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WEIGHTED SUB-BERGMAN HILBERT SPACES IN THE UNIT DISK

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Abstract. We study sub-Bergman Hilbert spaces in the weighted Bergman space A^2_α. We generalize the results already obtained by Kehe Zhu for the standard Bergman space A^2.

Keywords: weighted Bergman space, sub-Bergman Hilbert space, weighted Toeplitz operator, reproducing kernel

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1. Introduction

Let $\mathbb{D}$ denote the unit disk in the complex plane. For $\alpha > -1$ we define the weighted Bergman space A^2_α as the space of all analytic functions f in $\mathbb{D}$ such that

$$\int_{\mathbb{D}} |f(z)|^2 \,\mathrm{d}A_\alpha(z) < +\infty$$

where $\mathrm{d}A_\alpha(z) = (\alpha+1)\pi^{-1}(1-|z|^2)^\alpha \,\mathrm{d}x\,\mathrm{d}y$ denotes the normalized area measure. It is well-known that A^2_α is a Hilbert space of analytic functions. The weighted Bergman projection $P_\alpha\colon\ L^2(\mathbb{D},\mathrm{d}A_\alpha) \to A^2_\alpha$ is defined by

$$P_\alpha f(z) = \int_{\mathbb{D}} f(w) K_\alpha(z,w)\,\mathrm{d}A_\alpha(w),$$

where

$$K_\alpha(z,w) = \frac{1}{(1-z\overline{w})^{2+\alpha}}, \quad (z,w) \in \mathbb{D}\times\mathbb{D}$$

is the reproducing kernel for the space A^2_α. For $\varphi \in L^\infty(\mathbb{D})$, the weighted Toeplitz operator on A^2_α is defined by

$$T^\alpha_\varphi f = P_\alpha(\varphi f).$$

When $\alpha = 0$, we omit the superscript and simply write T_φ instead of T_φ^0; using this convention, P, $\mathrm{d}A$, and $K(z,w)$ stand respectively for P_α, $\mathrm{d}A_\alpha$, and $K_\alpha(z,w)$ in the standard (unweighted) Bergman space case $\alpha = 0$.

Let H_1 and H_2 be two Hilbert spaces, and let $T\colon\ H_1 \to H_2$ be a bounded operator. The range of T with the inner product

$$\langle Tx, Ty\rangle_{H_2} = \langle x, y\rangle_{H_1}, \qquad x, y \in H_1 \ominus \ker T,$$

is denoted by $\mathcal{M}(T)$. The Hilbert space

$$\mathcal{H}(T) = \mathcal{M}((I - TT^*)^{1/2})$$

is called the complemented space to $\mathcal{M}(T)$.

Recall that $H^\infty = H^\infty(\mathbb{D})$ denotes the Banach space of all bounded analytic functions on the unit disk; we denote its unit ball by $(H^\infty)_1$. We consider a function $\varphi \in (H^\infty)_1$ and study the spaces $\mathcal{H}(T_\varphi^\alpha)$ and $\mathcal{H}(T_{\overline{\varphi}}^\alpha)$. These are Hilbert spaces in the weighted Bergman space A_α^2, and are called *sub-Bergman Hilbert spaces*. For simplicity, we denote them by $\mathcal{H}_\alpha(\varphi)$ and $\mathcal{H}_\alpha(\overline{\varphi})$ respectively. For $\alpha = 0$, these spaces were studied by Kehe Zhu in his two subsequent papers [5] and [6]. Indeed, Zhu's work was inspired by the pioneering work of Donald Sarason in introducing the phrase "*sub-Hardy Hilbert spaces*" in [2]. For the history and importance of the sub-Hardy and sub-Bergman Hilbert spaces we refer the reader to the just mentioned papers.

