrm{int}(T(v_0, p_0, z))$ such that we have both $|F(w(T, E_0, v_0)) - k| < 2\varepsilon$ and $|F(w'(T, E_0, v_0)) - k| < 2\varepsilon$. Clearly,

$$T(v_0, w(T, E_0, v_0), w'(T, E_0, v_0)) \cap (K \cup K') = \emptyset.$$

– As a second case, suppose that both approaches are tangential. Just as $\alpha(\theta)$ denotes the closest point to v_0 of the intersection of $\partial(K)$ with the ray $r(v_0, \theta)$, we let $\alpha'(\theta)$ denote the closest point to v_0 of the intersection of $\partial(K') \cap r(v_0, \theta)$. In the present case there are two sequences of angles $\{\varphi_n\}$ decreasing to 0 and $\{\varphi_n'\}$ increasing to 0 such that each $\alpha(\varphi_n)$ and $\alpha'(\varphi_n')$ are in C. Due to the tangential approaches and the convexity of K and K', we may further assume that both the sequences $\{\|\alpha(\varphi_n)\|\}$ and $\{\|\alpha'(\varphi_n')\|\}$ increase to c. Without loss of generality, we shall further assume that $\alpha'(\varphi_1') \in \mathrm{int}(T(v_0, v_1, u_1'))$ and $\arg(p_0 - \alpha'(\varphi_1')) < \pi/4$.
Now, there is an n_1 such that both $\alpha(\varphi_{n_1}) \in \mathrm{int}(T(v_0, v_1, u_1))$ and $2\|\alpha(\varphi_{n_1})\| > \|\alpha'(\varphi_1')\| + c$. Since $F \in \mathrm{SA}$ there is a $w(T, E_0, v_0) \in T(v_0, \alpha(\varphi_{n_1}), x_0)$, where $x_0 = \mathrm{proj}(\alpha(\varphi_{n_1}), x\text{-axis})$, such that we have $|F(w(T, E_0, v_0)) - k| < 2\varepsilon$. The line $l(w(T, E_0, v_0), x_0)$ intersects $\partial(K')$ at a point within the triangle $T(v_0, p_0, \alpha'(\varphi_1'))$ since the segment $[p_0, \alpha'(\varphi_1')] \subseteq K'$. Let x_1 denote this point of intersection. If $x_1 \in C$, set $x^* = x_1$. If $x_1 \notin C$, then there are two points c_0 and c_1 of C such that $x_1 \in [c_0, c_1] \subseteq \partial(K')$, where c_0 is further from p_0 than is c_1. It follows from the convexity of K' that $c_0 \in T(v_0, \alpha'(\varphi_1'), p_0)$. In this case, set $x^* = c_0$.
Now, in whichever way x^* is chosen, we have $T(v_0, w(T, E_0, v_0), x^*) \cap K' = \{x^*\}$ and since F is sectorially approachable, there is a point $w'(T, E_0, v_0) \in T(v_0, b(\theta_0), x^*) \cap S'$ such that $|F(w'(T, E_0, v_0)) - k| < 2\varepsilon$. Finally, as $T(v_0, \alpha(\varphi_{n_1}), x_0) \cap K = \emptyset$, it follows that the triangle $T(v_0, w(T, E_0, v_0), x^*)$ does not intersect K, and so

$$T(v_0, w(T, E_0, v_0), w'(T, E_0, v_0)) \cap (K \cup K') = \emptyset.$$

– The final case is the mixed one; that is, suppose for specificity that we have nontangential approach above and tangential approach below. Then there is a point $z = (x, y) \in \mathrm{int}(T(v_0, v_1, u_1))$ such that $0 < x < c$, $\|z\| < c$ and $T(v_0, p_0, z) \cap K = \{p_0\}$. Since F is sectorially approachable, there is a point $w(T, E_0, v_0)$ in the interior of $T(v_0, p_0, z)$ such that $|F(w(T, E_0, v_0)) - k| < 2\varepsilon$. There is a sequence of angles $\{\varphi'_n\}$ increasing to 0 such that each $\alpha'(\varphi'_n) \in C$. Due to the tangential approach and the convexity of K', we may further assume that $\{\|\alpha'(\varphi'_n)\|\}$ increases to c. Choose n_1 so that $\alpha'(\varphi'_n) \in \mathrm{int}(T(v_0, v_1, u'_1))$ and $\|\alpha'(\varphi'_n)\| > \|w(T, E_0, v_0)\|$. Now, $T(v_0, w(T, E_0, v_0), \alpha'(\varphi'_n)) \cap (K \cup \mathcal{K}') = \emptyset$ and because $F \in \mathrm{SA}$ there is a point $w'(T, E_0, v_0)$ belonging to the set $T(v_0, w(T, E_0, v_0), \alpha'(\varphi'_n)) \cap S'$ such that $|F(w'(T, E_0, v_0)) - k| < 2\varepsilon$. Clearly,

