Czechoslovak Mathematical Journal

Romi F. Shamoyan; Olivera R. Mihić

On some inequalities in holomorphic function theory in polydisk related to diagonal mapping

Czechoslovak Mathematical Journal, Vol. 60 (2010), No. 2, 351–370

Persistent URL: `http://dml.cz/dmlcz/140574`

Czechoslovak Mathematical Journal, 60 (135) (2010), 351–370

ON SOME INEQUALITIES IN HOLOMORPHIC FUNCTION THEORY IN POLYDISK RELATED TO DIAGONAL MAPPING

Romi F. Shamoyan, Bryansk, Olivera R. Mihić, Belgrade

(Received October 22, 2008)

Abstract. We present a description of the diagonal of several spaces in the polydisk. We also generalize some previously known contentions and obtain some new assertions on the diagonal map using maximal functions and vector valued embedding theorems, and integral representations based on finite Blaschke products. All our results were previously known in the unit disk.

Keywords: polydisk, diagonal mapping, Hardy classes, holomorphic spaces

MSC 2010: 32A18

1. Introduction and main definitions

Let $n \in \mathbb{N}$ and $\mathbb{C}^n = \{z = (z_1, \ldots, z_n)\colon\ z_k \in \mathbb{C},\ 1 \leqslant k \leqslant n\}$ be the n-dimensional space of complex coordinates. We denote the unit polydisk by

$$U^n = \{z \in \mathbb{C}^n\colon\ |z_k| < 1,\ 1 \leqslant k \leqslant n\}$$

and the distinguished boundary of U^n by

$$T^n = \{z \in \mathbb{C}^n\colon\ |z_k| = 1,\ 1 \leqslant k \leqslant n\}.$$

By m_{2n} we denote the volume measure on U^n and by m_n we denote the normalized Lebesgue measure on T^n. Let $H(U^n)$ be the space of all holomorphic functions on U^n. When $n = 1$, we simply denote U^1 by U, T^1 by T, m_{2n} by m_2, m_n by m. We refer to [5] and [11] for further details.

The first author was supported by RFFI Grant 09-01-97-517. The second author was supported by MNZŽS Serbia, Project 144010.

The Hardy spaces, denoted by $H^p(U^n)$ $(0 < p \leqslant \infty)$, are defined by $H^p(U^n) = \{f \in H(U^n)\colon \sup\limits_{0\leqslant r<1} M_p(f,r) < \infty\}$, where

$$M_p^p(f,r) = \int_{T^n} |f(r\xi)|^p \,\mathrm{d}m_n(\xi), \quad M_\infty(f,r) = \max_{\xi\in T^n} |f(r\xi)|,\ r \in (0,1),\ f \in H(U^n).$$

As usual, we denote by $\vec{\alpha}$ the vector $(\alpha_1,\dots,\alpha_n)$.

For $\alpha_j > -1$, $j = 1,\dots,n$, $0 < p < \infty$, recall that the weighted Bergman space $A^p_{\vec{\alpha}}(U^n)$ consists of all holomorphic functions on the polydisk such that

$$\|f\|^p_{A^p_{\vec{\alpha}}} = \int_{U^n} |f(z)|^p \prod_{i=1}^n (1-|z_i|^2)^{\alpha_i} \,\mathrm{d}m_{2n}(z) < \infty.$$

When $\alpha_1 = \ldots = \alpha_n = \alpha$ then we use notation $A^p_\alpha(U^n)$.

Let $\mathbb{Z}^n_+ = \{(k_1,\dots,k_n)\colon k_j \in \mathbb{Z}_+ = \mathbb{N}\cup\{0\}\}$,

$$\mathbb{Z}^n_- = \{(k_1,\dots,k_n)\colon k_j \in \mathbb{Z}_-,\ j=1,\dots,n\}.$$

If u is n-harmonic (harmonic in each variable), then as usual

$$u(r_1\mathrm{e}^{\mathrm{i}\varphi_1},\dots,r_n\mathrm{e}^{\mathrm{i}\varphi_n}) = \sum_{k_1,\dots,k_n=-\infty}^{\infty} C_{k_1,\dots,k_n} \prod_{j=1}^n r_j^{|k_j|}\mathrm{e}^{\mathrm{i}k_j\varphi_j}.$$

We define the fractional derivative of order α of an n-harmonic function in the usual way as follows

$$\mathcal{D}^\alpha u(\vec{r}\mathrm{e}^{\mathrm{i}\vec{\varphi}}) = \sum_{k_1,\dots,k_n=-\infty}^{\infty} C_{k_1,\dots,k_n} \frac{\Gamma(\alpha+|k|+1)}{\Gamma(\alpha+1)\Gamma(|k|+1)} \prod_{j=1}^n r_j^{|k_j|}\mathrm{e}^{\mathrm{i}k_j\varphi_j},$$

$$\frac{\Gamma(\alpha+|k|+1)}{\Gamma(\alpha+1)\Gamma(|k|+1)} = \prod_{j=1}^n \frac{\Gamma(\alpha_j+|k_j|+1)}{\Gamma(\alpha_j+1)\Gamma(|k_j|+1)}, \quad \alpha_j > -1.$$

Let further

$$h^p(\vec{\alpha}) = \left\{u \text{ is } n\text{-harmonic: } \int_{U^n} \prod_{k=1}^n (1-|z_k|)^{\alpha_k} |u(z_1,\dots,z_n)|^p \,\mathrm{d}m_{2n}(z) < \infty\right\},$$

$0 < p < \infty$, $\alpha_k > -1$, $k = 1,\dots,n$.

Note that it is easy to check that $\mathcal{D}^\alpha u$ is n-harmonic if u is n-harmonic.

We will call a function u pluriharmonic if $u = \mathrm{Re}(f)$, $f \in H(U^n)$ and we denote

$$\tilde{h}^p(\vec{\alpha}) = \{u \text{ is pluriharmonic: } \|u\|_{h^p(\vec{\alpha})} < \infty\}.$$

If $1 \leqslant p_j < \infty$, $j = 1, \ldots, n$, both $h^p(\alpha)$ and $\tilde{h}^p(\vec{\alpha})$ are Banach spaces, for $0 < p \leqslant 1$, $j = 1, \ldots, n$, $h^p(\alpha)$ and $\tilde{h}^p(\vec{\alpha})$ are quasinormed spaces.

Throughout the paper, we write C (sometimes with indices) to denote a positive constant which might be different at each occurrence (even in a chain of inequalities) but is independent of the functions or variables being discussed.

The notation $A \asymp B$ means that there is a positive constant C, such that $B/C \leqslant A \leqslant CB$. We will write for two expressions $A \lesssim B$ if there is a positive constant C such that $A < CB$.

Let us reiterate the main definition.

Definition 1 (see [5]). Let $\mathcal{X} \subset H(U)$, $\mathcal{Y} \subset H(U^n)$ be subspaces of $H(U)$ and $H(U^n)$. We say that the diagonal of $\mathcal{Y}$ coincides with $\mathcal{X}$ if for any function f, $f \in \mathcal{Y}$, $f(z, \ldots, z) \in \mathcal{X}$, and the reverse is also true: for every function g from $\mathcal{X}$ there is an extension $f(z_1, \ldots, z_n)$, $f \in \mathcal{Y}$ so that $f(z, \ldots, z) = g(z)$.

Note when $\operatorname{Diag}(\mathcal{Y}) = \mathcal{X}$, then $\|f\|_{\mathcal{X}} \asymp \inf_{\Phi} \|\Phi(f)\|_{\mathcal{Y}}$, where $\Phi(f)$ is an arbitrary analytic extension of f from the diagonal of the polydisk to the polydisk. The problem to studying the diagonal map and its applications was suggested for the first time by W. Rudin in [11]. Later on, several papers appeared where complete solutions were given for classical holomorphic spaces such as the Hardy and Bergman classes, see [5], [6], [8], [12], [16] and the references there. Recently the complete answer was given for the so-called mixed norm spaces in [10]. For many other classes the answer is unknown. The aim of this note is to add various new results in this research area.

Theorems on the diagonal map have numerous applications in the theory of holomorphic functions (see, for example, [2], [14]). Analogues of the diagonal map problem, the so-called trace problems in various functional spaces in $\mathbb{R}^n$, are well-known (see, for example, [9]).

This paper is organized as follows. In the second section we collect preliminary assertions. In the third and fourth sections we present various new results concerned with the so-called diagonal map operator in the polydisk; practically all results from these sections were previously known for particular values of the parameters or are obvious for the case of the unit disk.

2. Preliminaries

We need the following lemmas.

