Czechoslovak Mathematical Journal, 60 (135) (2010), 371–379

NON-LINEAR MAPS PRESERVING IDEALS ON A PARABOLIC SUBALGEBRA OF A SIMPLE ALGEBRA

Dengyin Wang, Haishan Pan, Xuansheng Wang, Xuzhou

(Received October 27, 2008)

Abstract. Let $\mathcal{P}$ be an arbitrary parabolic subalgebra of a simple associative F-algebra. The ideals of $\mathcal{P}$ are determined completely; Each ideal of $\mathcal{P}$ is shown to be generated by one element; Every non-linear invertible map on $\mathcal{P}$ that preserves ideals is described in an explicit formula.

Keywords: simple associative F-algebra, ideals, maps preserving ideals

MSC 2010: 15A04, 15A27

1. Introduction

If F is a field, then an F-*algebra* (associative) is a set $\mathcal{A}$ with a ring structure and an F-vector space structure that share the same addition operation, and with the additional property that $(ax)y = a(xy) = x(ay)$ for $a \in F$ and $x, y \in \mathcal{A}$. An F-algebra is called finite-dimensional if it has finite dimension as an F-vector space. A subspace I of an F-algebra $\mathcal{A}$ is called an *ideal* (two-sided) if xI and Ix are contained in I for any $x \in \mathcal{A}$. We say that the F-algebra $\mathcal{A}$ is *semisimple* if all non-zero $\mathcal{A}$-modules are semisimple. $\mathcal{A}$ is said to be *simple* if its only two-sided ideals are itself and the zero ideal. An algebra $\mathcal{D}$ is said to be a *division algebra* if the non-zero elements of $\mathcal{D}$ form a group under multiplication. One of the most fundamental results on the structure of F-algebra is due to Wedderburn, which says that a finite-dimensional algebra $\mathcal{A}$ is semisimple iff it is isomorphic to a direct sum of matrix algebras over finite-dimensional division algebras, and $\mathcal{A}$ is simple iff it is isomorphic to a matrix algebra over a finite-dimensional division algebra. In view of Wedderburn's theorem, the study on finite-dimensional simple algebras can be reduced to that on the matrix algebra $M_{n\times n}(\mathcal{D})$ consisting of $n \times n$ matrices over a finite-dimensional division algebra $\mathcal{D}$. Note that the F-dimension of $M_{n\times n}(\mathcal{D})$

is $n^2 \cdot \dim_F \mathcal{D}$. Let $T_n(\mathcal{D})$ (resp., $S_n(\mathcal{D})$) be the subalgebra of $M_{n\times n}(\mathcal{D})$ of all upper triangular matrices (resp., strictly upper triangular matrices). Subalgebras of $M_{n\times n}(\mathcal{D})$ that contain $T_n(\mathcal{D})$ are called *parabolic subalgebras* of $M_{n\times n}(\mathcal{D})$.

More recently some authors devoted to determine ad-nilpotent ideals of certain special subalgebras of simple Lie algebras. In [1]–[8], the authors determined ad-nilpotent ideals of the Borel subalgebras of a complex simple Lie algebra, and in [9] Righi extent the results to the parabolic subalgebras of a simple Lie algebra. In [10], Panyushev determined the normalizers of ad-nilpotent ideals of a Borel subalgebra in a complex simple Lie algebra. Some other authors are interested in describing invertible transformations on linear Lie algebras that preserves certain special subalgebras. For instance, Radjavi and Šemrl [11] determined non-linear invertible transformations on the general linear algebra $\mathrm{gl}(n,\mathbb{C})$ that preserves solvable subalgebras. Motivated by these papers, we in this note dedicate to determine all ideals of $\mathcal{P}$ and to characterize non-linear invertible maps on $\mathcal{P}$ that preserves ideals, where $\mathcal{P}$ is an arbitrary parabolic subalgebra of a simple F-algebra.

2. Ideals of a parabolic subalgebra of a simple algebra

Now we consider that a parabolic subalgebra of $M_{n\times n}(\mathcal{D})$ consists of matrices of what form. Let $\Phi = \{(i,j) \mid 1 \leqslant j \leqslant i \leqslant n\}$. For $\alpha = (i,j)$, $\beta = (k,l) \in \Phi$, we define $\alpha \preceq \beta$ if $k \leqslant i$ and $j \leqslant l$. Thus $(\Phi, \preceq)$ becomes a partially ordered set. A nonempty subset Ψ of Φ is said to be an *upper set*, if for $\alpha \in \Psi, \beta \in \Phi$, $\alpha \preceq \beta$ implies $\beta \in \Psi$. Let Ψ be an upper subset of Φ, Δ the subset of Ψ consisting of all minimal elements in Ψ (relative to $\preceq$). Then we have that

$$\Psi = \{\beta \in \Phi \mid \alpha \preceq \beta \text{ for cetain } \alpha \in \Delta\}.$$

Now let $\mathcal{P}$ be a parabolic subalgebra of $M_{n\times n}(\mathcal{D})$. By E we mean the $n \times n$ identity matrix. For $\beta = (i,j) \in \Phi$, E_β means the $n \times n$ matrix unit $E_{i,j}$. Define $\Psi(\mathcal{P})$ to be the subset of Φ consisting of $\beta \in \Phi$ for which $E_\beta \in \mathcal{P}$, and $\Delta(\mathcal{P})$ the subset of $\Psi(\mathcal{P})$ of minimal elements in $\Psi(\mathcal{P})$.