$$T(v_0, w(T, E_0, v_0), w'(T, E_0, v_0)) \cap (K \cup K') = \emptyset.$$

- If $K \cap E_0 \neq \emptyset$ and $v_1 \notin K \cap E_0$, then we associate a pair of bridge points $w(T, E_0, v_1), w'(T, E_0, v_1)$ with E_0. This pair will have the following properties.
  – $w(T, E_0, v_1) \in \mathrm{int}(T)$, and $w'(T, E_0, v_1) \in \mathrm{int}(T')$, where $T' \in \mathcal{T}_m$ and shares the edge E_0 with T. More specifically, we require both that $w(T, E_0, v_1) \in \mathrm{int}(T(v_0, v_1, u_0))$, and $w'(T, E_0, v_0) \in \mathrm{int}(T(v_0, v_1, u'_0))$, where u'_0 is the midpoint of the edge of T' opposite v_0.
  – Both $|F(w(T, E_0, v_1)) - k| < 2\varepsilon$ and $|F(w'(T, E_0, v_1)) - k| < 2\varepsilon$.
  – $T(v_1, w(T, E_0, v_1), w'(T, E_0, v_1)) \cap (K \cup K') = \emptyset$, where $K' = \mathrm{hull}(C \cap T')$.
  This pair of points is obtained in the same manner as in the previous case.

We proceed analogously to define bridge point pairs associated with sides E_1 and E_2. This completes our discussion of bridge points associated with a basic triangle T. However, we should point out that we will work our way through all basic triangles lexicographically so that if a basic triangle T^* has an edge that had a bridge pair (or pairs) associated with one of its edges previously due to the fact that $T^* = T'$ for some T that occurred earlier in the ordering, that edge will be associated with the same pair(s).

2.3. Proof of Lemma 2.1 in the general case

Handling the general situation, i.e., where C intersects the boundary of a basic triangle $T \in \mathcal{B}_m$ is a little more involved, but fairly routine. Some of our triangulation triangles intersecting such a T will not entirely lie in T, but will intersect a neighboring basic triangle as well. Care will need to be taken so that such an overlapping triangle will be compatible with how we deal with the neighboring triangle. Indeed, this care will be provided in that the only such overlapping triangles will be ones having two vertices being a pair of bridge points associated with an edge of T and the third vertex being either an endpoint of that edge or an endpoint of the line segment where the convex hull of $C \cap T$ intersected that edge. Let's begin by seeing how this would work in a few very simple situations:

First, suppose that $C \cap T = \{v_1\}$. Then from the previous section we have one pair of bridge points $w(T, E_0, v_0)$, $w'(T, E_0, v_0)$ associated with edge E_0, and one pair of bridge points $w(T, E_1, v_1)$, $w'(T, E_1, v_1)$ associated with edge E_1. The triangles for our triangulation of $\mathbb{R}^2$ which arise from T would first consist of the four "bridge triangles":

$$T(v_0, w(T, E_0, v_0), w'(T, E_0, v_0)), \quad T(v_1, w(T, E_0, v_0), w'(T, E_0, v_0)),$$
$$T(v_0, w(T, E_2, v_0), w'(T, E_2, v_0)), \quad \text{and} \quad T(v_2, w(T, E_1, v_1), w'(T, E_1, v_1)).$$

The region of T not covered by these four triangles is polygonal and can readily be triangulated.

Next, suppose that $K \cap T = [v_1, v_2]$, where $K = \text{hull}(C \cap T)$. In this instance we have one pair of bridge points $w(T, E_0, v_0)$, $w'(T, E_0, v_0)$ associated with edge E_0, and one pair of bridge points $w(T, E_2, v_0)$, $w'(T, E_2, v_0)$ associated with edge E_2.The triangles for our triangulation of $\mathbb{R}^2$ which arise from T would first consist of the four "bridge triangles":

$$T(v_0, w(T, E_0, v_0), w'(T, E_0, v_0)), \quad T(v_1, w(T, E_0, v_0), w'(T, E_0, v_0)),$$
$$T(v_1, w(T, E_1, v_1), w'(T, E_1, v_1)), \quad \text{and} \quad T(v_2, w(T, E_2, v_0), w'(T, E_2, v_0)).$$

Again, the region of T not covered by these four triangles is polygonal and can readily be triangulated.

Of course, if all three vertices of T are in $C \cap T$, then there are no bridge pairs and we simply take T to be our triangulation of T.