Lemma 1 [1]**.** *Let $0 < p < q < \infty$, and let μ be a positive Borel measure in the unit disk. If $\mu\{z \in U\colon |z-\xi| < r\} < Cr^{q/p}$ for each $\xi \in T$ and each $r > 0$, then*

$$\int_U \|f(z)\|_X^q \,\mathrm{d}\mu(z) \leqslant C\|f\|_{H^p(X)},$$

where X is a Banach space in U or a quasinormed space.

Lemma 2. *Let $t > -1$, $\beta_j > (t+1)/n$, $j = 1, \ldots, n$. Then*

$$J = \int_0^1 \frac{(1-|w|)^t d|w|}{\prod\limits_{j=1}^n |1-|w|\mathrm{e}^{\mathrm{i}\varphi} z_j|^{\beta_j}} \leqslant \frac{C}{\prod\limits_{j=1}^n |1-z_j \mathrm{e}^{\overline{\mathrm{i}\varphi}}|^{\beta_j - \frac{t+1}{n}}}, \quad z_j \in U,\ \mathrm{e}^{\mathrm{i}\varphi} \in T.$$

P r o o f. We restrict the proof to the case of $n = 2$. The general case can be considered similarly. We have obviously

$$J = \int_0^{R_0} \frac{(1-|w|)^t d|w|}{|1-|w|\mathrm{e}^{\overline{\mathrm{i}\varphi}} z_1|^{\beta_1} |1-|w|\mathrm{e}^{\mathrm{i}\varphi} z_2|^{\beta_2}} + \int_{R_0}^1 \frac{(1-|w|)^t d|w|}{|1-|w|\mathrm{e}^{\overline{\mathrm{i}\varphi}} z_1|^{\beta_1} |1-|w|\mathrm{e}^{\mathrm{i}\varphi} z_2|^{\beta_2}},$$

where $R_0 = \max(|z_1|, |z_2|)$. Hence

$$\begin{aligned}\int_{R_0}^1 \frac{(1-|w|)^t d|w|}{|1-|w|\mathrm{e}^{\overline{\mathrm{i}\varphi}} z_1|^{\beta_1} |1-|w|\mathrm{e}^{\mathrm{i}\varphi} z_2|^{\beta_2}} &\leqslant \frac{C(1-R_0)^{\frac{t+1}{2}}(1-R_0)^{\frac{t+1}{2}}}{|1-z_1\mathrm{e}^{\overline{\mathrm{i}\varphi}}|^{\beta_1} |1-z_2\mathrm{e}^{\overline{\mathrm{i}\varphi}}|^{\beta_2}} \\ &\leqslant \frac{C}{\prod\limits_{j=1}^2 |1-z_j\mathrm{e}^{\overline{\complement\varphi}}|^{\beta_j - \frac{t+1}{2}}}.\end{aligned}$$

Thus

$$\begin{aligned}\int_0^{R_0} \frac{(1-|w|)^t d|w|}{|1-|w|\mathrm{e}^{\overline{\mathrm{i}\varphi}} z_1|^{\beta_1} |1-|w|\mathrm{e}^{\mathrm{i}\varphi} z_2|^{\beta_2}} &\leqslant \int_0^{R_0} \frac{(1-R|z|)^{\frac{t+1}{2}}(1-R)^{\frac{t}{2}-\frac{1}{2}}\,\mathrm{d}R}{|1-R\mathrm{e}^{\mathrm{i}\varphi}\,\overline{z_1}|^{\beta_1}|1-R\mathrm{e}^{\mathrm{i}\varphi}\,\overline{z_2}|^{\beta_2}} \\ &\leqslant \frac{C}{\prod\limits_{j=1}^2 |1-z_j\mathrm{e}^{\overline{\mathrm{i}\varphi}}|^{\beta_j - \frac{t+1}{2}}}.\end{aligned}$$

The proof is complete. $\square$

Lemma 3. *Let* $\mathrm{d}\mu(z)=(1-|z|)^{1/p-2}\,\mathrm{d}m_2(z)$, $0<p<1$. *Then*

$$\mu\{z\in U\colon\ |z-\xi|<r\}<Cr^{1/p},\quad r>0,\ \xi\in T.$$

Proof. Let $K_r=\{z\colon\ 0<|z-\xi|<r\}$, $z=\xi+r\mathrm{e}^{\mathrm{i}\varphi}=\varrho\mathrm{e}^{\mathrm{i}\varphi}$, $\xi=\mathrm{e}^{\mathrm{i}\theta}$. Then

$$|\mathrm{e}^{\mathrm{i}\theta}-\mathrm{e}^{\mathrm{i}\varphi}|<r,\quad 1-r<\varrho<1,\quad \left|\sin\frac{\theta-\varphi}{2}\right|<r.$$

Hence we have

$$\begin{aligned}\int_{K_r}\mathrm{d}\mu(z)&=\int_{1-r}^{1}\int_{|\theta-\varphi|}(1-\varrho)^{1/p-2}\varrho\,\mathrm{d}\varrho\,\mathrm{d}\varphi\\&=\int_{1-r}^{1}\int_{\theta-2r}^{\theta+2r}(1-\varrho)^{1/p-2}\varrho\,\mathrm{d}\varrho\,\mathrm{d}\varphi\leqslant\frac{4r}{1/p-1}r^{1/p-1}.\end{aligned}$$

The proof is complete. □

3. Vector valued embedding theorems, maximal operators and diagonal mapping

The goal of this section is to show how vector valued embeddings and estimates for maximal functions can be used for the study of the operator of diagonal mapping. In this section we also completely describe traces of some holomorphic classes in the polydisk on the diagonal $(z,\ldots,z)$.

In [6] authors gave a partial solution of Rudin's problem providing, in particular, the following assertion.

Theorem A. *Let* $1<p<q<\infty$, $n\geqslant 2$. *Then*

$$\int_U|g(z)|^q(1-|z|)^{nq/p-2}\,\mathrm{d}m_2(z)\leqslant C\|f\|_{H^p(U^n)},$$

where $g(z)=(\mathcal{D}f)(z)=f(z,\ldots,z)$.

We provide below generalizations of this result.

Theorem 1. *Let $0 < p_i < 1$, $i = 1, \ldots, n$. Then*

$$\int_{U^n} |f(z_1, \ldots, z_n)| \prod_{k=1}^{n} (1 - |z_k|)^{1/p_k - 2} \,\mathrm{d}m_{2n}(z)$$
$$\leqslant C \sup_{0 \leqslant r < 1} \left(\int_T \left(\ldots \int_T \left(\int_T |f(r\xi)|^{p_1} \,\mathrm{d}m_1(\xi_1) \right)^{\frac{p_2}{p_1}} \mathrm{d}m_1(\xi_2) \ldots \right)^{\frac{p_n}{p_{n-1}}} \mathrm{d}m_1(\xi_n) \right)^{\frac{1}{p_n}}$$
$$= C\|f\|_{H^{\vec{p}}} < \infty \quad \textit{and} \quad \|\mathcal{D}f\|_{A^q_{(\sum_{i=1}^n \frac{1}{p_i})q-2}} \leqslant C\|f\|_{H^{\vec{p}}},\ q \geqslant 1.$$

P r o o f. We use an idea which appeared in [1] for the study of Hardy classes in the polydisk. Let $n = 2$. The general case can be considered similarly by a small modification of the proof we provide below. Let $f\colon U \to L^p(X)$, where X is a quasinormed space, $0 < p < \infty$, so for every $z \in U$, $\int_U \|f(z)\|_X^p \,\mathrm{d}m_2(z) < \infty$, and $H^p(X)$ is the closure of polynomials depending on z with values in X

$$P_A(x) = \sum_{k=0}^{n} x_k z^k, \quad x_k \in X, \qquad \|f\|^p_{L^p(X)} = \int_T \|f(\xi)\|^p \,\mathrm{d}\xi,\ 0 < p < \infty.$$

Note that if $X = \mathbb{C}$ then $H^p(X)$ is the ordinary Hardy class. If $X = H^{p_1}$ then

$$H^{p_2}(X) = \left\{ f \in H(U^2)\colon \sup_{r<1} \left(\int_T \left(\int_T |f(r\xi_1, r\xi_2)|^{p_1} \,\mathrm{d}\xi_1 \right)^{\frac{p_2}{p_1}} \mathrm{d}\xi_2 \right)^{\frac{1}{p_2}} < \infty \right\},$$

$0 < p_1, p_2 < \infty$. Put in Lemma 1 $q = 1$, $p = p_2$, $H^p = H^{p_2}$, $X = H^{p_1}$. Then $H^p(X) = H^{p_2}(H^{p_1})$. From Lemma 1 we have

$$\int_U \|f(z)\|_{H^{p_1}} \,\mathrm{d}\mu(z) \leqslant C\|f\|_{H^{p_1,p_2}}. \tag{1}$$

Since by Lemma 3 the measure $\mu(z_2) = (1 - |z_2|)^{1/p_2} \,\mathrm{d}m_2(z_2)$ can be used in Lemma 1 we have

$$\int_U \left(\int_T |f(\xi, z_2)|^{p_1} \,\mathrm{d}\xi \right)^{1/p_1} (1 - |z_2|)^{1/p_2 - 2} \,\mathrm{d}m_2(z_2) \leqslant C\|f\|_{H^{p_1,p_2}},$$
$$\int_U |f(z_1, z_2)|(1 - |z_1|)^{1/p_1 - 2} \,\mathrm{d}m_2(z_1) \leqslant C\left(\int_T |f(\xi, z_2)|^{p_1} \,\mathrm{d}\xi \right)^{1/p_1},$$

and we have the estimate we need from the last two inequalities.