In [5], Zhu proved that $\mathcal{H}(\varphi)$ equals $\mathcal{H}(\overline{\varphi})$ and that both the spaces contain H^∞. He then was able to show that if $\varphi = B$ is a finite Blaschke product, then $\mathcal{H}(B) = H^2$, the Hardy space on the unit disk (see [6]). Here we will see that

$$H^\infty \subset \mathcal{H}_\alpha(\varphi) = \mathcal{H}_\alpha(\overline{\varphi}),$$

for α positive, moreover, if φ equals a finite Blaschke product B, then

$$\mathcal{H}_\alpha(B) = \mathcal{H}_\alpha(\overline{B}) = A_{\alpha-1}^2.$$

We should mention that S. Sultanic in a recent paper obtained the same results by using a very computational method (see [4]).

2. The spaces $\mathcal{H}_\alpha(\varphi)$ and $\mathcal{H}_\alpha(\overline{\varphi})$

This section is devoted to the proof of the fact that the sub-Bergman Hilbert spaces $\mathcal{H}_\alpha(\varphi)$ and $\mathcal{H}_\alpha(\overline{\varphi})$ coincide as sets, and that their norms are equivalent. Moreover, both the spaces contain H^∞.

Proposition 2.1. *Let $\varphi \in (H^\infty)_1$ and $\alpha > -1$. The reproducing kernels of $\mathcal{H}_\alpha(\varphi)$ and $\mathcal{H}_\alpha(\overline{\varphi})$ are given, respectively, by*

$$K^\alpha_\varphi(z,w) = \frac{1-\varphi(z)\overline{\varphi(w)}}{(1-z\overline{w})^{\alpha+2}}, \qquad z,w \in \mathbb{D}$$

and

$$K^\alpha_{\overline{\varphi}}(z,w) = \int_{\mathbb{D}} \frac{1-|\varphi(u)|^2}{(1-z\overline{u})^{\alpha+2}(1-u\overline{w})^{\alpha+2}} \,\mathrm{d}A_\alpha(u).$$

Proof. Suppose that for $w \in \mathbb{D}$, K^α_w are the reproducing kernels of A^2_α. According to I–3 of [2], the reproducing kernels of $\mathcal{H}_\alpha(\varphi)$ are given by

$$(I - T^\alpha_\varphi T^\alpha_{\overline{\varphi}})K^\alpha_w, \qquad w \in \mathbb{D}.$$

Note that for every $z \in \mathbb{D}$ we have

$$\begin{aligned} T^\alpha_{\overline{\varphi}} K^\alpha_w(z) &= \int_{\mathbb{D}} K_\alpha(z,u)\overline{\varphi(u)} K^\alpha_w(u)\,\mathrm{d}A_\alpha(u) \\ &= \overline{\int_{\mathbb{D}} K^\alpha_z(u)\varphi(u)K_\alpha(w,u)\,\mathrm{d}A_\alpha(u)} \\ &= \overline{T^\alpha_\varphi K^\alpha_z(w)} = \overline{\varphi(w)}K^\alpha_w(z) \end{aligned}$$

so that $T^\alpha_{\overline{\varphi}} K^\alpha_w = \overline{\varphi(w)}K^\alpha_w$, and hence

$$\begin{aligned} K^\alpha_\varphi(z,w) &= (I - T^\alpha_\varphi T^\alpha_{\overline{\varphi}})K^\alpha_w(z) \\ &= (1-\overline{\varphi(w)}\varphi)K^\alpha_w(z) \\ &= \frac{1-\varphi(z)\overline{\varphi(w)}}{(1-z\overline{w})^{\alpha+2}}, \qquad z,w \in \mathbb{D}. \end{aligned}$$

As for the second part, we note that according to I–3 of [2], the reproducing kernel of $\mathcal{H}_\alpha(\overline{\varphi})$ has the form

$$K^\alpha_{\overline{\varphi},w} = (I - T^\alpha_{\overline{\varphi}} T^\alpha_\varphi)K^\alpha_w = T_{1-|\varphi|^2}K^\alpha_w.$$