Proposition.

(i) *$\Psi(\mathcal{P})$ is an upper subset of Φ.*

(ii) $\mathcal{P} = S_n(\mathcal{D}) + \sum\limits_{\beta \in \Psi(\mathcal{P})} \mathcal{D}E_\beta$.

(iii) $\mathcal{P} = S_n(\mathcal{D}) + \sum\limits_{\alpha \in \Delta(\mathcal{P})} \Big(\sum\limits_{\alpha \preceq \beta} \mathcal{D}E_\beta\Big)$.

P r o o f. For (i), $(1,1) \in \Psi(\mathcal{P})$, thus $\Psi(\mathcal{P})$ is nonempty. Suppose that $\alpha = (i,j) \in \Psi(\mathcal{P})$, $\beta = (k,l) \in \Phi$ and $\alpha \preceq \beta$. Then $k \leqslant i$, $j \leqslant l$. We have, by $E_\beta = E_{k,i} \cdot E_\alpha \cdot E_{j,l}$, that $E_\beta \in \mathcal{P}$. So $\beta \in \Psi(\mathcal{P})$. Hence $\Psi(\mathcal{P})$ is an upper subset of Φ.

For (ii), denote $S_n(\mathcal{D}) + \sum\limits_{\beta\in\Psi(\mathcal{P})} \mathcal{D}E_\beta$, by $\mathcal{Q}$. If $\beta \in \Psi(\mathcal{P})$, then $E_\beta \in \mathcal{P}$. We have by $dE_\beta = (dE) \cdot E_\beta$, that $dE_\beta \in \mathcal{P}$ for all $d \in \mathcal{D}$. Thus $\mathcal{Q} \subseteq \mathcal{P}$. Conversely, for any $x \in \mathcal{P}$, write it in the form: $x = s_x + \sum\limits_{\beta\in\Phi} a_\beta E_\beta$, where $s_x \in S_n(\mathcal{D}), a_\beta \in \mathcal{D}$. Suppose $a_\gamma \neq 0$ for certain $\gamma = (i,j) \in \Phi$. Then by $E_\gamma = (a_\gamma^{-1}E) \cdot E_{i,i} \cdot x \cdot E_{j,j} \in \mathcal{P}$ we have that $\gamma \in \Psi(\mathcal{P})$. This implies that $x \in \mathcal{Q}$. Therefore, $\mathcal{P} = \mathcal{Q}$.

(iii) is evident. $\square$

Let $\mathcal{P}$ be an arbitrary parabolic subalgebra of a simple F-algebra $M_{n\times n}(\mathcal{D})$, where $\mathcal{D}$ is a finite-dimensional division F-algebra. Suppose $\Delta(\mathcal{P}) = \{(i_1,j_1),(i_2,j_2),\ldots,(i_s,j_s)\}$, where $1 \leqslant i_1 < i_2 < \ldots < i_s \leqslant n$ and $1 \leqslant j_1 < j_2 < \ldots < j_s \leqslant n$. Let $n_k = i_k - j_k + 1$ for $k = 1,2,\ldots,s$, then $\sum\limits_{k=1}^{s} n_k = n$. By (iii) of the above proposition, one will see that $\mathcal{P}$ actually consists of all upper triangular block matrices of the form

$$\begin{pmatrix} A_{11} & A_{12} & \ldots & A_{1s} \\ 0 & A_{22} & \ldots & A_{2s} \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & A_{ss} \end{pmatrix}, \text{where } A_{ij} \in M_{n_i\times n_j}(\mathcal{D}),\ \sum_{i=1}^{s} n_i = n.$$

If $s = 1$, then $\mathcal{P}$, as just stated, exactly is $M_{n\times n}(\mathcal{D})$ itself. Conversely, if $s = n$, then $\mathcal{P}$ exactly is $T_n(\mathcal{D})$. Now we describe the ideals of $\mathcal{P}$. One will see that an ideal of $\mathcal{P}$ is closed under $\mathcal{D}$-scalar multiplication (not only closed under F-scalar multiplication). Each element $x \in \mathcal{P}$ will be written in the form $x = (X_{ij})_{s\times s}$ for brevity, where $X_{ij} \in M_{n_i\times n_j}(\mathcal{D})$ is in the (i,j)-position. Let S be a subset of $\mathcal{P}$. The minimal ideal of $\mathcal{P}$ containing S is denoted by $\langle S\rangle$, called the ideal of $\mathcal{P}$ generated by S. Actually, $\langle S\rangle$ just is the intersection of all ideals of $\mathcal{P}$ which contains S. For $1 \leqslant k \leqslant l \leqslant s$ and $1 \leqslant i_k \leqslant n_k$, $1 \leqslant j_l \leqslant n_l$, we denote by $E^{k,l}_{i_k,j_l}$ the matrix unit of $M_{n\times n}(\mathcal{D})$ with 1 in the $\Bigl(i_k + \sum\limits_{p=1}^{k-1} n_p, j_l + \sum\limits_{q=1}^{l-1} n_q\Bigr)$-position and 0 elsewhere. For $1 \leqslant k \leqslant l \leqslant s$, we denote by $\mathcal{E}_{kl}$ the subset of $\mathcal{P}$ consisting of all block matrices of the form