To get the second estimate of the theorem we use the fact that (see [5])

$$\int_U |f(z, \ldots, z)|^p (1 - |z|)^{\sum_{j=1}^{n} \alpha_j + 2n - 2} \,\mathrm{d}m_2(z) \leqslant C\|f\|_{A^p_{\vec{\alpha}}(U^n)}, \quad 0 < p < \infty,\ \alpha_j > -1,$$

for $p=1$ and the estimate (see [5])

$$\left(\int_{U^n}|\Phi(z)|\prod_{k=1}^{n}(1-|z|)^{\alpha}\,\mathrm{d}m_2(z)\right)^{s}\leqslant C\int_{U^n}|\Phi(z)|^{s}\prod_{k=1}^{n}(1-|z|)^{\alpha s+2s-2}\,\mathrm{d}m_2(z),$$

$s\leqslant 1$, $\Phi\in H(U^n)$. The theorem is proved. □

Remark 1. The first assertion in Theorem 1 for $p_1=\ldots=p_n$ is well-known (see [1]) and plays a crucial role in the description of the dual of Hardy classes H^p, $p<1$ on the polydisk (see [1]).

We give now another extension of a theorem of P. L. Duren and A. L. Shields using the classical interpolation theorem.

Theorem 2. *Let* $1\leqslant p<q<\infty$. *Then* $M_q(\mathcal{D}f,r)(1-r^2)^{\frac{n}{p}-\frac{1}{q}}\leqslant CM_p(f,r)$, $n>1$, *and* $\int_0^1(1-r)^{\frac{-p}{q}+n-1}\left(\int_T|(\mathcal{D}f)(r\xi)|^q\,\mathrm{d}\xi\right)^{\frac{p}{q}}\,\mathrm{d}r\leqslant C\|f\|_{H^p(U^n)}$, $n>1$.

Remark 2. Note that since $M_p(f,r)$ is increasing (see [5]) we have

$$\begin{aligned}&\int_U|g(z)|^q(1-|z|)^{\frac{q}{p}\alpha-2}\,\mathrm{d}m_2(z)\\&\qquad\leqslant C\int_0^1(1-r)^{\frac{-p}{q}+\alpha-1}\left(\int_T|g(r\xi)|^q\,\mathrm{d}\xi\right)^{\frac{p}{q}}\mathrm{d}r,\quad p\leqslant q,\ \alpha\geqslant 1.\end{aligned}$$

So our theorem improves Theorem A from [6].

P r o o f of Theorem 2. We will show Theorem 2 for $n=2$. The general case can be obtained similarly. Let $f\in H^p(U^2)$. Then by the Poisson integral representation

$$|f(z,z)|=C\left|\int_{T^2}\frac{f(\xi_1,\xi_2)(1-|z|)^2\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2}{|1-\langle\overline{\xi_1},z\rangle|^2|1-\langle\overline{\xi_2},z\rangle|^2}\right|,\quad z\in U.$$

We choose $s>1$, $1-1/s=1/p-1/q$ and put $c=p/(p-1)$, $b=s/(s-1)$, $a=q$. Then $1/a+1/b+1/c=1$. Let $\Phi(\xi,z)=\prod_{k=1}^{2}|1-\langle\overline{\xi_k},z_k\rangle|^{-2}$, $\xi_k\in T$, $z_k\in U$, $k=1,2$. Then by Hölder's inequality

$$\begin{aligned}\frac{|f(z,z)|}{(1-|z|)^2}&\leqslant C\left(\int_{T^2}|\Phi(\xi,z)|^s|f(\xi)|^p\,\mathrm{d}\xi\right)^{\frac{1}{a}}\left(\int_{T^2}|f(\xi)|^p\,\mathrm{d}\sigma(\xi)\right)^{\frac{1}{b}}\left(\int_{T^2}|\Phi(\xi,z)|^s\,\mathrm{d}\xi\right)^{\frac{1}{c}}\\&\leqslant C_1(A(f))(B(f))\frac{1}{(1-|z|)^{(2s-1)2/c}}\end{aligned}$$

and

$$\int_T |f_R(z,z)|^q (1-|z|)^{\frac{2q}{c}(2s-1)-2q}\,\mathrm{d}\varphi$$

$$\leqslant C\biggl(\int_T\int_{T^2} |\Phi(\xi,z)|^s |f_R(\xi)|^p\,\mathrm{d}\xi\,\mathrm{d}\varphi\biggr)\|f_R\|^{pq/b}_{H^p(T^2)}.$$

Finally $\int_T \left(|f_R(z,z)|^q\,\mathrm{d}\varphi\right)^{\frac{1}{q}}(1-R)^\alpha \leqslant C\|f_R\|_{H^p(T^2)}$, $R\in(0,1)$, $\alpha=(4s-1-2q+2qC^{-1}(2s-1))/q$, $s\in[1,q)$, $t>0$. Hence $\|(\mathcal{D}f_R)\|_{L^q}(1-R)^{2/p-1/q}\leqslant C\|f_R\|_{H^p}$. Note that the last estimate is well-known for $n=1$: $M_q(f,r)\leqslant C(1-r^2)^{1/q-1/p}M_p(f,r)$, $1\leqslant p<q$.

To get the estimate, we need we apply the classical Marcinkiewicz interpolation theorem to the following function Φ,

$$(\Phi_f)(r)=(1-r^2)^{-1/q}\biggl(\int_T |f(r\xi,r\xi)|^q\,\mathrm{d}\xi\biggr)^{1/q},\quad 0<r<1.$$

Since $\{r\in(0,1)\colon \Phi_f(r)>t\}\subset\{r\in(0,1)\colon (1-r^2)^2\leqslant \left(C_s t^{-1}\|f\|_s\right)^s\}$, for any $s\in[1,r)$, $t>0$, we have for all t

$$\mu(\Phi_f(r)>t)\leqslant\left(C_s t^{-1}\|f\|_s\right)^s,\quad \mu(r)=4r(1-r^2)\,\mathrm{d}r.$$

Hence by putting $s=\frac{1}{2}(p+q)$, $s=1$ and applying the Marcinkiewicz interpolation theorem we get the estimate we need

$$\int_0^1 (1-r^2)^{1-p/q}\left(M_q(\mathcal{D}f,r)\right)^p\,\mathrm{d}r\leqslant C\|f\|^p_{H^p(U^2)}.$$

The proof of the theorem is now complete. □

Below we provide a complete description of the diagonal of some analytic classes based on Lorentz spaces on the unit circle.

Theorem 3. *Let $q>1$ and $\alpha\in(-1,1)$. Then $\operatorname{Diag}(L(\alpha,q))=L_1(\alpha,q)$, where*

$$L(\alpha,q)=\biggl\{f\in H(U^n)\colon \sum_{k_1,\dots,k_n\geqslant 0} 2^{\frac{-k_1}{n}(\alpha+1)}\dots 2^{\frac{-k_n}{n}(\alpha+1)}\sup_{\vec r\in I_{\vec k}}\|f(r_1\xi,\dots,r_n\xi)\|_{L^{q,\infty}(T)}<\infty\biggr\},$$

$$L_1(\alpha,q)=\biggl\{f\in H(U)\colon \int_0^1\|f_r\|_{L^{q,\infty}(T)}(1-r)^\alpha\,\mathrm{d}r<\infty\biggr\},$$

$$I_{\vec k}=\prod_{i=1}^n(1-2^{-k_i},1-2^{-k_i-1}].$$

P r o o f. We have

$$\begin{aligned}\int_0^1 &\|f_r\|_{L^{p,q}(T)}(1-r)^\alpha\,\mathrm{d}r\\ &\leqslant C\sum_{k\geqslant 0}\sup_{r\in I_k}\|f_r\|_{L^{p,q}(T)}2^{-k\alpha}2^{-k}\\ &\leqslant C\sum_{k_1,\dots,k_n\geqslant 0}2^{\frac{-k_1}{n}(\alpha+1)}\dots 2^{\frac{-k_n}{n}(\alpha+1)}\sup_{r_1\in I_{k_1},\dots,r_n\in I_{k_n}}\|f(r_1\xi,\dots,r_n\xi)\|_{L^{p,q}(T)},\end{aligned}$$

where $I_k=(1-2^{-k},1-2^{-k-1}]$.