Let's see how we could handle another very specific, but somewhat more complicated situation: Suppose that $T(v_0, v_1, v_2)$ is a basic triangle from $\mathcal{T}_m$ and that $[v_0, v_1]$ is horizontal, that $\emptyset \neq K \setminus [v_0, v_1] \subset \text{int}(T)$, and that $\emptyset \neq K \cap [v_0, v_1] \subset (v_0, v_1)$, where $K = \text{hull}(C \cap T)$. Thus, $K \cap [v_0, v_1]$ is a line segment (possibly degenerate) $[p_0, q_0]$ where we take p_0 to be the endpoint closer to v_0. Using the previous subsection, we associate with edge $E_0 = [v_0, v_1]$ the pair of bridge points $w(T, E_0, v_0)$, $w'(T, E_0, v_0)$. This pair has the following properties.

- $w(T, E_0, v_0) \in \text{int}(T)$, and $w'(T, E_0, v_0) \in \text{int}(T')$, where $T' \in \mathcal{T}_m$ and shares edge E_0 with T. More specifically, we shall require that $w(T, E_0, v_0) \in \text{int}(T(v_0, v_1, u_1))$, and $w'(T, E_0, v_0) \in \text{int}(T(v_0, v_1, u_1'))$, where u_1' is the midpoint of the edge of T' opposite v_1.

- $|F(w(T, E_0, v_0)) - k| < 2\varepsilon$ and $|F(w'(T, E_0, v_0)) - k| < 2\varepsilon$.
- $T(v_0, w(T, E_0, v_0), w'(T, E_0, v_0)) \cap (K \cup K') = \emptyset$, where $K' = \text{hull}(C \cap T')$.

Likewise, a second pair of bridge points is associated with E_0. These are $w(T, E_0, v_1)$ and $w'(T, E_0, v_1)$ with E_0. This pair has the following properties.

- $w(T, E_0, v_1) \in \text{int}(T)$, and $w'(T, E_0, v_1) \in \text{int}(T')$, where $T' \in \mathcal{T}_m$ and shares edge E_0 with T. More specifically, we shall require that $w(T, E_0, v_1) \in \text{int}(T(v_0, v_1, u_0))$, and $w'(T, E_0, v_0) \in \text{int}(T(v_0, v_1, u_0'))$, where u_0' is the midpoint of the edge of T' opposite v_0.
- Both $|F(w(T, E_0, v_1)) - k| < 2\varepsilon$ and $|F(w'(T, E_0, v_1)) - k| < 2\varepsilon$.
- $T(v_1, w(T, E_0, v_1), w'(T, E_0, v_1)) \cap (K \cup K') = \emptyset$, where $K' = \text{hull}(C \cap T')$.

We place the following four triangles into our triangulation of $\mathbb{R}^2$:

$$T(v_0, w(T, E_0, v_0), w'(T, E_0, v_0)), \quad T(p_0, w(T, E_0, v_0), w'(T, E_0, v_0)),$$
$$T(v_1, w(T, E_0, v_1), w'(T, E_0, v_1)), \quad \text{and} \quad T(q_0, w(T, E_0, v_1), w'(T, E_0, v_1)).$$

These will be the only triangles that intersect the interiors of both T and T', and we refer to these triangles as the bridge triangles for T and T'.

Next, we triangulate the portion of T not covered by the interiors of the bridge triangles, basically as in the special case of Section 2.1. Unless otherwise noted, we shall use the same terminology as in that section. Again, we begin by examining the situation from v_0. As before, we let $\theta_0^* = \arg(q_2 - v_0)$. However, we let $\theta_0 = \arg(w(T, E_0, v_0) - v_0)$. For every $\theta \in (\theta_0, \theta_0^*]$ we proceed exactly as previously to find two points $a(\theta)$, $b(\theta)$ in $W_{2\varepsilon}$ such that $\arg(a(\theta) - v_0) < \theta < \arg(b(\theta) - v_0)$ and $T(v_0, a(\theta), b(\theta)) \cap K = \emptyset$. Then we set $a(\theta_0) = w(T, E_0, v_0)$. Since $T(v_0, a(\theta_0), w'(T, E_0, v_0)) \cap (K \cup K') = \emptyset$, there is a point z such that $\|z\| = \|a(\theta_0)\|$, $\arg(a(\theta_0) - v_0) < \arg(z - v_0) < \arg(q_2 - v_0)$, and $T(v_0, z, a(\theta_0)) \cap K = \emptyset$. Using the sectorial approachability of F, we find a point $b(\theta_0) \in T(v_0, z, a(\theta_0)) \cap W_{2\varepsilon}$. Clearly, $T(v_0, a(\theta_0), b(\theta_0)) \cap K = \emptyset$.