Since for every $z \in \mathbb{D}$ we have

$$\begin{aligned} K^\alpha_{\overline{\varphi}}(z,w) = K^\alpha_{\overline{\varphi},w}(z) &= T_{1-|\varphi|^2}K^\alpha_w(z) \\ &= \int_{\mathbb{D}} \frac{1-|\varphi(u)|^2}{(1-z\overline{u})^{\alpha+2}(1-u\overline{w})^{\alpha+2}}\,\mathrm{d}A_\alpha(u), \end{aligned}$$

the result follows. □

Proposition 2.1. *Let $\varphi \in (H^\infty)_1$ and $\alpha > -1$. Then every element of $\mathcal{H}_\alpha(\overline{\varphi})$ has the representation*

$$f(z) = \int_{\mathbb{D}} \frac{1-|\varphi(w)|^2}{(1-z\overline{w})^{\alpha+2}}\, g(w)\,\mathrm{d}A_\alpha(w),$$

where g is an analytic function satisfying

$$\int_{\mathbb{D}} |g(z)|^2 \left(1-|\varphi(z)|^2\right)\,\mathrm{d}A_\alpha(z) < +\infty.$$

Proof. Put $\mathrm{d}A_{\alpha,\varphi}(z) = (1-|\varphi(z)|^2)\,\mathrm{d}A_\alpha(z)$, and let $A^2_{\alpha,\varphi}$ be the subspace of $L^2(\mathbb{D}, \mathrm{d}A_{\alpha,\varphi})$ consisting of all analytic functions. Define an operator

$$S^\alpha_\varphi\colon\ A^2_{\alpha,\varphi} \to A^2_\alpha$$

by $S^\alpha_\varphi g = P_\alpha((1-|\varphi|^2)g)$. It follows that $\|S^\alpha_\varphi\|_{A^2_\alpha} \leqslant \|g\|_{A^2_{\alpha,\varphi}}$, moreover, for every $f \in A^2_\alpha$ and every $g \in A^2_{\alpha,\varphi}$ we have

$$\begin{aligned}\langle (S^\alpha_\varphi)^* f, g\rangle_{A^2_{\alpha,\varphi}} &= \langle f, P_\alpha\left((1-|\varphi|^2)g\right)\rangle_{A^2_\alpha}\\ &= \langle f, (1-|\varphi|^2)g\rangle_{L^2(\mathbb{D},\mathrm{d}A_\alpha)} = \langle f, g\rangle_{A^2_{\alpha,\varphi}}.\end{aligned}$$

This means that $(S^\alpha_\varphi)^*$ is the inclusion operator. Note that for every $w \in \mathbb{D}$ we have $S^\alpha_\varphi(S^\alpha_\varphi)^* K^\alpha_w \in \mathcal{M}(S^\alpha_\varphi)$. On the other hand, given $f \in \mathcal{M}(S^\alpha_\varphi)$, there exists $g \in A^2_{\alpha,\varphi} \ominus \ker S^\alpha_\varphi$ such that $S^\alpha_\varphi g = f$. Therefore

$$\begin{aligned}\langle f, S^\alpha_\varphi(S^\alpha_\varphi)^* K^\alpha_w\rangle_{\mathcal{M}(S^\alpha_\varphi)} &= \langle g, (S^\alpha_\varphi)^* K^\alpha_w\rangle_{A^2_{\alpha,\varphi}}\\ &= \langle f, K^\alpha_w\rangle_{A^2_\alpha}\\ &= f(w),\end{aligned}$$

which means that $S^\alpha_\varphi(S^\alpha_\varphi)^* K^\alpha_w$ are the reproducing kernels of $\mathcal{M}(S^\alpha_\varphi)$. It now follows that for every $z, w \in \mathbb{D}$ we have

$$\begin{aligned}S^\alpha_\varphi(S^\alpha_\varphi)^* K^\alpha_w(z) &= P_\alpha\left((1-|\varphi|^2)K^\alpha_w\right)(z)\\ &= \int_{\mathbb{D}} \frac{1-|\varphi(u)|^2}{(1-z\overline{u})^{\alpha+2}(1-u\overline{w})^{\alpha+2}}\,\mathrm{d}A_\alpha(u).\end{aligned}$$

This together with Proposition 2.1 implies that $S^\alpha_\varphi(S^\alpha_\varphi)^* K^\alpha_w$ are the reproducing kernels of $\mathcal{H}_\alpha(\overline{\varphi})$, too. Now, from the uniqueness property we conclude that $\mathcal{M}(S^\alpha_\varphi) = \mathcal{H}_\alpha(\overline{\varphi})$. In particular, for every $f \in \mathcal{H}_\alpha(\varphi)$ there is a $g \in A^2_{\alpha,\varphi}$ such that $f = S^\alpha_\varphi g$. □

The next proposition now follows from I–8 and I–9 of [2].