$$\begin{pmatrix} 0 & \ldots & 0 & A_{1l} & \ldots & A_{1s} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & A_{kl} & \ldots & A_{ks} \\ 0 & \ldots & 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & 0 & \ldots & 0 \end{pmatrix}, \quad \begin{array}{l} \text{where } A_{ij} \in M_{n_i\times n_j}(\mathcal{D}) \\ \text{for } 1 \leqslant i \leqslant k,\ l \leqslant j \leqslant s. \end{array}$$

It's easy to see that all $\mathcal{E}_{kl}$ for $1 \leqslant k \leqslant l \leqslant s$ are ideals of $\mathcal{P}$.

Lemma 2.1. *Let $1 \leqslant k \leqslant l \leqslant s$, $x = (X_{ij})_{s\times s} \in \mathcal{P}$. Then $\langle x\rangle = \mathcal{E}_{kl}$ if and only if $x \in \mathcal{E}_{kl}$ and the (k,l)-entry X_{kl} of x is not zero. Particularly, $\langle E^{k,l}_{i_k,j_l}\rangle = \mathcal{E}_{kl}$ for any i_k, j_l satisfying $1 \leqslant i_k \leqslant n_k$, $1 \leqslant j_l \leqslant n_l$.*

Proof. $\Leftarrow$: By assumption on x, the beginning $l-1$ columns and the last $s-k$ rows of x are all zero, and $X_{kl} \neq 0$. Suppose the (i_k, j_l)-entry of X_{kl} is not zero, then $E^{k,l}_{i_k,j_l}$, being a nonzero multiple of $E^{k,k}_{i_k,i_k} \cdot x \cdot E^{l,l}_{j_l,j_l}$, belongs to $\langle x\rangle$ (note that any ideal of $\mathcal{P}$ is closed under $\mathcal{D}$-scalar multiplication). Then we have

$$E^{k,l}_{p_k,q_l} = E^{k,k}_{p_k,i_k} \cdot E^{k,l}_{i_k,j_l} \cdot E^{l,l}_{j_l,q_l} \in \langle x\rangle,$$

for any pair (p_k, q_l) satisfying $1 \leqslant p_k \leqslant n_k$, $1 \leqslant q_l \leqslant n_l$. Furthermore, we have

$$E^{k,m}_{p_k,j_m} = E^{k,l}_{p_k,1} \cdot E^{l,m}_{1,j_m} \in \langle x\rangle$$

for $l \leqslant m \leqslant s$ and $1 \leqslant p_k \leqslant n_k$, $1 \leqslant j_m \leqslant n_m$. Now for any pair (t, m) satisfying $1 \leqslant t \leqslant k$, $l \leqslant m \leqslant s$ and for any pair (i_t, j_m) satisfying $1 \leqslant i_t \leqslant n_t$, $1 \leqslant j_m \leqslant n_m$, we have that

$$E^{t,m}_{i_t,j_m} = E^{t,k}_{i_t,1} \cdot E^{k,m}_{1,j_m} \in \langle x\rangle.$$

Note that the set

$$\{E^{t,m}_{i_t,j_m} \mid 1 \leqslant t \leqslant k,\ l \leqslant m \leqslant s, 1 \leqslant i_t \leqslant n_t,\ 1 \leqslant j_m \leqslant n_m\}$$

forms a $\mathcal{D}$-basis of $\mathcal{E}_{kl}$, so $\mathcal{E}_{kl} \in \langle x\rangle$. Obviously, $\langle x\rangle \in \mathcal{E}_{kl}$. Finally, $\langle x\rangle = \mathcal{E}_{kl}$.

$\Rightarrow$: If $\langle x\rangle = \mathcal{E}_{kl}$, then $x \in \mathcal{E}_{kl}$. It's easy to see that the subset J of $\mathcal{E}_{kl}$ consisting of the elements whose (k,l)-entry is zero forms an ideal of $\mathcal{P}$. If the (k,l)-entry of x is zero, then $x \in J$, forcing $\mathcal{E}_{kl} = \langle x\rangle \subseteq J$, absurd. $\square$

Let S be a subset of $\mathcal{P}$. Write every $x \in S$ in the form $x = (X_{ij})_{s\times s}$, where X_{ij} is in the (i,j)-position, set

$$\Sigma_S = \left\{(k,l) \mid \begin{array}{c} X_{kl} \neq 0 \quad \text{for some } x \in S, \text{and } Y_{kq} = Y_{pl} = 0 \\ \text{for all } y \in S,\ \text{and for } q = 1,2,\ldots,l-1,\ p = k+1,\ldots,s \end{array}\right\};$$

and set

$$G_S = \sum_{(k,l)\in\Sigma_S} E^{k,l}_{1,1}.$$

Lemma 2.2. *Let S be a subset of $\mathcal{P}$. Then $\langle S\rangle = \langle G_S\rangle = \sum\limits_{(k,l)\in\Sigma_S} \mathcal{E}_{kl}$.*