For each $\theta \in [\theta_0, \theta_0^*]$, let $\psi_a(\theta) = \arg(a(\theta) - v_0)$ and $\psi_b(\theta) = \arg(b(\theta) - v_0)$. Then the collection $\{[\psi(a(\theta_0)), \psi(b(\theta_0))\} \cup \{(\psi_a(\theta), \psi_b(\theta))\colon \theta_0 < \theta \leqslant \theta_0^*\}$ is an open cover for $[\theta_0, \theta_0^*]$ and as such contains a finite chain cover, say $\{[\psi_a(\theta_0), \psi_b(\theta_0))\} \cup \{(\psi_a(\theta^i), \psi_b(\theta^i))\colon i = 1, 2, \ldots, P_0\}$, where $\psi_a(\theta^{i+1}) < \psi_b(\theta^i) < \psi_a(\theta^{i+2})$ for $i = 1, 2, \ldots, P_0 - 2$ and $\psi_a(\theta^1) < \psi_b(\theta_0) < \psi_a(\theta^2)$. Using this, it is a straightforward matter to see that there is a set of triangles $T_j(v_0, A_{0,j}, A_{0,j+1})$, $j = 1, 2, \ldots, M_0 \leqslant P_0$ such that $\arg(A_{0,1} - v_0) = \psi_a(\theta^1)$, $\arg(A_{0,M_0+1} - v_0) = \psi_b(\theta^{P_0})$, $T(v_0, A_{0,j}, A_{0,j+1}) \cap K = \emptyset$, for each $j = 1, 2, \ldots, M_0$, each $A_{0,j} \in \{a(\theta^i), b(\theta^i)\colon i = 1, 2, \ldots, P_0\}$, and the polygon, H_0, having vertices

$$a(\theta_0) = w(T, E_0, v_0),\ A_{0,1},\ A_{0,2},\ \ldots,\ A_{0,M_0},\ q_2$$

is convex. We triangulate H_0 by connecting q_2 to each vertex of H_0. We then add the triangle $T(q_2, p_0, w(T, E_0, v_0))$ to our collection.

We then proceed to vertex v_1 where our work is analogous to what was just completed for v_0. Here, q_0 and p_1 will play the roles analogous to p_0 and q_2, respectively, for the v_0 case. Finally, our work from vertex v_2 is identical to what was done in the special case of Section 2.1. The portion of K not yet covered by this collection of triangles is either empty or is a polygon, which we may readily cover with a triangulation, all of whose triangles lie in K, and whose vertices are in C. At this point, our assembled triangulating triangles cover K.

We next supplement this collection of triangles as follows: We add all triangles of the form $T(v_0, A_{0,j}, A_{0,j+1})$, $j = 1, 2, \ldots M_0 - 1$. Recall that none of these triangles intersect K and each has two vertices in $W_{2\varepsilon}$ while v_0 is the third vertex. We then proceed exactly as in the special case to triangulate any remaining regions of T. At that point we will have completed the triangulation process for T. Every triangulating triangle is a subset of T except for the four bridge triangles which overlap T'. When T' is considered, these same four bridge triangles will result.

So, at this point we can say that if every basic triangle $T \in \mathcal{B}_m$ was one of the following types, our triangulation of $\mathbb{R}^2$ would be complete:

1. $T \cap C = \emptyset$.
2. $T \cap C$ consists of one or two vertices of T.
3. All three vertices of T belong to C.
4. $T \cap C \subset \operatorname{int}(T)$.
5. $K = \operatorname{hull}(C \cap T)$ intersects exactly one edge of T and contains neither vertex of that edge.

Of course, numerous other possibilities remain. However, all can be handled with the strategies demonstrated to this point.

3. The Crinkling Lemma

The goal of this section is to show how to obtain an enhancement of the triangulation inherited from Lemma 2.1 that will preserve the approximating properties of that lemma but also have arbitrarily small F-graph mesh.

3.1. The triangulating procedure

As a tool toward that goal we establish a procedure that we will use repeatedly in the proof of the Crinkling Lemma.

Proposition 3.1. *Suppose T is a triangle with a specified vertex V and opposite angles ψ_1 and ψ_2. Suppose that N is a natural number, $0 < \eta < 1/2$, and that* *angles* $0 < \alpha_1 < \psi_1$ *and* $0 < \alpha_2 < \psi_2$ *are given. We define a process by which convex quadrilaterals,* Q_n*,* $n = 1, 2, \ldots, N$ *are inductively described.*

The Construction:

A specific case: Let $a > 0$ and label three points $O = (0,0)$, $A = (a,0)$, and $V = (0,d)$, where both a and d are positive. We first consider the triangle $T = T(O,A,V)$. Set $A' = (a(1-\eta),0)$ and $O' = (a\eta,0)$.

Step 1: Two vertices of Q_1 are O and A. Let p_1 be any point in $T(A,A',V)$ such that

$$\pi - \psi_1 < \arg(p_1 - A) < \pi - \alpha_1,$$

and let q_1 be any point of $T(O,O',V)$ such that

$$\max(\arg(p_1), \alpha_2) < \arg(q_1) < \pi/2.$$

Set $Q_1 = \operatorname{hull}(\{O, A, p_1, q_1\})$.