Proposition 2.3. *Let $\varphi \in (H^\infty)_1$, $\alpha > -1$ and $f \in A^2_\alpha$. Then*
(a) *$f \in \mathcal{H}_\alpha(\varphi)$ if and only if $T^\alpha_{\overline{\varphi}} f \in \mathcal{H}_\alpha(\overline{\varphi})$ and in this case*

$$\|f\|^2_{\mathcal{H}_\alpha(\varphi)} = \|f\|^2_{A^2_\alpha} + \|T^\alpha_{\overline{\varphi}} f\|^2_{\mathcal{H}_\alpha(\overline{\varphi})},$$

(b) *$f \in \mathcal{H}_\alpha(\overline{\varphi})$ if and only if $T^\alpha_{\varphi} f \in \mathcal{H}_\alpha(\varphi)$ and in this case*

$$\|f\|^2_{\mathcal{H}_\alpha(\overline{\varphi})} = \|f\|^2_{A^2_\alpha} + \|T^\alpha_{\varphi} f\|^2_{\mathcal{H}_\alpha(\varphi)},$$

(c) *$\mathcal{M}(T^\alpha_\varphi) \cap \mathcal{H}_\alpha(\varphi) = \varphi \mathcal{H}_\alpha(\overline{\varphi})$.*

Proposition 2.4. *Let $\varphi \in (H^\infty)_1$ and $\alpha > 0$. Then every $\psi \in H^\infty$ is a multiplier on both $\mathcal{H}_\alpha(\varphi)$ and $\mathcal{H}_\alpha(\overline{\varphi})$, moreover, $\|T^\alpha_\psi\| \leqslant \|\psi\|_\infty$.*

P r o o f. Assume that $\|\psi\|_\infty = 1$. By Proposition 2.1, the functions

$$\frac{1 - \psi(z)\overline{\psi(w)}}{(1 - z\overline{w})^{1+\alpha/2}}, \qquad \frac{1 - \varphi(z)\overline{\varphi(w)}}{(1 - z\overline{w})^{1+\alpha/2}}$$

are reproducing kernels of $\mathcal{H}_{\alpha/2-1}(\psi)$ and $\mathcal{H}_{\alpha/2-1}(\varphi)$, respectively. According to Lemma 3.11 of [5] the product

$$\begin{aligned} K(z,w) &= \frac{(1 - \psi(z)\overline{\psi(w)})(1 - \varphi(z)\overline{\varphi(w)})}{(1 + z\overline{w})^{\alpha+2}} \\ &= (1 - \psi(z)\overline{\psi(w)}) K^\alpha_\varphi(z,w) \end{aligned}$$

is again a reproducing kernel on $\mathbb{D}$. It now follows from a theorem of Beatrous and Burbea (see [3], or Theorem 2.2 of [5]) that ψ is a contractive multiplier on $\mathcal{H}_\alpha(\varphi)$. To see that ψ is a multiplier on $\mathcal{H}_\alpha(\overline{\varphi})$ we assume $f \in \mathcal{H}_\alpha(\overline{\varphi})$. According to Proposition 2.4, $\varphi f \in \mathcal{H}_\alpha(\varphi)$ and hence $\psi(\varphi f) \in \mathcal{H}_\alpha(\varphi)$. Thus $\psi f \in \mathcal{H}_\alpha(\overline{\varphi})$, by Proposition 2.4. Finally, we note that

$$\begin{aligned} \|\psi f\|^2_{\mathcal{H}_\alpha(\overline{\varphi})} &= \|\psi f\|^2_{A^2_\alpha} + \|\psi\varphi f\|^2_{\mathcal{H}_\alpha(\varphi)} \\ &= \|\psi\|^2_\infty (\|f\|^2_{A^2_\alpha} + \|\varphi f\|^2_{\mathcal{H}_\alpha(\varphi)}) \\ &= \|f\|^2_{\mathcal{H}_\alpha(\overline{\varphi})}. \end{aligned}$$