Proof. By construction of Σ_S and G_S, one easily sees that

$$S \subseteq \sum_{(k,l)\in\Sigma_S} \mathcal{E}_{kl}, \quad G_S \in \sum_{(k,l)\in\Sigma_S} \mathcal{E}_{kl},$$

which follow that

$$\langle S\rangle \subseteq \sum_{(k,l)\in\Sigma_S} \mathcal{E}_{kl}, \quad \langle G_S\rangle \subseteq \sum_{(k,l)\in\Sigma_S} \mathcal{E}_{kl}.$$

If $(k,l)\in\Sigma_S$, then there exists some $x = (X_{ij})_{s\times s} \in S$ for which $X_{kl} \neq 0$. Then the $s\times s$ block matrix, denoted by $[X_{k,l}]$, with X_{kl} in the (k,l)-position and 0 elsewhere, being equal to $\left(\sum\limits_{i=1}^{n_k} E_{i,i}^{k,k}\right)\cdot x\cdot\left(\sum\limits_{j=1}^{n_l} E_{j,j}^{l,l}\right)$, naturally belongs to $\langle S\rangle$. By Lemma 2.1 we have

$$\mathcal{E}_{kl} = \langle [X_{kl}]\rangle \subseteq \langle S\rangle, \quad \text{and} \quad \sum_{(k,l)\in\Sigma_S} \mathcal{E}_{kl} \subseteq \langle S\rangle.$$

Then we have $\sum\limits_{(k,l)\in\Sigma_S} \mathcal{E}_{kl} = \langle S\rangle$. For $(k,l)\in\Sigma_S$, we have $E_{1,1}^{k,l} = E_{1,1}^{k,k}\cdot G_S\cdot E_{1,1}^{l,l} \in \langle G_S\rangle$, which follows that $\mathcal{E}_{kl} = \langle E_{1,1}^{k,l}\rangle \subseteq \langle G_S\rangle$. Thus $\sum\limits_{(k,l)\in\Sigma_S} \mathcal{E}_{kl} \subseteq \langle G_S\rangle$. This further forces $\sum\limits_{(k,l)\in\Sigma_S} \mathcal{E}_{kl} = \langle G_S\rangle$. $\square$

Applying Lemma 2.2, we are ready to assert the main results in this subsection.

Theorem 2.3. *Every ideal I of $\mathcal{P}$ can be generated by just one element. More concretely, $I = \langle G_I\rangle = \sum\limits_{(k,l)\in\Sigma_I} \mathcal{E}_{kl}$.*

Theorem 2.3 shows that each ideal of $\mathcal{P}$ takes the form

$$\mathcal{E}_{k_1,l_1} + \mathcal{E}_{k_2,l_2} + \ldots + \mathcal{E}_{k_t,l_t},$$

where $k_i \leqslant l_i$ for $i = 1,\ldots,t$, $1 \leqslant k_1 < k_2 < \ldots < k_t \leqslant s$ and $1 \leqslant l_1 < l_2 < \ldots < l_t \leqslant s$. Now we can count the number of ideals of $\mathcal{P}$. Set $C_m^i = m\cdot(m-1)\ldots(m-i+1)/i\cdot(i-1)\cdot\ldots\cdot 2\cdot 1$ for $1\leqslant i\leqslant m$. Set

$$D_s^1 = C_s^1; \quad D_s^2 = \sum_{k=2}^{s} D_k^1;$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$D_s^i = \sum_{k=i}^{s} D_k^{i-1}, \quad \text{where } i > 1;$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$D_s^{s-1} = D_s^{s-2} + D_{s-1}^{s-2}; \quad D_s^s = D_s^{s-1}.$$

Then by Theorem 2.3, we have:

Theorem 2.4. *The number of nonzero ideals of $\mathcal{P}$ is $\sum\limits_{i=1}^{s} D_s^i$.*