Note that if $-\alpha_1 < \arg(z - A) < \pi - \psi_1$, then

$$T(p_1, A, z) \cap Q_1 = [A, p_1].$$

Also if $\pi/2 < \arg(z) < \pi + \alpha_2$, then

$$T(q_1, O, z) \cap Q_1 = [O, q_1].$$

Inductive step: Suppose $n < N-1$ and convex quadrilaterals $Q_1, Q_2, \ldots, Q_n$ have been defined such that

1. $\arg(p_{n-1} - V) > \arg(p_n - V) > \arg(A' - V)$,
2. $\arg(V - p_{n-1}) < \arg(p_n - p_{n-1}) < \arg(p_{n-1} - A)$,
3. $\arg(O' - V) > \arg(q_n - V) > \arg(q_{n-1} - V)$,
4. $\arg(V - q_{n-1}) > \arg(q_n - q_{n-1}) > \arg(q_{n-1})$.

The conditions on p_n entail that the set $E \equiv \{z\colon \arg(p_n - V) > \arg(z - V) > \arg(A' - V) \text{ and } \arg(V - p_n) < \arg(z - p_n) < \arg(p_n - A)\} \neq \emptyset$. Let p_{n+1} be any point of E not in Q_n. Moreover, the inductive assumption also entails that the set $F \equiv \{z\colon \arg(O' - V) > \arg(z - V) > \arg(q_n - V) \text{ and } \arg(V - q_n) > \arg(z - q_n) > \arg(q_n)\} \neq \emptyset$. Choose q_{n+1} to be any point of F not in $\operatorname{hull}(\{p_{n+1}, p_n, q_n\})$. This completes the definition of the quadrilateral Q_{n+1} having vertices p_n, q_n, p_{n+1}, q_{n+1}.

Finally, in the case $n = N-1$ we let Q_N be the (degenerate) quadrilateral $\operatorname{hull}(\{q_{N-1}, p_{N-1}, V\})$.

The general case: The general case can be reduced to the first case by use of a suitable affine transformation. Specifically, for the case of a general triangle T with given parameters N, η, ψ_1, ψ_2, α_1, and α_2, first coordinatize so that the vertices of T are $(0,0)$, $(a,0)$, and $V=(c,d)$, where $a>0$ and $d>0$. Let $L\colon \mathbb{R}^2\to\mathbb{R}^2$ be given by $L((x_1,x_2))=(x_1-cx_2/d,x_2)$ and define $Q_n=L^{-1}(Q'_n)$ where Q'_n is the nth quadrilateral specified above for the triangle $T((0,0),(a,0),(0,d))$ with specified angles $\psi'_1=\tan^{-1}(d/a)$, $\psi'_2=\pi/2$, $\alpha'_1=\arg((-\sin\alpha_1-cd^{-1}\cos\alpha_1,\cos\alpha_1))-\arg((-a,d))$ and $\alpha'_2=\frac{1}{2}\pi-\arg((\cos(\psi_1-\alpha_1)-cd^{-1}\sin(\psi_1-\alpha_1),\sin(\psi_1-\alpha_1))$.

This completes our discussion of the triangulation procedure.

3.2. The Crinkling Lemma.

We now prove the following "Crinkling Lemma," one of the two basic pieces of machinery to be used in the proof of the main result.

Lemma 3.1 (The Crinkling Lemma). *Suppose $f\in\mathrm{SA}$, that $F\colon\mathbb{R}^2\to\mathbb{R}$ is its symmetric periodic extension, and $\Theta=\{T_n\colon n=1,2,\dots\}$ is a triangulation of $\mathbb{R}^2$. Suppose too that $\{\mathrm{v}-\mathrm{osc}(F,T)\colon T\in\Theta\}$ is bounded. Then for each $\varepsilon>0$ there is an enhancement Θ^* of Θ such that $\mathrm{v}-\mathrm{osc}(F,T)<\varepsilon$ for every $T\in\Theta^*$. Moreover, for every $T'\in\Theta$*

$$\sup\{\mathrm{v}-\mathrm{osc}(F,T)\colon T\in\Theta^*\colon T\cap T'\neq\emptyset\}<\mathrm{v}-\mathrm{osc}(F,T')+\frac{\varepsilon}{2}.$$

Proof. First, let B denote an upper bound for $\{\mathrm{v}-\mathrm{osc}(F,T)\colon T\in\Theta\}$ and let $N\in\mathbb{N}$ be sufficiently large that $N\varepsilon>2B$. The enhancement discussed in the statement of the lemma is accomplished in two steps.