We put $q=\infty$ to get a half of the theorem.

Let us prove the reverse. Let $n=2$. The general case can be obtained by a small modification. Let $\widetilde{\alpha}$ be sufficiently large positive number. We put

$$f(|z_1|\xi,|z_1|\xi)=C_\alpha\int_U\frac{f(w)(1-|w|)^{\widetilde{\alpha}}\,\mathrm{d}m_2(w)}{(1-\langle\overline{w},z_1\rangle)^{\frac{\widetilde{\alpha}+2}{2}}(1-\langle\overline{w},z_2\rangle)^{\frac{\widetilde{\alpha}+2}{2}}},$$

where C_α is the Bergman projection constant. Obviously

$$f(|z|\xi,|z|\xi)=f(z),\quad z=|z|\xi,\ z\in U.$$

And moreover by a well-known characterization of the $L^{q,\infty}$ classes (see [7])

$$\begin{aligned}\|f\|_{L^{q,\infty}(T)}&\asymp\sup_{I\subset T}\frac{1}{|I|^{1-1/q}}\int_I|f(|z_1|\xi,|z_1|\xi)|\,\mathrm{d}m(\xi)\\ &\leqslant C\int_0^1\sup_{I\subset T}\frac{1}{|I|^{1-1/q}}\int_I\int_T\frac{f(w)(1-|w|)^{\widetilde{\alpha}}\,\mathrm{d}m_2(w)\,\mathrm{d}\xi}{\prod\limits_{k=1}^{2}(1-\langle w,z_k\rangle)^{\frac{\widetilde{\alpha}+2}{2}}}.\end{aligned}$$

Note that our integral over the unit circle is a convolution of two functions. Hence we can apply the well-known Young's inequality for convolutions in Lorentz clasess (see [7])

$$\|f*g\|_{L^{q,\infty}(T)}\leqslant C\|f\|_{L^{q,\infty}(T)}\|g\|_{L^1(T)},\quad q>1,$$

where $g(\xi)=1/(1-\langle|\overline{w|\xi},z\rangle)^{\frac{\widetilde{\alpha}+2}{2}}$, $w=|w|\xi,\ z,w\in U,\ \xi\in T$. Hence we have

$$\begin{aligned}&\sum_{k_1,k_2\geqslant 0}\sup_{r_1\in I_{k_1},r_2\in I_{k_2}}\|f_r\|_{L^{q,\infty}(T)}(1-r_1)^{\frac{\alpha+1}{2}}(1-r_2)^{\frac{\alpha+1}{2}}\\ &\leqslant C\sup_{r_1\in I_{k_1},r_2\in I_{k_2}}\sum_{k_1,k_2\geqslant 0}\Bigl(\prod_{j=1}^{2}2^{-k_j\frac{\alpha+1}{2}}\Bigr)\\ &\quad\times\int_0^1\|f_{|w|}\|_{L^{q,\infty}(T)}(1-|w|)^{\widetilde{\alpha}}\left\|\frac{1}{(1-\langle|\overline{w|\xi},z_1\rangle)^{\frac{\widetilde{\alpha}+2}{2}}(1-\langle|\overline{w|\xi},z_2\rangle)^{\frac{\widetilde{\alpha}+2}{2}}}\right\|_{L^1(T)}d|w|,\end{aligned}$$

where $\widetilde{\alpha}$ is big enough.

Note now

$$\text{(2)}\qquad \sum_{n\geqslant 1}\frac{2^{-n\beta}}{1-\lambda_n|z|}\leqslant\frac{C}{(1-|z|)^{1-\beta}},\quad \lambda_n=1-2^{-n},\ \beta\in(0,1),\ |z|\in(0,1).$$

Further for $\widetilde{\alpha}>0$

$$\left\|\frac{1}{(|1-\langle\overline{|w|\xi},z_1\rangle|\ |1-\langle\overline{|w|\xi},z_2\rangle|)^{\widetilde{(\alpha+2)/2}}}\right\|_{L^1(T)}\leqslant C\,\frac{(1-|w|)^{-\widetilde{\alpha}/2}(1-|w|)^{1+\widetilde{\alpha}/2}}{\prod\limits_{j=1}^{n}(1-(1-2^{k_j})|w|)}.$$

Therefore by using (2) twice for $\beta=\frac{1}{2}(\alpha+1)$, $\alpha\in(-1,1)$ we get finally what we need. The theorem is proved. $\square$

Remark 3. We note also that in Theorem 3 we can replace the $L^{p,\infty}$ classes by any functional Banach space X on T so that $\|f*g\|_{X(T)}\leqslant C\|f\|_{X(T)}\|g\|_{L^1(T)}$.

Since it is known that (see [12]) $\int_U|(\mathcal{D}f)(z)|^p(1-|z|)^{n-2}\,\mathrm{d}m_2(z)\leqslant C\|f\|_{H^p(U^n)}$, $n>1,\ 0<p<\infty$, then we can apply it to the slice function $f_\varrho(z)=f(\varrho z),\ z\in U^n$, $\varrho\in(0,1)$ we have

$$\begin{aligned}\sup_{\varrho<1}(1-\varrho)^\alpha\int_U|f(\varrho z,\dots,\varrho z)|^p(1-|z|)^{n-2}\,\mathrm{d}m_2(z)\\ \leqslant C\sup_{\varrho<1}M_p(f,\varrho)(1-\varrho)^\alpha,\quad \alpha\geqslant 0,\ n>1.\end{aligned}$$

Moreover if

$$(T_{n,\widetilde{\alpha}}f)(z)=F(z_1,\dots,z_n)=C\int_U\frac{f(z)(1-|z|)^{\widetilde{\alpha}}\,\mathrm{d}m_2(z)}{\prod\limits_{k=1}^{n}(1-\langle\overline{z},z_k\rangle)^{\frac{\widetilde{\alpha}+2}{n}}},\quad z_j\in U,\ j=1,\dots,n$$

then $(\mathcal{D}F)=f$, $\|F\|_{H^p}\leqslant C\|f\|_{A^p_{n-2}}$, $0<p<\infty$, $n>1$ (see [12]) and $T_{n,\widetilde{\alpha}}(f_\varrho)=F_\varrho$. The last equality follows from

$$\int_T f(\varrho tz)g(r\overline{t})\,\mathrm{d}t=\int_T f(tz)g(\varrho r\overline{t})\,\mathrm{d}t,\quad f,g\in H(U),\ z\in U,\ \varrho,r\in(0,1).$$

Collecting the arguments we gave above, in the following theorem we provide a description of the diagonal of weighted Hardy spaces. Note that this description was missing in the recent paper [10].

Theorem 4. *Let $n > 1$, $0 < p < \infty$, $\alpha \geqslant 0$. Then* $\operatorname{Diag}(A^{p,\infty}_\alpha) = A^{p,n}_\alpha$, *where*

$$A^{p,\infty}_\alpha = \{f \in H(U^n)\colon \sup_{\varrho<1} M_p(f,\varrho)(1-\varrho)^\alpha < \infty\},\ M_p^p(f,r) = \int_{T^n} |f(r\xi)|^p \,\mathrm{d}\xi,$$
$$A^{p,n}_\alpha = \Big\{f \in H(U)\colon \sup_{\varrho<1} (1-\varrho)^\alpha \int_U |f(\varrho z)|^p (1-|z|)^{n-2} \,\mathrm{d}m_2(z) < \infty\Big\}.$$

We will provide now the description of the diagonal of the $A^{p,\infty}_\alpha$ classes, also for $p = \infty$. Let S be a set of positive measurable functions w on $(0,1)$ such that $m_w \leqslant w(\lambda r)/w(r) \leqslant M_w$, for all $r \in (0,1)$, $\lambda \in [q_w, 1]$ and some fixed M_w, m_w, q_w such that $m_w, q_w \in (0,1)$, $M_w > 0$. The classes S were studied in [15]. Let $\alpha_w = \log m_w/\log q_w$, then it can be shown that $w(x) \in [x^{\alpha_w}, x^{-\beta_w}]$, $x \in (0,1)$, $\beta_w = \ln M_w/\ln(1/q_w)$, $w \in S$, (see [13], [15]).

Let further $\Lambda_{w_1,\dots,w_n,\beta} = \{f \in H(U^n)\colon \sup\limits_{z_j\in U, j=1,\dots,n} |f(z_1,\dots,z_n)| \prod\limits_{k=1}^n (w_k(1-|z_k|) \times (1-|z_k|)^{\beta/n}) < \infty\}$, $\beta > 0$.