Example. In case that $s=4$, the number of nonzero ideals of $\mathcal{P}$ is 41. We list these ideals in the following.
$\mathcal{E}_{1,4}$, $\mathcal{E}_{1,3}$, $\mathcal{E}_{1,2}$, $\mathcal{E}_{1,1}$, $\mathcal{E}_{2,4}$, $\mathcal{E}_{1,3}+\mathcal{E}_{2,4}$, $\mathcal{E}_{1,2}+\mathcal{E}_{2,4}$, $\mathcal{E}_{1,1}+\mathcal{E}_{2,4}$,
$\mathcal{E}_{2,3}$, $\mathcal{E}_{1,2}+\mathcal{E}_{2,3}$, $\mathcal{E}_{1,1}+\mathcal{E}_{2,3}$, $\mathcal{E}_{2,2}$, $\mathcal{E}_{1,1}+\mathcal{E}_{2,2}$,
$\mathcal{E}_{3,4}$, $\mathcal{E}_{1,3}+\mathcal{E}_{3,4}$, $\mathcal{E}_{1,2}+\mathcal{E}_{3,4}$, $\mathcal{E}_{1,1}+\mathcal{E}_{3,4}$, $\mathcal{E}_{2,3}+\mathcal{E}_{3,4}$,
$\mathcal{E}_{1,2}+\mathcal{E}_{2,3}+\mathcal{E}_{3,4}$, $\mathcal{E}_{1,1}+\mathcal{E}_{2,3}+\mathcal{E}_{3,4}$, $\mathcal{E}_{2,2}+\mathcal{E}_{3,4}$, $\mathcal{E}_{1,1}+\mathcal{E}_{2,2}+\mathcal{E}_{3,4}$,
$\mathcal{E}_{3,3}$, $\mathcal{E}_{1,2}+\mathcal{E}_{3,3}$, $\mathcal{E}_{1,1}+\mathcal{E}_{3,3}$, $\mathcal{E}_{2,2}+\mathcal{E}_{3,3}$, $\mathcal{E}_{1,1}+\mathcal{E}_{2,2}+\mathcal{E}_{3,3}$,
$\mathcal{E}_{4,4}$, $\mathcal{E}_{1,3}+\mathcal{E}_{4,4}$, $\mathcal{E}_{1,2}+\mathcal{E}_{4,4}$, $\mathcal{E}_{1,1}+\mathcal{E}_{4,4}$, $\mathcal{E}_{2,3}+\mathcal{E}_{4,4}$,
$\mathcal{E}_{1,2}+\mathcal{E}_{2,3}+\mathcal{E}_{4,4}$, $\mathcal{E}_{1,1}+\mathcal{E}_{2,3}+\mathcal{E}_{4,4}$, $\mathcal{E}_{2,2}+\mathcal{E}_{4,4}$, $\mathcal{E}_{1,1}+\mathcal{E}_{2,2}+\mathcal{E}_{4,4}$,
$\mathcal{E}_{3,3}+\mathcal{E}_{4,4}$, $\mathcal{E}_{1,2}+\mathcal{E}_{3,3}+\mathcal{E}_{4,4}$, $\mathcal{E}_{1,1}+\mathcal{E}_{3,3}+\mathcal{E}_{4,4}$, $\mathcal{E}_{2,2}+\mathcal{E}_{3,3}+\mathcal{E}_{4,4}$, $\mathcal{E}_{1,1}+\mathcal{E}_{2,2}+\mathcal{E}_{3,3}+\mathcal{E}_{4,4}$.

3. Non-linear maps on $\mathcal{P}$ that preserves ideals

$\mathcal{P}$ is as in Section 2. Denote by $\Phi_{\mathcal{P}}$ the set of all nonzero ideals of $\mathcal{P}$. An invertible map φ on $\mathcal{P}$ is called *preserving ideals* if it maps every ideal of $\mathcal{P}$ to another such ideal of the same dimension. To show relationship between ideals of $\mathcal{P}$, we in this subsection investigate the non-linear maps on $\mathcal{P}$ preserving ideals. It is easy to see that the product of two such maps and the inverse of one such map also are such maps. This shows that all such maps on $\mathcal{P}$ form a group under multiplication of maps. Let I be an ideal of $\mathcal{P}$, the number of nonzero ideals of $\mathcal{P}$ which is contained in I is called the *length* of I, and we denote it by $l(I)$. For example, $l(\mathcal{E}_{1,s})=1$, $l(\mathcal{E}_{2,s-1})=4$.

Lemma 3.1. *Let φ be an invertible map on $\mathcal{P}$ preserving ideals, then φ preserves the length of ideals, namely, $l(\varphi(I))=l(I)$ for any $I\in\Phi_{\mathcal{P}}$.*

Proof. Suppose $l(I)=m$ and $I_1,\ldots,I_m$ are the distinct nonzero ideals of $\mathcal{P}$ contained in I. Then $\varphi(I_1),\varphi(I_2),\ldots,\varphi(I_m)$ also are distinct nonzero ideals of $\mathcal{P}$ contained in $\varphi(I)$. This shows that $l(\varphi(I))\geqslant l(I)$. Considering the action of φ^{-1} on the ideal $\varphi(I)$ we have that $l(I)=l(\varphi^{-1}(\varphi(I)))\geqslant l(\varphi(I))$. Hence $l(\varphi(I))=l(I)$. $\square$

Lemma 3.2. *Let φ be an invertible map on $\mathcal{P}$ preserving ideals, I an ideal of $\mathcal{P}$ with decomposition $I=I_1+I_2+\ldots+I_t$, where all I_i are ideals of $\mathcal{P}$. If φ stabilizes each I_i for $i=1,2,\ldots,t$, Then φ stabilizes I.*

Proof. By assumption, $I_i=\varphi(I_i)\subseteq\varphi(I)$ for $i=1,2,\ldots,t$. Thus $I=\sum\limits_{i=1}^{t} I_i\subseteq\varphi(I)$. Comparing dimensions of I and $\varphi(I)$ we have $\varphi(I)=I$. $\square$

Lemma 3.3. *Let φ be an invertible map on $\mathcal{P}$ preserving ideals, $s \geqslant 2$. If φ stabilizes $\mathcal{E}_{1,s-1}$ and $\mathcal{E}_{2,s}$ respectively, then φ stabilizes each ideal of $\mathcal{P}$.*