Step 1. Let $T\in\Theta$ be fixed and identify one vertex as "V." Denote the angles opposite V as ψ_1 and ψ_2 and the corresponding vertices as A_1 and A_2. As Θ is a triangulation, there are two additional triangles $T_i\equiv T(V,A_i,B_i)\in\Theta$ that contain the edges $[V,A_i]$ respectively. To simplify notation we suppose that V lies above both A_1 and A_2, that $[A_1,A_2]$ is horizontal, with A_2 left of A_1. Let $\alpha_1=-\arg(B_1-A_1)$ and $\alpha_2=\arg(A_2-B_2)$. Let $\eta=\frac{1}{10}$. We are now in position to use the triangulating procedure 3.1 on T, but the selection of the points p_i and q_i is no longer quite so arbitrary. Here we set $p_0=A_1$, $q_0=A_2$ and use the fact that $f\in\mathrm{SA}$ (and more specifically that F takes closed convex sets with nonempty interiors to intervals) to select the p_{i+1} and q_{i+1} so that for $i=0,1,\dots,N-1$

$$F(p_{i+1})=\frac{(N-i-1)F(A_1)+(i+1)F(V)}{N}\quad\text{and}$$
$$F(q_{i+1})=\frac{(N-i-1)F(A_2)+(i+1)F(V)}{N}.$$

In particular, we have both $|F(p_{i+1})-F(p_i)|<\frac{1}{2}\varepsilon$ and $|F(q_{i+1})-F(q_i)|<\frac{1}{2}\varepsilon$. It follows from the triangulating procedure that the region of T contiguous to $T(V, A_1, B_1)$ (or to $T(V, A_2, B_2)$) but outside of $\bigcup_{n=1}^{N} Q_n$ is a convex region. We triangulate T by adding edges $[V, p_i]$ and $[V, q_i]$ and also one of the two diagonals to each Q_i for $i = 1, 2, \ldots, N$ to the original edges of T. Note that at this point, we have defined a refinement triangulation of T with the property that any point of T is in a refinement triangle with two of its vertices mapping to within $\frac{1}{2}\varepsilon$ of each other and such that all new vertices map to convex combinations of the values of f at the vertices of T.

Step 2. Let T now denote a triangle from the refinement triangulation described in Step 1 above. Two vertices of T map to within $\frac{1}{2}\varepsilon$ of each other and these we label with A_1 and A_2 with the remaining vertex, called the chosen vertex, labeled by V. We will assume the same general position of these vertices as in Step 1 above. The base angles of T are again denoted by ψ_1 and ψ_2 and the angles, α_1 and a_2 are again determined by the adjacent triangles of the refinement triangulation exactly as in Step 1 above. Denote the two neighboring triangles (from the refinement) that contain the edges $[V, A_i]$ by $T_i \equiv T(V, A_i, B_i) \in \Theta$ respectively. Note that these neighboring triangles need not be contained in the same parent triangle that contained T, but could be a subset of a neighboring parent triangle. We apply the Triangulating procedure to T with $\eta = \frac{1}{10}$, again insisting that the p_{i+1} and q_{i+1} are selected so that for $i = 0, 1, \ldots, N-1$

$$F(p_{i+1}) = \frac{(N-i-1)F(A_1) + (i+1)F(V)}{N} \quad \text{and}$$
$$F(q_{i+1}) = \frac{(N-i-1)F(A_2) + (i+1)F(V)}{N}.$$

Diagonals of the corresponding regions, Q_i for $i = 1, 2, \ldots, N$ are again used to create some of the new triangles of the enhancement. The region of T outside of the $\bigcup_{n=1}^{N} Q_n$ consists of two separate pieces, P_1 containing the points p_i and contiguous to $T(V, A_1, B_1)$ and P_2 containing the points q_i and contiguous to $T(V, A_2, B_2)$. What is done with P_i depends on the nature of the triangle $T(V, A_i, B_i)$, $i = 1, 2$. For specificity we consider the case of $T(V, A_1, B_1)$.

1. The chosen vertex for $T(V, A_1, B_1)$ is also V.
   In this instance, $|F(A_1) - F(B_1)| < \frac{1}{2}\varepsilon$ and quadrilaterals inserted into $T(V, A_1, A_2)$ are oriented in the same direction as the quadrilaterals inserted into $T(V, A_1, B_1)$ and we denote the left vertices of those quadrilaterals by $\{q'_i \colon i = 1, 2, \ldots, N\}$. The Triangulation procedure entails that $P_1 \cup T(V, A_1, B_1)$ is convex and we insert the segments $[p_i, q'_i]$ to form a column of quadrilaterals spanning $T(V, A_1, A_2)$ and $T(V, A_1, B_1)$. We then insert a diagonal in each of these quadrilaterals to obtain the desired triangles for the

enhancement. It is easy to verify that for each of the enhancement triangles, T^*, we have $\mathrm{v}-\mathrm{osc}(F, T^*) < \varepsilon$.