□

Theorem 2.5. *Let $\varphi \in (H^\infty)_1$ and $\alpha > 0$. Then $\mathcal{H}_\alpha(\varphi) = \mathcal{H}_\alpha(\overline{\varphi})$ with equivalence of norms.*

Proof. Assume that $\varphi \neq 0$, otherwise $\mathcal{H}_\alpha(\varphi) = \mathcal{H}_\alpha(\overline{\varphi}) = A^2_\alpha$. By the preceding proposition, $\varphi\mathcal{H}_\alpha(\varphi) \subset \mathcal{H}_\alpha(\varphi)$. On the other hand, $\varphi\mathcal{H}_\alpha(\varphi) \subset \varphi A^2_\alpha = \mathcal{M}(T^\alpha_\varphi)$. It now follows from Proposition 2.3 that

$$\varphi\mathcal{H}_\alpha(\varphi) \subset \mathcal{M}(T^\alpha_\varphi) \cap \mathcal{H}_\alpha(\varphi) = \varphi\mathcal{H}_\alpha(\overline{\varphi}).$$

This implies that $\mathcal{H}_\alpha(\varphi) \subset \mathcal{H}_\alpha(\overline{\varphi})$. As for the reverse inclusion, let T denote the operator of multiplication by φ on $L^2(\mathbb{D}, \mathrm{d}A_\alpha)$. It is well-known that T is bounded and $T^*f = \overline{\varphi}f$. Now for every f and g in $L^2(\mathbb{D}, \mathrm{d}A_\alpha)$ we have

$$\begin{aligned}\langle T^*Tf, g\rangle &= \int_{\mathbb{D}} \varphi(z)f(z)\overline{\varphi(z)}\,\overline{g(z)}\,\mathrm{d}A_\alpha(z)\\ &= \langle \overline{\varphi}f, \overline{\varphi}g\rangle\\ &= \langle TT^*f, g\rangle.\end{aligned}$$

This shows that T is a normal operator, from which it follows that its restriction to A^2_α is subnormal:

$$T^\alpha_\varphi T^\alpha_{\overline{\varphi}} = T^\alpha_\varphi (T^\alpha_\varphi)^* \leqslant (T^\alpha_\varphi)^* T^\alpha_\varphi = T^\alpha_{\overline{\varphi}} T^\alpha_\varphi.$$

This implies the inclusion $\mathcal{H}_\alpha(\overline{\varphi}) \subset \mathcal{H}_\alpha(\varphi)$ from which the equality $\mathcal{H}_\alpha(\varphi) = \mathcal{H}_\alpha(\overline{\varphi})$ follows. Finally, let $I_1\colon \mathcal{H}_\alpha(\varphi) \to \mathcal{H}_\alpha(\overline{\varphi})$ and $I_2\colon \mathcal{H}_\alpha(\overline{\varphi}) \to \mathcal{H}_\alpha(\varphi)$ denote the identity operators. By Proposition 2.3, both I_1 and I_2 are bounded, so that the norms on $\mathcal{H}_\alpha(\varphi)$ and $\mathcal{H}_\alpha(\overline{\varphi})$ are equivalent. □

Theorem 2.6. *Let $\varphi \in (H^\infty)_1$ and $\alpha > 0$. Then $H^\infty \subset \mathcal{H}_\alpha(\varphi) = \mathcal{H}_\alpha(\overline{\varphi})$.*

Proof. According to the preceding theorem, it remains to verify that $H^\infty \subset \mathcal{H}_\alpha(\overline{\varphi})$. To this end, it suffices to show that $\mathcal{H}_\alpha(\overline{\varphi})$ contains a nonzero constant function (see Proposition 2.4). Let E denote the proper subspace of $A^2_{\alpha,\varphi}$ generated by $\{z^n\}_{n\geqslant 1}$. Consider $g \in A^2_{\alpha,\varphi} \ominus E$ with $\|g\|_{A^2_{\alpha,\varphi}} = 1$. Put

$$f(z) = \langle g, 1\rangle_{A^2_{\alpha,\varphi}} = \int_{\mathbb{D}} g(u)(1 - |\varphi(u)|^2)\,\mathrm{d}A_\alpha(u).$$