In our next theorem we describe the diagonal of Bloch type classes with general weights in the polydisk.

Theorem 5. *Let $w_j^{-n} \in S$, $\tau > \alpha_{\tilde w}$, $\tilde w = w_j^{-n}$, $j = 1,\dots,n$. Then*

$$\operatorname{Diag}(\Lambda_{w_1,\dots,w_n,\tau}) = \{f \in H(U)\colon \sup_{z\in U} |f(z)| \prod_{k=1}^n (w_k(1-|z|)(1-|z|)^\tau) < \infty\}.$$

P r o o f. One part of the theorem is trivial. To prove the reverse we need the following estimate (see [13])

$$\int_U \frac{w(1-|z|)\chi^p_\gamma(z)\,\mathrm{d}m_2(z)}{|1-\langle z,\overline{z_1}\rangle|^{\alpha+2}} \leqslant C\frac{w(1-|z_1|)}{(1-|z_1|)^\alpha}\chi^p_\gamma(z_1), \tag{3}$$

where $z_1 \in U$, $w \in S$, $\alpha > \alpha_w$, $\gamma \in [0, \min(1-\beta_w)]$, $\chi_\gamma(z) = 1/(1-|z|)^{\gamma/pq}$, $\gamma \in [0,\infty)$, $1/p + 1/q = 1$, $p > 1$. We have as above for any positive τ

$$F(z_1,\dots,z_n) = C(\tau)\int_U \frac{f(z)(1-|z|)^\tau\,\mathrm{d}m_2(z)}{\prod\limits_{k=1}^n (1-\langle\overline{z},z_k\rangle)^{\frac{\tau+2}{n}}},\quad \mathcal{D}F = f.$$

Hence using (3) for $\gamma = 0$ and Hölder's inequality for m functions we have

$$|F(z_1, \ldots, z_n)| \lesssim \int_U \frac{\prod_{k=1}^n (w_k(1-|z|))^{-1}\,\mathrm{d}m_2(z)}{\prod_{k=1}^n (1-\langle \overline{z}, z_k\rangle)^{\frac{\tau+2}{n}}}$$

$$\lesssim \prod_{j=1}^n \Big(\int_U \frac{(w_j(1-|z|))^{-n}\,\mathrm{d}m_2(z)}{|1-\langle \overline{z}, z_j\rangle|^{\tau+2}}\Big)^{\frac{1}{n}} \lesssim \prod_{j=1}^n \frac{w_j(1-|z|)^{-1}}{(1-|z_j|)^{\frac{\tau}{n}}}.$$

The proof of the theorem is complete. □

Remark 4. The corresponding theorem for little Bloch-type classes can be proved similarly.

The expanded Bergman projection

$$(T_{n,\alpha} f)(w) = C(n,\alpha) \int_U \frac{f(z)(1-|z|)^\alpha}{\prod_{k=1}^n (1-\langle \overline{z}, w_k\rangle)^{\frac{\alpha+2}{n}}}\,\mathrm{d}m_2(z), \quad \alpha > -1,$$

where $w = (w_1, \ldots, w_n) \in U^n$, $C(n,\alpha)$ is the Bergman constant from the Bergman representation formula, plays a crucial role in the study of the diagonal map (see [5], [8], [10], [16] and the references there).

We will now provide new estimates for this operator using, in particular, Stein-type maximal functions from [18]. At the same time we extend previously known estimates.

Theorem 6. (a) *Let* $\Gamma_\gamma(\xi) = \{z \in U\colon\ |1-\overline{\xi}z| < \gamma(1-|z|)\}$, $\gamma > 1$, $\xi \in T$. *Let* $\beta \in (0, \frac{1}{2})$, $\alpha > \beta$, $n = 2$. *Then*

$$\int_T \Big(\sup_{z_1 \in \Gamma_\gamma(\xi)} \sup_{z_2 \in \Gamma_\gamma(\xi)} |\mathcal{D}^\alpha_{z_2} T_{n,0}(f)(z_1, z_2)|(1-|z_1|)^{\alpha-\beta}(1-|z_2|)^\beta\,\mathrm{d}m(\xi)\Big)^2 \leqslant C\|f\|^2_{H^2(U)}.$$

(b) *Let* $p > 2$, $1/p + 1/q = 1$, $t \in (-2,-1)$, $\alpha > \max(t + 2/q, 0)$, $n = 2$. *Then*

$$\sup_{z_1, z_2 \in U} |T_{n,\alpha} f(z_1, z_2)|(1-|z_1|)^{t+2}(1-|z_2|)^{\frac{\alpha-t}{2}-\frac{1}{q}}$$
$$\leqslant C\Big(\int_T \sup_{z \in \Gamma_\gamma(\xi)} |f(z)|(1-|z|)^{\frac{\alpha}{2}}\Big)^p\,\mathrm{d}m(\xi).$$

Remark 5. Putting $n = 1$, $\alpha = 0$, $\beta = 0$ in the first estimate of Theorem 6 we get the following well-known estimate for H^p classes (see [5], Chapter 1)

$$\int_T \sup_{z \in \Gamma_\gamma(\xi)} |\Phi(z)|^2\,\mathrm{d}m(\xi) \leqslant C\|\Phi\|^2_{H^2(U)}.$$

Putting $n=1$, $\alpha=0$, $t=-2$ in the second statement of Theorem 6 we get the well-known estimate (see [1], Theorem 2.5, [4], [5])

$$\sup_{|z|<1}|\Phi(z)|(1-|z|)^{\frac{1}{p}} \leqslant C\int_T \sup_{w\in\Gamma_\gamma(\xi)}|\Phi(w)|^p\,\mathrm{d}m(\xi)=\|\Phi\|^p_{H^p}.$$

P r o o f of Theorem 6. Let $T_{2,0}(f)=\Phi(z_1,z_2)$. Then, by using Hölder's inequality we obtain

$$\begin{aligned}|\Phi(z_1,z_2)| &\lesssim C\int_U \frac{|f(w)|}{|1-\langle\overline{w},z_1\rangle||1-\langle\overline{w},z_2\rangle|}\,\mathrm{d}m_2(w)\\ &\leqslant C\biggl(\int_U \frac{|f(w)|^2(1-|w|)^{2\beta}}{|1-\langle\overline{w},z_1\rangle|^2}\,\mathrm{d}m_2(w)\biggr)^{\frac12}\biggl(\int_U \frac{(1-|w|)^{-2\beta}}{|1-\langle\overline{w},z_2\rangle|^2}\,\mathrm{d}m_2(w)\biggr)^{\frac12}.\end{aligned}$$

Hence since $\beta\in(0,\frac12)$ and $\alpha>\beta$,

$$\begin{aligned}|\mathcal{D}^\alpha_{z_2}\Phi(z_1,z_2)| \lesssim C\biggl(\int_U \frac{|f(w)|^2(1-|w|)^{-2\beta}}{|1-\langle\overline{w},z_1\rangle|^2}\,\mathrm{d}m_2(w)\biggr)^{\frac12}\\ \times\biggl(\int_U \frac{(1-|w|)^{2\beta}}{|1-\langle\overline{w},z_2\rangle|^{2+2\alpha}}\,\mathrm{d}m_2(w)\biggr)^{\frac12},\end{aligned}$$

then we have

$$\begin{aligned}&\sup_{z_1,z_2\in\Gamma_\gamma(\xi)}|\mathcal{D}^\alpha_{z_2}\Phi(z_1,z_2)|(1-|z_2|)^{\alpha-\beta}(1-|z_1|)^\beta\\ &\leqslant C\sup_{z_1\in\Gamma_\gamma(\xi)}\biggl(\int_U \frac{|f(w)|^2(1-|w|)^{-2\beta}}{|1-\langle\overline{w},z_1\rangle|^2}\,\mathrm{d}m_2(w)(1-|z_1|)^{2\beta}\biggr)^{\frac12}=G_1(f),\ \text{(see [4])}.\end{aligned}$$

Note that

$$(|1-\langle|\lambda|\overline{\xi},z\rangle|)\asymp(|1-\langle\overline{\lambda},z\rangle|),\quad z\in U,\ \lambda\in\Gamma_\beta(\xi).$$

Hence

$$G_1(f)(\xi)\lesssim C\sup_{0<r<1}\biggl(\int_U \frac{(1-|z|)^{-2\beta}|f(z)|^2(1-r)^{2\beta}}{|1-\langle r\overline{\xi},z\rangle|^2}\,\mathrm{d}m_2(z)\biggr)^{\frac12}=\widetilde{G_1}(f,\xi,\beta).$$

Obviously for $\gamma\in(1,2-2\beta)$, $\beta\in(0,\frac12)$,

$$\widetilde{G_1}(f,\xi,\beta)\lesssim C\sup_{0<r<1}\biggl(\int_T \frac{(1-r)^{\gamma-1}|f(r\xi)|^2}{|1-\langle r\xi,\varphi\rangle|^\gamma}\,\mathrm{d}m(\xi)\biggr)^{\frac12}.$$

Therefore it is enough to use the estimates for Stein-type maximal functions [18]

$$\Big\| \sup_{0<r<1} \Big(\int_T \frac{(1-r)^{\alpha-1}|f(r\varphi)|^p}{|1-\langle r\overline{\varphi},\xi\rangle|^\alpha} \,\mathrm{d}\varphi \Big)^{\frac{1}{p}} \Big\|_{L^p} \leqslant C\|f\|_{H^p},$$

$f \in H^p$, $p>1$, $\beta \in (0,1/p)$, $\alpha \in (1, 2-\beta p)$, to get what we need. So the proof of the first estimate is complete.