2. The chosen vertex for $T(V, A_1, B_1)$ is A_1.
   This case is analogous to the first case with the exception that the inserted segments are $[p_i, q'_{N-i}]$. Again for each of the enhancement triangles, T^*, it readily follows that $\mathrm{v}-\mathrm{osc}(F, T^*) < \varepsilon$.
3. The remaining case occurs when the chosen vertex for $T(V, A_1, B_1)$ is B_1.
   In this instance, the quadrilaterals for $T(V, A_1, B_1)$ are inserted roughly perpendicular to those in $T(V, A_1, A_2)$. We let the initial "p" vertex for $T(V, A_1, B_1)$ be denoted by p^* so that $T(V, A_1, p^*)$ is one of the refinement triangles from Step 1. We add the segments $[p^*, p_i]$ for $i = 1, 2, \ldots, N$. Since $|F(A_1) - F(V)| < \frac{1}{2}\varepsilon$ due to the definition of $T(V, A_1, B_1)$ and since $|F(p^*) - F(V)| < \frac{1}{2}\varepsilon$ by the choice of p^*, it follows that $\mathrm{v}-\mathrm{osc}(F, T(p^*, p_i, p_{i+1})) < \varepsilon$ for $i = 0, 1, \ldots, N$.

This completes the description of the enhancement required by the lemma. □

4. Main theorem

With the Crinkling Lemma in hand, we are ready to proceed to the proof of our main result:

Theorem 4.1. *If $f\colon \mathbb{I}^2 \to \mathbb{R}$ is sectorially approachable, then f is strongly polygonally approximable.*

Proof. Let $f\colon \mathbb{I}^2 \to \mathbb{R}$ be sectorially approachable, and let $F\colon \mathbb{R}^2 \to \mathbb{R}$ be the symmetric periodic extension of f. According to Theorem 3 on p. 388 in [7] there is a sequence $\{h_n\}$ of Baire class one functions from $\mathbb{R}^2$ to $\mathbb{R}$ converging uniformly to F with the property that for each n, the range of h_n is a discrete set $\{d_1^n, d_2^n, \ldots\}$. We shall assume that for each $x \in I^2$, $|h_n(x) - F(x)| < 1/n$. Note that for every n and every i, the set $h_n^{-1}(d_i^n)$ is of ambiguous class one, i.e., is both a $\mathcal{G}_\delta$ and an $\mathcal{F}_\sigma$.

Let ν denote a finite sequence of natural numbers, i.e., $\nu \in \mathbb{N}^{<\mathbb{N}}$. We represent the length of ν by $|\nu|$. The kth term of ν is denoted $\nu(k)$, and if ν has length at least n, then the truncated sequence $\{\nu(1), \nu(2), \ldots, \nu(n)\}$ is denoted $\nu|_n$. If $\tau = \nu|_n$ for some n we say that ν is an extension of τ. Also, let

$$\mathcal{I}_k = \{\nu \in \mathbb{N}^{<\mathbb{N}} : |\nu| \leqslant k \text{ and } \max\{\nu(1), \nu(2), \ldots \nu(|\nu|)\} \leqslant k\}.$$

For each n and each ν of length n we let M_ν denote the $\mathcal{F}_\sigma$ set

$$M_\nu = h_1^{-1}(d^1_{\nu(1)}) \cap h_2^{-1}(d^2_{\nu(2)}) \cap \ldots \cap h_n^{-1}(d^n_{\nu(n)}),$$

and we note the following properties of these sets:

1. For each n, $\bigcup_{|\nu|=n} M_\nu = \mathbb{R}^2$.
2. If μ is an extension of ν, then $M_\mu \subseteq M_\nu$.
3. If neither μ nor ν is an extension of the other, then $M_\mu \cap M_\nu = \emptyset$.

Next, using a simple diagonalization argument, we may find a collection $\{C_\nu^n\colon \nu \in \mathbb{N}^{<\mathbb{N}}, n \in \mathbb{N}\}$ of closed sets such that:

a. For each ν, $\bigcup_{n=1}^{\infty} C_\nu^n = M_\nu$.
b. If $n < m$, then $C_\nu^n \subseteq C_\nu^m$.
c. If μ is an extension of ν, then $C_\mu^n \subseteq C_\nu^n$.

Now, fix $k \in \mathbb{N}$ and choose $m_k \in \mathbb{N}$ sufficiently large that $\mathcal{B}_{m_k}$ separates each of the k collections of disjoint closed sets

$$\mathcal{C}_l = \{C_\nu^k\colon \nu \in \mathcal{I}_k \text{ and } |\nu| = l\} \quad \text{where } l = 1, 2, \ldots, k.$$

(The astute reader will note that we defined the notion of "separates" for a *finite* collection of *compact* disjoint sets in $\mathbb{R}^2$ and our collection is not finite and the sets are closed, not necessarily compact. However, because of the periodic nature of F, our sets C_ν^k can be taken to be periodic in nature, as well. Thus, if we can separate them restricted to, say $[-4,4] \times [-4,4]$, then we will automatically have them separated in the plane.)