According to Proposition 2.2, the constant function f belongs to $\mathcal{H}_\alpha(\overline{\varphi})$. However, f does not vanish identically, otherwise we get

$$\langle g, 1\rangle_{A^2_{\alpha,\varphi}} = 0, \qquad g \in E^\perp$$

from which we obtain $1 \in E$, a contradiction. □

3. Finite Blaschke products

In this section we intend to describe $\mathcal{H}_\alpha(B)$ and $\mathcal{H}_\alpha(\overline{B})$ where B is a finite Blaschke product. For the standard Bergman space A^2_α, this was done by Zhu in [6]. He proved that $\mathcal{H}_\alpha(B) = \mathcal{H}_\alpha(\overline{B}) = H^2$, the Hardy space. The following theorem says that for $\alpha > 0$, the spaces $\mathcal{H}_\alpha(B)$ and $\mathcal{H}_\alpha(\overline{B})$ equal $A^2_{\alpha-1}$, the Hilbert space associated with the reproducing kernel

$$K^{\alpha-1}_w(z) = \frac{1}{(1 - z\overline{w})^{\alpha+1}}.$$

Note that for $\alpha = 0$, the function $(1 - z\overline{w})^{-1}$ is the reproducing kernel for the Hardy space.

Theorem 3.1. *Let B be a finite Blaschke product and $\alpha > 0$. Then*

$$\mathcal{H}_\alpha(B) = \mathcal{H}_\alpha(\overline{B}) = A^2_{\alpha-1}.$$

Proof. We first verify that $\mathcal{H}_\alpha(\overline{B}) \subset A^2_{\alpha-1}$. Let $f \in \mathcal{H}_\alpha(\overline{B})$. By Proposition 2.2 we have

$$f(z) = Tg(z) = \int_{\mathbb{D}} \frac{1 - |B(w)|^2}{(1 - z\overline{w})^{\alpha+2}}\, g(w)\,\mathrm{d}A_\alpha(w),$$

where g is an analytic function satisfying

$$\int_{\mathbb{D}} |g(z)|^2 \left(1 - |B(z)|^2\right)\, \mathrm{d}A_\alpha(z) < +\infty.$$

According to Lemma 1 of [5], there exists a $C > 0$ such that

$$1 - |B(z)|^2 \leqslant C(1 - |z|^2), \qquad z \in \mathbb{D},$$

from which it follows that $g \in A^2_{\alpha+1}$. Moreover, for every $z \in \mathbb{D}$ we have

$$(1 - |z|^2)^{-1}|f(z)| \leqslant C(1 - |z|^2)^{-1} \int_{\mathbb{D}} \frac{(1 - |w|^2)^{\alpha+1}}{|1 - z\overline{w}|^{\alpha+2}} |g(w)|\,\mathrm{d}A(w).$$

Put $\mathrm{d}\mu(z) = (1 - |z|^2)^{\alpha+1}\,\mathrm{d}A(z)$. By Theorem 1.9 of [1] the operator

$$\Lambda g(z) = (1 - |z|^2)^{-1} \int_{\mathbb{D}} \frac{(1 - |w|^2)^{\alpha+1}}{|1 - z\overline{w}|^{\alpha+2}}\, g(w)\,\mathrm{d}A(w)$$

is bounded on $L^2(\mathbb{D}, \mathrm{d}\mu)$. Therefore we can find a constant C_1 such that

$$\int_{\mathbb{D}} |f(z)|^2(1 - |z|^2)^{-2}\,\mathrm{d}\mu(z) \leqslant C_1\|g\|_{L^2(\mathbb{D},\mathrm{d}\mu)} = \frac{C_1}{\alpha + 2}\|g\|_{A^2_{\alpha+1}}.$$