Let us prove the second estimate. First, we have the following chain of known estimates (see for example [4]).

$$\int_U \mathrm{d}\mu(z) \lesssim C \int_T \int_{\Gamma_t(\xi)} \frac{\mathrm{d}\mu(z)}{1-|z|} \,\mathrm{d}m(\xi), \tag{4}$$

$$\int_T |M_{H-L}f(\xi)|^p \,\mathrm{d}\xi \leqslant C \int_T |f(\xi)|^p \,\mathrm{d}\xi, \quad p>1, \tag{5}$$

where $M_{H-L}f$ is the classical Hardy-Littlewood maximal operator, and

$$\int_U |f(z)|^{\tilde{p}} \,\mathrm{d}m_2(z) \lesssim C \int_T (\sup_{z\in\Gamma_t(\xi)} |f(z)|)^{\tilde{p}} C(\mu)(\xi)\,\mathrm{d}\xi, \tag{6}$$

where μ is a positive Borel measure, $0<\tilde{p}<\infty$, f is measurable in U and as usual $C(\mu)(\xi) = \sup\limits_{\xi\in I} |I|^{-1}\int_{\triangle I} \mathrm{d}\mu(\xi)$, $\triangle I = \{z = r\xi,\ \xi \in I,\ 1-|z|<r<1\}$, $I \subset T$. Using (4) we have

$$|\Phi(z_1,z_2)| \lesssim C(\alpha) \int_U \frac{|f(w)|(1-|w|)^\alpha}{(1-\langle z_1,\overline{w}\rangle)^{\frac{\alpha+2}{2}}(1-\langle z_2,\overline{w}\rangle)^{\frac{\alpha+2}{2}}} \,\mathrm{d}m_2(w),$$

where $z_1, z_2 \in U$ and $C(\alpha)$ is the Bergman projection constant. Further on, by using (4) and applying Hölder's inequality twice, we get

$$\begin{aligned} |\Phi(z_1,z_2)| &\lesssim C\Big(\int_T \Big(\int_{\Gamma_\alpha(\xi)} \frac{|f(w)|^2(1-|w|)^{2\alpha-t}\,\mathrm{d}m_2(w)}{(1-|w|)^2|1-\langle z_1,\overline{w}\rangle|^{\alpha+2}} \Big)^{\frac{p}{2}} \mathrm{d}\xi \Big)^{\frac{1}{p}} \\ &\quad \times \Big(\int_T \Big(\int_{\Gamma_\alpha(\xi)} \frac{(1-|w|)^t\,\mathrm{d}m_2(w)}{|1-\langle z_2,\overline{w}\rangle|^{\alpha+2}} \Big)^{\frac{q}{2}} \mathrm{d}m(\xi) \Big)^{\frac{1}{p}} \lesssim B(f)(1-|z_2|)^{-(\frac{\alpha-t}{2}-\frac{1}{q})}, \end{aligned}$$

$p>2$, $\alpha > t+2/q$, $t\in(-2,-1)$, $\alpha > t/2$, $1/p+1/q=1$.

Using Fubini's theorem and a duality argument

$$\begin{aligned} B(f) &= \sup_{\|\varphi\|_{L(\frac{p}{2})'}} \int_T \int_{\Gamma_\alpha(\xi)} \frac{|f(w)|^2(1-|w|)^{2\alpha-t}\,\mathrm{d}m_2(w)}{(1-|w|)^2|1-\langle z_1,\overline{w}\rangle|^{\alpha+2}} |\psi(\xi)|\,\mathrm{d}m(\xi) \\ &= \sup_{\|\varphi\|_{L(\frac{p}{2})'}} \int_U \frac{|f(w)|^2(1-|w|)^{2\alpha-t}}{|1-\langle z_1,\overline{w}\rangle|^{\alpha+2}} \int_T |\psi(\xi)|\chi_{\Gamma_\alpha(\xi)}(z)\,\mathrm{d}\xi \frac{\mathrm{d}m_2(w)}{(1-|w|)^2}. \end{aligned}$$

Hence using (6) and the estimate

$$\sup_{z\in\Gamma_\eta}\frac{1}{1-|z|}\int_T|\varphi(\xi)|\chi_{\Gamma_\tau(\xi)}(z)\,\mathrm{d}m(\xi)\leqslant CM_{H-L}(\varphi)(\xi),\quad \text{(see [4])},$$

we have ($\tilde f=f(1-|w|)^{\alpha/2}$)

$$\begin{aligned}B(f)&\lesssim\sup_\varphi\int_T(A_\infty(\tilde f)(\xi))^2M_{H-L}(\varphi)(\xi)C\left(\frac{(1-|w|)^{\alpha-t-1}}{|1-\langle z,w\rangle|^{\alpha+2}}\right)(\xi)\,\mathrm{d}\xi\\&\lesssim\sup_\varphi\int_T(A_\infty(\tilde f)(\xi))^2M_{H-L}(\varphi)(\xi)\,\mathrm{d}\xi\sup_{\tilde w\in U}\int_U\frac{(1-|w|)^{\alpha-t-1}(1-|\tilde w|)^N\,\mathrm{d}m_2(w)}{|1-\langle z_1,\tilde w\rangle|^{\alpha+2}|1-\langle\tilde w,w\rangle|^{N+1}},\end{aligned}$$

where M_{H-L} is the Hardy-Littlewood maximal function. We used the fact that

$$\|C(F)\|_{L^\infty}=\sup_{\tilde w\in U}\int_U\frac{|F(z)|\,\mathrm{d}m_2(z)}{|1-\langle\tilde w,z\rangle|^N}(1-|\tilde w|)^{N-1},\ N>1.$$

From the last estimate, Hölder's inequality and (5) we finally get

$$|\Phi(z_1,z_2)|(1-|z_1|)^{t+2}|(1-|z_2|)^{\frac{\alpha-t}{2}-\frac{1}{q}}\leqslant C\|\tilde f\|_{L^p},$$

$t\in(-2,-1)$, $p>2$. The proof of Theorem 6 is complete. □

Sharp diagonal mapping theorems can be obtained in various spaces of harmonic functions. We give an example.

Theorem 7. *Let $0<p<\infty,\ \alpha_j>-1,\ j=1,\dots,n$. Then*

$$\mathrm{Diag}(\tilde h^p(\vec\alpha))=h^p\bigg(\sum_{j=1}^n\alpha_j+2n-2\bigg),\quad n\geqslant1.$$

Remark 6. Since the $\tilde h^p(\vec\alpha)$ classes contain the holomorphic Bergman spaces in the polydisk, Theorem 7 can be considered as an extension of the theorem on the diagonal map in $A^p_\alpha(U^n)$—the Bergman classes in the polydisk (see [5]).

P r o o f of Theorem 7. The proof of Theorem 7 almost repeats the corresponding proof for the classical Bergman classes in the polydisk (see, for example, [12]). We add some needed remarks. For every harmonic function v such that

$$\int_U|v(z)|^p(1-|z|)^\alpha\,\mathrm{d}m_2(z)<\infty,\quad \alpha>1,\ 0<p<\infty,$$
$$v(z)=C(\beta)\int_Uv(w)(1-|w|^2)^{n(\beta+2)-2}\operatorname{Re}\Big(\frac{1}{1-\langle\overline w,z\rangle}\Big)^{n(\beta+2)}\,\mathrm{d}m_2(w).$$

Let $u(z_1,\dots,z_n) = C(\beta)\int_U v(w)(1-|w|^2)^{n(\beta+2)-2}\operatorname{Re}\Big(\prod_{k=1}^{n}(1-\langle z_k,\overline{w}\rangle)^{\beta+2}\Big)^{-1}\times$ $\mathrm{d}m_2(w)$.