Proof. Recalling Theorem 2.3 and Lemma 3.2, we only need to show that $\varphi(\mathcal{E}_{k,l}) = \mathcal{E}_{k,l}$ for all pairs k, l $(1 \leqslant k \leqslant l \leqslant s)$. First we show by induction on j that φ stabilizes all $\mathcal{E}_{1,j}, j = 1, 2, \ldots, s$. Obviously, $\mathcal{E}_{1,s}$ is the only ideal of $\mathcal{P}$ which has length 1, so it is stable under φ. By assumption we have $\varphi(\mathcal{E}_{1,s-1}) = \mathcal{E}_{1,s-1}$. Assume that $\varphi(\mathcal{E}_{1,k+1}) = \mathcal{E}_{1,k+1}$, where $1 \leqslant k \leqslant s-2$. $\varphi(\mathcal{E}_{1,k})$ is an ideal of $\mathcal{P}$ of length $s+1-k$ which contains $\mathcal{E}_{1,k+1}$. But, only $\mathcal{E}_{1,k}$ and $\mathcal{E}_{1,k+1} + \mathcal{E}_{2,s}$ satisfy such conditions. Thus φ maps $\mathcal{E}_{1,k}$ onto $\mathcal{E}_{1,k}$ or onto $\mathcal{E}_{1,k+1} + \mathcal{E}_{2,s}$. However, φ stabilizes $\mathcal{E}_{1,k+1}$ and $\mathcal{E}_{2,s}$, respectively, thus φ stabilizes $\mathcal{E}_{1,k+1} + \mathcal{E}_{2,s}$ (apply Lemma 3.2). If $\varphi(\mathcal{E}_{1,k}) = \mathcal{E}_{1,k+1} + \mathcal{E}_{2,s}$, then $\varphi^{-1}(\mathcal{E}_{1,k+1} + \mathcal{E}_{2,s}) = \mathcal{E}_{1,k}$, absurd. So φ stabilizes $\mathcal{E}_{1,k}$. Hence φ stabilizes $\mathcal{E}_{1,j}$ for $j = 1, 2, \ldots, s$.

Similar discussions show that $\varphi(\mathcal{E}_{k,s}) = \mathcal{E}_{k,s}$ for $k = 1, 2, \ldots, s$.

Now we show by induction on k that φ stabilizes $\mathcal{E}_{k,k}, \mathcal{E}_{k,k+1}, \ldots, \mathcal{E}_{k,s}$, respectively, for $k = 1, 2, \ldots, s$. In case $k = 1$, the assertion holds already. Assume that $\varphi(\mathcal{E}_{i,j}) = \mathcal{E}_{i,j}$ for $j = i, \ldots, s$, where $1 \leqslant i \leqslant s-1$. We desire to prove that $\varphi(\mathcal{E}_{i+1,j}) = \mathcal{E}_{i+1,j}$, $j = i+1, \ldots, s$. To achieve this goal, we will use induction on j. When $j = s$, $\varphi(\mathcal{E}_{i+1,s}) = \mathcal{E}_{i+1,s}$ has been proved. Assume that $\varphi(\mathcal{E}_{i+1,q+1}) = \mathcal{E}_{i+1,q+1}$, where $i+1 \leqslant q \leqslant s-1$. We need to show that $\varphi(\mathcal{E}_{i+1,q}) = \mathcal{E}_{i+1,q}$. Note that $\varphi(\mathcal{E}_{i+1,q})$ is an ideal of $\mathcal{P}$ containing $\mathcal{E}_{i+1,q+1} + \mathcal{E}_{i,q}$ and it has length just one more than that of $\mathcal{E}_{i+1,q+1} + \mathcal{E}_{i,q}$. Theorem 2.3 shows that only three ideals, namely, $\mathcal{E}_{i+1,q}$, $\mathcal{E}_{1,q-1} + \mathcal{E}_{i+1,q+1} + \mathcal{E}_{i,q}$ and $\mathcal{E}_{i+1,q+1} + \mathcal{E}_{i,q} + \mathcal{E}_{i+2,s}$ satisfy such conditions. By induction hypothesis and applying Lemma 3.2 we know that φ stabilizes $\mathcal{E}_{1,q-1} + \mathcal{E}_{i+1,q+1} + \mathcal{E}_{i,q}$ and $\mathcal{E}_{i+1,q+1} + \mathcal{E}_{i,q} + \mathcal{E}_{i+2,s}$, respectively. If $\varphi(\mathcal{E}_{i+1,q}) = \mathcal{E}_{1,q-1} + \mathcal{E}_{i+1,q+1} + \mathcal{E}_{i,q}$, then $\varphi^{-1}(\mathcal{E}_{1,q-1} + \mathcal{E}_{i+1,q+1} + \mathcal{E}_{i,q}) = \mathcal{E}_{i+1,q}$, which is absurd. Thus φ fails to send $\mathcal{E}_{i+1,q}$ to $\mathcal{E}_{1,q-1} + \mathcal{E}_{i+1,q+1} + \mathcal{E}_{i,q}$. Similarly, φ fails to send $\mathcal{E}_{i+1,q}$ to $\mathcal{E}_{i+1,q+1} + \mathcal{E}_{i,q} + \mathcal{E}_{i+2,s}$. Hence φ stabilizes $\mathcal{E}_{i+1,q}$. Then φ stabilizes each $\mathcal{E}_{i,j}$ for $1 \leqslant i \leqslant j \leqslant s$ by induction. Now by Lemma 3.2 we see that φ stabilizes all ideals of $\mathcal{P}$. $\square$