This argument shows that $f \in A^2_{\alpha-1}$, or $\mathcal{H}_\alpha(\overline{B}) \subset A^2_{\alpha-1}$. So far we have proved that $\mathcal{H}_\alpha(\overline{B})$ equals the range of the operator $T\colon\ A^2_{\alpha,\,B} \to A^2_{\alpha-1}$. We now consider the operator $S\colon\ A^2_{\alpha-1} \to A^2_{\alpha,B}$ defined by

$$h(z) = Sf(z) = \int_{\mathbb{D}} \frac{f(w)}{(1-z\overline{w})^{\alpha+2}}\,\mathrm{d}A_{\alpha-1}(w).$$

Note that for $f \in A^2_{\alpha-1}$ we have

$$\begin{aligned} f(z) + \frac{zf'(z)}{\alpha+1} &= \int_{\mathbb{D}} \frac{f(w)}{(1-z\overline{w})^{\alpha+1}}\,\mathrm{d}A_{\alpha-1}(w) + \frac{z}{\alpha+1}\int_{\mathbb{D}} \frac{(\alpha+1)\overline{w}f(w)}{(1-z\overline{w})^{\alpha+2}}\,\mathrm{d}A_{\alpha-1}(w) \\ &= \int_{\mathbb{D}} \frac{f(w)}{(1-z\overline{w})^{\alpha+2}}\,\mathrm{d}A_{\alpha-1}(w) = Sf(z), \end{aligned}$$

from which it follows that for $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$, we have

$$Sf(z) = \sum_{n=0}^{\infty} \frac{n+\alpha+1}{\alpha+1} a_n z^n.$$

By Lemma 1 of [5] we know that $1 - |B(z)|^2 \asymp 1 - |z|^2$, so that

$$\begin{aligned} \|Sf\|^2_{A^2_{\alpha,B}} \asymp \|Sf\|^2_{A^2_{\alpha+1}} &= \sum_{n=0}^{\infty} \frac{n!\Gamma(\alpha+3)(n+\alpha+1)^2}{\Gamma(n+\alpha+3)(\alpha+1)^2}|a_n|^2 \\ &\geqslant \sum_{n=0}^{\infty} \frac{n!\Gamma(\alpha+1)}{\Gamma(n+\alpha+1)}|a_n|^2 = \|f\|^2_{A^2_{\alpha-1}}, \end{aligned}$$

which means that S is bounded from below. Since S is invertible, the image of the unit ball of $A^2_{\alpha-1}$ under S contains a ball of radius $r > 0$ centered at zero. Therefore for every unit vector $g \in A^2_{\alpha,B}$ we have

$$\begin{aligned} \|Tg\|_{A^2_{\alpha-1}} &= \sup\{|\langle Tg, f\rangle_{A^2_{\alpha-1}}|\colon\ \|f\|_{A^2_{\alpha-1}} \leqslant 1\} \\ &= \sup\left\{\left|\int_{\mathbb{D}} g(w)\overline{Sf(w)}(1-|B(w)|^2)\,\mathrm{d}A_\alpha(w)\right|\colon\ \|f\|_{A^2_{\alpha-1}} \leqslant 1\right\} \\ &\geqslant \sup\left\{\left|\int_{\mathbb{D}} g(w)\overline{h(w)}(1-|B(w)|^2)\,\mathrm{d}A_\alpha(w)\right|\colon\ \|h\|_{A^2_{\alpha,B}} \leqslant r\right\} \\ &\geqslant \sup\left\{\left|\int_{\mathbb{D}} g(w)\overline{h(w)}(1-|B(w)|^2)\,\mathrm{d}A_\alpha(w)\right|\colon\ \|h\|_{A^2_{\alpha,B}} = r\right\} \\ &= r\|g\|_{A^2_{\alpha,B}} \\ &= r. \end{aligned}$$

This means that T is bounded from below so that its range, $\mathcal{H}_\alpha(\overline{B})$, is closed in $A^2_{\alpha-1}$. Since $\mathcal{H}_\alpha(\overline{B})$ contains H^∞ by Theorem 2.6 and H^∞ is dense in the weighted Bergman space $A^2_{\alpha-1}$, we conclude that $\mathcal{H}_\alpha(\overline{B}) = A^2_{\alpha-1}$. □

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