Note that$u(z,\dots,z) = v(z)$, $z\in U^n$ and u is a pluriharmonic function. Indeed to prove the last assertion we have

$$v(w) = \sum_{k=-\infty}^{\infty} C_k \varrho^{|k|}\mathrm{e}^{\mathrm{i}k\theta},\quad w=\varrho\mathrm{e}^{\mathrm{i}\theta},\ z=r\mathrm{e}^{\mathrm{i}\varphi}$$

$$u(z_1,\dots,z_n) = C(\beta)\int_0^1\int_{-\pi}^{\pi}\sum_{k=-\infty}^{\infty} C_k\varrho^{|k|}\mathrm{e}^{\mathrm{i}k\theta},$$

$$(1-\varrho^2)^{n(\beta+2)-2}\sum_{\substack{(k_1,\dots,k_n)\\ \in\mathbb{Z}_+^n\cup\mathbb{Z}_-^n}}\frac{\Gamma(\beta+|k|+2)}{\Gamma(\beta+2)\Gamma(|k|+1)} r_1^{|k_1|}\dots r_n^{|k_n|}\prod_{j=1}^{n}\mathrm{e}^{\mathrm{i}k_j\varphi_j}\varrho^{|k_j|}e^{-ik_j\theta}\varrho\,\mathrm{d}\varrho\,\mathrm{d}\theta$$

$$= C(\beta)\sum_{\substack{(k_1,\dots,k_n)\\ \in\mathbb{Z}_+^n\cup\mathbb{Z}_-^n}} C_k\frac{\Gamma(\beta+|k|+2)}{\Gamma(\beta+2)\Gamma(|k|+1)}\prod_{j=1}^{n} r_j^{|k_j|}\mathrm{e}^{\mathrm{i}k_j\varphi_j}$$

$$\times\int_0^1 (1-\varrho^2)^{n(\beta+2)-2}\varrho^{2(\sum_{j=1}^{n}|k_j|)+1}\,\mathrm{d}\varrho$$

$$= \sum_{\substack{(k_1,\dots,k_n)\\ \in\mathbb{Z}_+^n\cup\mathbb{Z}_-^n}} C_{k_1,\dots,k_n} r_1^{|k_1|}\dots r_n^{|k_n|}\mathrm{e}^{\mathrm{i}k_1\varphi_1}\dots\mathrm{e}^{\mathrm{i}k_n\varphi_n},$$

hence u is pluriharmonic (see for example [11]). □

4. On some generalizations of diagonal mapping

The diagonal $(z,\dots,z)$ was generalized and studied relatively recently in [3]. Namely, in [3] Clark considered the map $(\mathcal{D}_B f) = f|_V = F$, where $f\in H(U^n)$, was based on finite Blaschke products,

$$V = \{(z_1,\dots,z_N)\in U^n\colon\ B_1(z_1) = \dots = B_N(z_N)\},$$
$$B_j(z) = \prod_{i=1}^{n_j}\frac{z-b_{ji}}{1-\overline{b_{ji}}z},\quad j=1,\dots,N,\ |b_{ji}|<1.$$

He also studied Bergman and Hardy spaces on the variety V, $A^p_\alpha(V)$ and $H^p(V)$ obtaining generalizations on known results of some diagonal mapping theorems for A^p_α, H^p for the $\mathcal{D}_B$ operator. Note that if $n_1 = \dots = n_N = 1$ and $b_{j1} = 0$ for all $j = 1,\dots,N$, then $(\mathcal{D}_B f) = f(z,\dots,z)$.

Our intention is to find new inequalities for holomorphic functions in U^n, which is its subvariety V. The following known estimate gives an example

$$\int_U |f(z,\dots,z)|^p(1-|z|)^{n-2}\,\mathrm{d}m_2(z) \leqslant C\|f\|_{H^p(U^n)},\quad 0<p<\infty,\ n>1 \text{ (see [12])}.$$

Obviously both sides of this estimate "tend" to $H^p(U)$ for $n=1$.

To get similar inequalities for the variety V, we will need some assertions from Clark's paper.

Any analytic function in V can be represented as

$$F(z_1,\dots,z_N)=\sum_{\nu=1}^{\infty} f_\nu(z_1,\dots,z_N)(B_1(z_1))^\nu,$$

$(z_1,\dots,z_N)\in V$, where f_ν belongs to an $n_1\dots n_N$ dimensional subspace of $H^2(U^N)$. Let

$$f_{[r]}(z_1,\dots,z_N)=\sum_{\nu=1}^{\infty} f_\nu(z_1,\dots,z_N)r^\nu(B_1(z_1))^\nu,\ r\in(0,1].$$

In [3] Clark defined Hardy space quasinorm on V as follows

$$\begin{aligned}\sup_{0\leqslant r<1}\|f_{[r]}\|^p_{H^p(U)} &= \sup_{0\leqslant r<1}\int_0^{2\pi}\int |f_{[r]}(w_1,\dots,w_N)|^p\,\mathrm{d}\mu_{\mathrm{e}^{\mathrm{i}\theta}}(w)\,\mathrm{d}\theta\\ &= \sup_{0\leqslant r<1}\int_T\sum_{k\geqslant 0}|f^k_{[r]}(\theta)|^p\left(\prod_{j=1}^N B_j'(\eta_{jk})\right)^{-1}\mathrm{d}\theta,\end{aligned}$$

where $\mathrm{d}\mu_{\mathrm{e}^{\mathrm{i}\theta}}$ is an appropriate atomic measure in the mentioned representation and $B_j(\eta(\lambda))=\lambda$, $j=1,\dots,N$, $\lambda=r\mathrm{e}^{\mathrm{i}\theta}\in U$, $B_j'(\eta_{jk})\neq 0$, $j=1,\dots,N$. For there are $n_1,\dots,n_N$ points $(\eta_k)=(\eta_{1k}(\lambda),\dots\eta_{Nk}(\lambda))$ such that $B_j(\eta_{jk})=\lambda$, $j=1,\dots,N$,

$$f^k_{[r]}(\theta)=f_{[r]}(\eta_{1k}(\lambda),\dots\eta_{Nk}(\lambda)) \text{ (see [3])}.$$

$\|f_{[r]}\|_{H^2(V)}$ is increasing and

$$\begin{aligned}(7)\qquad \|f_{[r]}\|^2_{H^2(V)} &\asymp \sum_k\int_0^{2\pi}|f(\eta_{1k}(r\mathrm{e}^{\mathrm{i}\theta}),\dots\eta_{Nk}(r\mathrm{e}^{\mathrm{i}\theta}))|^2\,\mathrm{d}\theta\\ &\asymp \int_0^{2\pi}\|h(z_1,\dots,z_N,r\mathrm{e}^{\mathrm{i}\theta})\|^2_{H^2(U^n)}\,\mathrm{d}\theta,\ 0\leqslant r<1,\ \text{(see [3])},\end{aligned}$$

where

$$h(z_1,\dots,z_N,\lambda)=\sum_k f(\eta_k(\lambda))\prod_{j=1}^N\frac{\lambda-B_j(z_j)}{(\eta_{jk}(\lambda)-z_j)}\frac{1}{|B_j'(\eta_{jk}(\lambda))|},$$

$z_j\in U,\ j=1,\dots,n,\ \lambda\in U$, where the sum on k is over all η_k.

Remark 7. Note that when $V = \{(z,\dots,z)\colon z\in U\}$ then

$$h(z_1,\dots,z_N,\lambda) = (h(z)) = f(z,\dots,z), \quad z\in U$$

and in (7) all expressions are equivalent to $\int_T |f(r\xi,\dots,r\xi)|^p \,\mathrm{d}\xi$.