Lemma 3.4. *φ and s is as in Lemma 3.3. If $\varphi(\mathcal{E}_{1,s-1}) = \mathcal{E}_{2,s}$, then $n_1 = n_s$, $n_2 = n_{s-1}$, …, $n_i = n_{s-i+1}$, …, $n_s = n_1$.*

Proof. Since $\mathcal{E}_{1,s-1}$ and $\mathcal{E}_{2,s}$ are the only ideals of $\mathcal{P}$ which has length 2. Then it follows from $\varphi(\mathcal{E}_{1,s-1}) = \mathcal{E}_{2,s}$ that $\varphi(\mathcal{E}_{2,s}) = \mathcal{E}_{1,s-1}$. Now assume that φ permutes $\mathcal{E}_{1,j}$ and $\mathcal{E}_{s-j+1,s}$, where $2 \leqslant j \leqslant s$. Note that $\varphi(\mathcal{E}_{1,j-1})$ contains $\mathcal{E}_{s-j+1,s}$ $(= \varphi(\mathcal{E}_{1,j}))$ and has one more length than that of $\mathcal{E}_{s-j+1,s}$. However, only the ideal $\mathcal{E}_{s-j+2,s}$ and $\mathcal{E}_{1,s-1} + \mathcal{E}_{s-j+1,s}$ satisfy these conditions. So $\varphi(\mathcal{E}_{1,j-1})$ coincides with either $\mathcal{E}_{s-j+2,s}$ or $\mathcal{E}_{1,s-1} + \mathcal{E}_{s-j+1,s}$. But the latter is the image of $\mathcal{E}_{2,s} + \mathcal{E}_{1,j}$ under φ, so $\varphi(\mathcal{E}_{1,j-1}) = \mathcal{E}_{n-j+2,s}$. Then we know by induction that $\varphi(\mathcal{E}_{1,k}) = \mathcal{E}_{s-k+1,s}$ for

$k = 1, 2 \ldots, s$. By comparing dimensions of $\mathcal{E}_{1,k}$ and that of $\mathcal{E}_{s-k+1,s}$, we have that

$$n_1(n_k + n_{k+1} + \ldots + n_s) = n_s(n_1 + n_2 + \ldots, +n_{s-k+1}), \quad k = 1, 2, \ldots, s.$$

This shows that $n_1 = n_s$, $n_2 = n_{s-1}$, ..., $n_k = n_{s-k+1}$, ..., $n_{s-1} = n_2$, $n_s = n_1$. $\square$

Suppose that $n_1 = n_s$, $n_2 = n_{s-1}$, ..., $n_k = n_{s-k+1}$, ..., $n_{s-1} = n_2$, $n_s = n_1$, and set

$$\omega = \begin{pmatrix} 0 & 0 & \ldots & 0 & E_{n_1} \\ 0 & 0 & \ldots & E_{n_2} & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & E_{n_{s-1}} & \ldots & 0 & 0 \\ E_{n_s} & 0 & \ldots & 0 & 0 \end{pmatrix},$$ where E_{n_i} means the $n_i \times n_i$ identity matrix.

Using ω we define the map $\tau_\omega\colon \mathcal{P} \to \mathcal{P}$, sending any $x \in \mathcal{P}$ to $\omega \cdot x' \cdot \omega$.

Lemma 3.5. *τ_ω, as just defined, is an invertible map on $\mathcal{P}$ preserving ideals.*

Proof. Obviously, τ_ω is a linear map and the square of τ_ω is the identity, so τ_ω is invertible. It is easy to see that $\tau_\omega(\mathcal{E}_{k,l}) = \mathcal{E}_{s-l+1,s-k+1}$ for all k, l satisfying $1 \leqslant k \leqslant l \leqslant s$. Then by Theorem 2.3, we see that τ_ω is an invertible linear map on $\mathcal{P}$ preserving ideals. $\square$

Define a relation $\sim$ in $\mathcal{P}$ in such a way: $x \sim y$ iff $\langle x\rangle = \langle y\rangle$. It is not hard to see that the relation $\sim$ is an equivalence relation, thus $\mathcal{P}$ is partitioned into the disjoint union of the equivalence classes relative to $\sim$. We say a map $\chi\colon \mathcal{P} \to \mathcal{P}$ *preserves lattice* (consulting a terminology from [11]) if it induces a permutation on each equivalence class. It should be pointed out that such a map need not be linear.