Let $\mathcal{W}_k$ denote the collection of all $T \in \mathcal{B}_{m_k}$ for which $T \cap \left(\bigcup_{\nu \in \mathcal{I}_k} C_\nu^k\right) \neq \emptyset$. For each $T \in \mathcal{W}_k$ we let $\mu \equiv \mu(T)$ denote the (necessarily unique) longest $\nu \in \mathcal{I}_k$ such that $T \cap C_\nu^k \neq \emptyset$ and set $l(T) = |\mu|$. We next define an equivalence relation $\approx$ on $\mathcal{W}_k$ by letting $T \approx S$ provided $C_{\mu(T)}^k = C_{\mu(S)}^k$. Let E_T denote the equivalence class of T and let $\mathcal{E}$ denote the collection of equivalence classes. We then choose $n_k > m_k$ so large that $\mathcal{T}_{n_k}$ strongly separates the collection of disjoint closed sets

$$\mathcal{K}_k = \left\{ W_{E_T} \equiv \bigcup_{S \in E_T} C_{\mu(T)}^k \cap S\colon\ E_T \in \mathcal{E} \right\},$$

where by "strongly separates" we mean that if W_{E_T} and $W_{E_{T'}}$ are disjoint, then every line segment from W_{E_T} to $W_{E_{T'}}$ intersects the interior of a triangle T in $\mathcal{B}_{n_k}$ for which T contains no point of any set in $\mathcal{K}_k$.

For each $T \in \mathcal{W}_k$, we let U_T denote the union of all basic triangles in $\mathcal{B}_{n_k}$ which intersect W_{E_T} and V_T denote the union of all basic triangles in $\mathcal{B}_{n_k}$ which intersect U_T. Let x_{E_T,n_k} denote the triangulation of $\mathbb{R}^2$ which is the enhancement of $\mathcal{B}_{n_k}$ obtained by applying Lemma 2.1 with $C = W_{E_T}$ and $\varepsilon = 1/k$. Let $H_{k,E_T}\colon \mathbb{R}^2 \to \mathbb{R}$ be the F-based continuous piecewise linear function supported by x_{E_T,n_k}.

For each basic triangle $\Delta \in \mathcal{B}_{n_k}$, let $L_{\Delta,k}\colon\ \Delta \to \mathbb{R}$ be the linear function determined by the values of F at the three vertices of Δ.

Then, let $L_k\colon\ \mathbb{R}^2 \to \mathbb{R}$ be given by

$$L_k(x) = \begin{cases} H_{k,E_T}(x) & \text{if } x \in V_T \text{ for some } T \in \mathcal{W}_k, \\ L_{\Delta,k}(x) & \text{if } x \text{ is not in any } V_T \text{ and } x \in \Delta \in \mathcal{T}_{n_k}. \end{cases}$$

Then L_k has the following properties:

- L_k is continuous and piecewise linear.
- If $x \in W_{E_T}$ then $|L_k(x) - h_{l(T)}| < 2/k$.

Let x_k denote the triangulation of $\mathbb{R}^2$ which supports L_k. Apply the Crinkling Lemma with $\varepsilon = 1/k$ to obtain a triangulation enhancement x_k^* of x_k. We have $\|S_k^*\|_F < 1/k$. Furthermore, if $L_k^*\colon\ \mathbb{R}^2 \to \mathbb{R}$ denotes the F-based continuous piecewise linear function supported by S_k^*, then for each $T \in x_k$, the range of L_k on T is within $2/k$ of the range of L_k^* on the union of all triangles in x_k^* which intersect T. Consequently, if $x \in W_{E_T}$ then $|L_k^*(x) - h_{l(T)}| < 4/k$.

This completes the definition of the sequence $\{L_k^*\}$ and what remains is to show that this sequence converges pointwise to F. To this end, fix $x \in \mathbb{R}^2$ and let $\varepsilon > 0$. Choose $K > 8/\varepsilon$ sufficiently large that $x \in C_\nu^K$ for some $\nu \in \mathcal{I}_K$ with $|\nu| > 2/\varepsilon$. For $k \geqslant K$, let $T \in \mathcal{T}_{m_k}$ contain x. Then, $l(T) \geqslant |\nu| > 2/\varepsilon$ since $C_\nu^k \cap T \neq \emptyset$. Hence,

$$\begin{aligned} |L_k^*(x) - F(x)| &< |L_k^*(x) - h_{l(T)}(x)| + |h_{l(T)}(x) - F(x)| \\ &< \frac{4}{k} + \frac{1}{l(T)} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

This completes the proof. $\square$

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