We define Bergman classes on V as follows (see [3])

$$A^{p,N}(V) = \left\{ f\in H(V)\colon \int_0^1 \|f_{[r]}(z_1,\dots,z_N)\|_{p,V}^p (1-r^2)^\alpha r\,\mathrm{d}r < \infty \right\},$$

$1\leqslant p<\infty$, $\alpha\geqslant 0$. For the diagonal case $(z,\dots,z)$ we have

$$\|f\|_{A^{p,N}}^p = \int_0^1\int_T |f(r\xi,\dots,r\xi)|^p (1-r)^\alpha \,\mathrm{d}r\,\mathrm{d}\xi.$$

Theorem 8. (a) *Let $\alpha > m-1$, $f\in H(V)$. Then*

$$\|f\|_{A^{2,n,V}}^2 \lesssim C\int_0^1\dots\int_0^1\int_T \|f_{[R_1,\dots,R_m]}\|_{2,V}^2 \prod_{j=1}^m (1-R_j)^{\frac{\alpha+1}{m}-1}\,\mathrm{d}R_1\dots\mathrm{d}R_m.$$

(b) *Let $F\in H(V)$ and $\int_0^{2\pi}\int \sup\limits_{r<1} |F_{[r]}(w)|\,\mathrm{d}\mu_{\mathrm{e}^{\mathrm{i}\theta}}(w)\,\mathrm{d}\theta<\infty$. Then there is a function $f\in H(U^N)$, $f|_V = F$ and*

$$|f(z_1,\dots,z_N)|(1-|z_1|)\dots(1-|z_N|)\lesssim C\int_0^{2\pi}\int \sup_{r<1}|F_{[r]}(w)|\,\mathrm{d}\mu_{\mathrm{e}^{\mathrm{i}\theta}}(w)\,\mathrm{d}\theta<\infty.$$

Proof. Since $\|f_{[r]}\|_{2,V}$ is increasing (see [3]) we have the following chain of estimates

$$\begin{aligned}
\|f\|_{A^{2,n,V}}^2 &= \int_0^1 \|f_{[r]}\|_{2,V}^2 (1-r)^\alpha r\,\mathrm{d}r = \sum_{k=0}^\infty \left(\int_{1-2^{-k}}^{1-2^{-(k+1)}} \|f_{[r]}\|_{2,V}^2 r\,\mathrm{d}r\right) 2^{-k\alpha}\\
&\leqslant C\sum_{k_1=0}^\infty\dots\sum_{k_m=0}^\infty (2^{\frac{-k_1\alpha}{m}}\dots 2^{\frac{-k_m\alpha}{m}})(2^{\frac{-k_1}{m}}\dots 2^{\frac{-k_m}{m}}) \sup_{r_j\in I_{k_j}} \|f_{[\tilde r]}\|_{2,V}^2\\
&\lesssim C\int_0^1\dots\int_0^1 \|f_{[\tilde r]}\|_{2,V}^2 \prod_{j=1}^m (1-r_j)^t\,\mathrm{d}r_1\dots\mathrm{d}r_m,
\end{aligned}$$

where $I_{k_j} = (1-2^{-k_j}, 1-2^{-(k_j+1)}]$, $[\tilde r] = [r_1^{1/m}\dots r_m^{1/m}]$, $t = (\alpha+1)/m-1$. Proof for the first part of the theorem is complete.

Let us prove the second estimate in our theorem. Since

$$\int_0^{2\pi}\int \sup_{r<1}|F_{[r]}(w)|\,\mathrm{d}\mu_{\mathrm{e}^{\mathrm{i}\theta}}(w)\,\mathrm{d}\theta<\infty,$$

F belongs to $A^{1,\alpha}(V)$ for any $\alpha\geqslant 0$, hence $F\in A^{1,N-2}(V)$ and we have the following integral representation

$$\begin{aligned}&f(z_1,\,\dots,z_N)\\&\quad=C(N)\int_0^1\int_0^{2\pi}\int F_{[r]}(w)\prod_{j=1}^{N}\frac{1-\overline{B_j}(w_j)B_j(z_j)}{1-\langle\overline{w_j},z_j\rangle}\frac{\mathrm{d}\mu_{\mathrm{e}^{\mathrm{i}\theta}}(w)(1-r^2)^{N-2}r\,\mathrm{d}r\,\mathrm{d}\theta}{1-r\overline{B_j}(w_j)B_j(z_j)},\end{aligned}$$

$z_j\in U$, $f\in H(U^N)$ and $f|_V=F$ (see [3]).

Moreover, we obviously have the following estimates using Lemma 2

$$\begin{aligned}|f(z_1,\dots,z_N)|&\leqslant C\int_0^{2\pi}\int\sup_{r<1}|F_{[r]}(w)|\prod_{j=1}^{N}\frac{|1-\overline{B_j}(w_j)B_j(z_j)|}{|1-\langle\overline{w_j},z_j\rangle|}\frac{\mathrm{d}\mu_{\mathrm{e}^{\mathrm{i}\theta}}(w)\,\mathrm{d}\theta}{|1-\overline{B_j}(w_j)B_j(z_j)|^{\frac{1}{N}}}\\&\lesssim C\int_0^{2\pi}\int\sup_{r<1}|F_{[r]}(w)|\,\mathrm{d}\mu_{\mathrm{e}^{\mathrm{i}\theta}}(w)\,\mathrm{d}\theta\frac{1}{\prod\limits_{k=1}^{N}(1-|z_k|)}.\end{aligned}$$

The theorem is proved. □

Remark 8. Both estimates in Theorem 8 were previously known for particular values of V, (see [5], [8], [17]). If $V=\{(z,\dots,z)\}$ then the first estimate gets the following form (see [17])

$$\begin{aligned}&\int_0^1\int_T|\mathcal{D}f(z)|^p(1-r)^{\alpha}\,\mathrm{d}r\,\mathrm{d}\xi\\&\qquad\leqslant C\int_0^1\dots\int_0^1\int_T|f(R_1\xi,\dots,R_n\xi)|^p\prod_{k=1}^{m}(1-R_k)^{\frac{\alpha+1}{m}-1}\,\mathrm{d}R_1\dots\mathrm{d}R_m\,\mathrm{d}\xi,\end{aligned}$$

$0<p<\infty$, $\alpha>-1$. The second estimate for $V=U$ coincides with the classical inequality of the theory of H^p classes $\sup\limits_{z\in U}|f(z)|(1-|z|)\leqslant C\|f\|_{H^1}$ (see [5]).

Acknowledgement. We thank Trieu Le for some valuable discussions.

References

[1] *A. V. Aleksandrov*: Essays on non locally convex Hardy classes. Lecture notes in Mathematics (V. V. Havin and N. K. Nikolski, eds.). Springer-Verlag, Complex Analysis and spectral theory, 1981, pp. 1–99.

[2] *E. Amar and C. Menini*: A counterexample to the corona theorem for operators on $H^2(D^n)$. Pacific J. Math. *206* (2002), 257–268.

[3] *D. Clark*: Restrictions of H^p functions in the polydisk. Amer. J. Math. *110* (1988), 1119–1152.

[4] *R. Coifman, Y. Meyer and E. Stein*: Some new functional spaces and their applications to harmonic analysis. J. Funct. Anal. *62* (1985), 304–335.

[5] *A. E. Djrbashian and F. A. Shamoyan*: Topics in the Theory of A^p_α Spaces. Leipzig, Teubner, 1988.

[6] *P. L. Duren and A. L. Shields*: Restriction of H^p functions on the diagonal of the polydisk. Duke Math. J. *42* (1975), 751–753.

[7] *L. Grafakos*: Classical and modern Fourier analysis. Prentice Hall, 2004.

[8] *M. Jevtic, M. Pavlovic and R. Shamoyan*: A note on diagonal mapping theorem in spaces of analytic functions in the unit polydisk. Publ. Math. Debrecen *74/1–2* (2009), 1–14.

[9] *V. Mazya*: Sobolev Spaces. Springer-Verlag, New York, 1985.

[10] *G. Ren and J. Shi*: The diagonal mapping in mixed norm spaces. Studia Math. *2* (2004), 103–117.

[11] *W. Rudin*: Function theory in polydisks. Benjamin, New York, 1969.

[12] *F. A. Shamoian*: Embedding theorems and characterization of traces in the spaces $H^p(U^n)$, $p \in (0, \infty)$. Mat. Sbornik, *3* (1978), 709–725. (In Russian.)

[13] *F. A. Shamoian*: Diagonal mapping and questions of representations in spaces of holomorphic functions in polydisk. Siberian Math. J. *31* (1990), 197–215. (In Russian.)

[14] *R. F. Shamoyan*: On the action of Hankel operators in bidisk and subspaces in H^∞ connected with inner functions in the unit disk. Doklady BAN, Bulgaria *60* (2007), 929–934.

[15] *E. Seneta*: Regularly varying functions. Springer-Verlag, New York, 1976.

[16] *J. Shapiro*: Mackey topologies, reproducing kernels and diagonal maps on the Hardy and Bergman spaces. Duke Math. J. *43* (1976), 187–202.

[17] *Songxiao Li and R. F. Shamoyan*: On some properties of a differential operator on the polydisk. Banach J. Math. Anal. (2009), 68–84.

[18] *I. Verbitsky*: Multipliers in spaces with fractional norms and inner functions. Siberian Math. J. *150* (1985), 51–72. (In Russian.)

Authors' addresses: Romi F. Shamoyan, Erevan State University, Armenia, e-mail: rshamoyan@yahoo.com; Olivera R. Mihić, Faculty of Organizational Sciences, University of Belgrade, Jove Ilića 154, Belgrade, Serbia, e-mail: oliveradj@fon.rs.