Lemma 3.6. *If $\chi\colon \mathcal{P} \to \mathcal{P}$ is a map preserving lattice, then it is invertible and preserves ideals.*

Proof. We shall show that χ actually stabilizes each ideal of $\mathcal{P}$. Let I be any given ideal of $\mathcal{P}$ and $x \in I$. Then by $\chi(x) \sim x$ we see that $\langle\chi(x)\rangle = \langle x\rangle \subseteq I$. Thus $\chi(x) \in I$, which follows that $\chi(I) \subseteq I$. Considering χ^{-1} we have that $\chi^{-1}(I) \subseteq I$, in other words, $I \subseteq \chi(I)$. Hence $\chi(I) = I$. $\square$

Theorem 3.7. *Let $\mathcal{P}$ be a parabolic subalgebra of $M_{n\times n}(\mathcal{D})$ taking the form as described as in the beginning part of Section 2, φ an invertible map on $\mathcal{P}$ preserving ideals. Then φ takes one of the following forms*

(i) *If $n_1 = n_s$, $n_2 = n_{s-1}$, ..., $n_k = n_{s-k+1}$, ..., $n_{s-1} = n_2$, $n_s = n_1$, then $\varphi\colon x \mapsto \omega^\delta \cdot \chi(x)' \cdot \omega^\delta$, where $\delta = 1$ or 2;*

(ii) *If $n_k \neq n_{s-k+1}$ for some k ($1 \leqslant k \leqslant s$), then $\varphi\colon x \mapsto \chi(x)$, where χ is an invertible map on $\mathcal{P}$ preserving lattice.*

P r o o f. If $n_k = n_{s-k+1}$ for $k = 1, 2, \ldots, s$, since $\mathcal{E}_{1,s-1}$ and $\mathcal{E}_{2,s}$ are the only ideals of $\mathcal{P}$ whose length is 2. So φ either permutes them or stabilizes each of them respectively. When the first case happens we set $\delta = 1$, otherwise, we set $\delta = 2$. Then we see that $\tau_\omega^\delta \cdot \varphi$ stabilizes $\mathcal{E}_{1,s-1}$ and $\mathcal{E}_{2,s}$, respectively. Then by Lemma 3.3, $\tau_\omega^\delta \cdot \varphi$, denoted by φ_1, stabilizes each ideal of $\mathcal{P}$. Now for any given $x \in \mathcal{P}$, since $\varphi_1(x) \in \varphi_1(\langle x \rangle) = \langle x \rangle$, we have that $\langle \varphi_1(x) \rangle \subseteq \langle x \rangle$. Considering φ_1^{-1} we have $\langle x \rangle = \langle \varphi_1^{-1}(\varphi_1(x)) \rangle \subseteq \langle \varphi_1(x) \rangle$. Thus $\varphi_1(x)$ and x respectively generates the same ideal, in other words, $\varphi_1(x) \sim x$. So φ_1 just is an invertible map on $\mathcal{P}$ preserving lattice. This proves (i).

As to (ii), if there exists some k such that $n_k \neq n_{s-k+1}$, then φ stabilizes $\mathcal{E}_{1,s-1}$ and $\mathcal{E}_{2,s}$, respectively (recall Lemma 3.4). Thus φ itself just is an invertible map on $\mathcal{P}$ preserving lattice. $\square$

Theorem 3.8. *The square of any invertible map on $\mathcal{P}$ preserving ideals stabilizes each ideal of $\mathcal{P}$.*

Acknowledgments. The authors would like to thank the referee for his/her valuable comments and suggestions.

References

[1] *L. Orsina and P Papi*: Enumeration of ad-nilpotent ideals of a Borel subalgebra in type A by class of nilpotence. C. R. Acad. Sci, Paris *330* (2000), 651–655.
[2] *D. Panyushev*: Ad-nilpotent ideals of a Borel subalgebra: generators and duality. J. Algebra *274* (2004), 822–846.
[3] *D. Panyushev*: Long Abelian ideals. Advances in Mathematics *186* (2004), 307–316.
[4] *D. Panyushev and G. Röhrle*: Spherical orbits and Abelian ideals. Advances in Mathematics *159* (2001), 229–246.
[5] *P. Cellini and P. Papi*: Abelian ideals of Borel subalgebras and affine Weyl groups. Advances in Mathematics *187* (2004), 320–361.
[6] *P. Cellini and P. Papi*: Ad-nilpotent ideals of a Borel subalgebra. J. Algebra *225* (2000), 130–141.
[7] *P. Cellini and P. Papi*: Ad-nilpotent ideals of a Borel subalgebra II. J. Algebra *258* (2002), 112–121.
[8] *C. Krattenthaler, L. Orsina and P. Papi*: Enumeration of ad-nilpotent $\mathfrak{b}$-ideals for simple Lie algebras. Advances in Applied Mathematics *28* (2002), 478–522.
[9] *Céline Righi*: Ad-nilpotent ideals of a parabolic subalgebra. J. Algebra *319* (2008), 1555–1584.
[10] *D. Panyushev*: Normalizers of ad-nilpotent ideals. European Journal of Combinatorics *27* (2006), 153–178.
[11] *H. Radjavi and P. Šemrl*: Non-linear maps preserving solvability. J. Algebra *280* (2004), 624–634.

Authors' address: Dengyin Wang, Haishan Pan, Xuansheng Wang, Department of Mathematics, China University of Mining and Technology, Xuzhou 221008, People's Republic of China, e-mail: `wdengyin@126.com`.